Algebra II
Department of Mathematics and Statistics
Problem sheet 9 (2 pages)
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1. Prove Lemma 8.14: Assume that R are S are rings, and that there exists a ring homomorphism $R \to S$. If $M = R^n$, then M_S and S^n are isomorphic as S-modules. (You can use Theorem 8.11.)

2. Let A be an associative and unital R-algebra. Show that there exists an R-algebra homomorphism $\varphi : R \to A$, such that $1_R \mapsto 1_A$. If R is a field, show that R can be embedded as a subalgebra of A.

3. Consider the quaternion algebra $\mathbb{H}$.
   a) Show that each element $x \in \mathbb{H} \setminus \{0\}$ has a multiplicative inverse.
   b) Suppose $x = x_1i+x_2j+x_3k$ and $y = y_1i+y_2j+y_3k$. Write the quaternion xy using the dot and cross products of the vectors (x_1, x_2, x_3) and (y_1, y_2, y_3).

4. Let R be a ring, and suppose M is an R-module. The *tensor algebra* $T(M)$ is defined as a direct sum
$$\bigoplus_{k=0}^{\infty} T_k(M),$$
where $T_0(M) = R$ and $T_{k+1}(M) = T_k(M) \otimes M$ for all $k \geq 0$. Identify each $T_k(M)$ with the corresponding submodule $\iota_k(T_k(M)) \subset T(M)$, where ι_k is the canonical injection. Identify also the modules $R \otimes M$ and M via the familiar isomorphism. Show that $(x, y) \mapsto x \otimes y$ is a well defined bilinear multiplication in the R-module $T(M)$, and that with respect to this multiplication, $T(M)$ is an associative and unital R-algebra.

   Hint: An arbitrary element of the module $T(M)$ has the form $a + \sum_{k=1}^{\infty} x_k$, where $a \in R$, $x_k = x_k^1 \otimes \cdots \otimes x_k^k$ for all $k \geq 1$, and $x_k \neq 0$ for only finitely many indices k.

5. Prove that there are only three non-isomorphic 2-dimensional unital algebras with real coefficients.

   Hint: Identify the subspace generated by the unit element with the scalar field $\mathbb{R}$ (cf. Question 2). Choose 1 as the other basis vector and some $b \notin \mathbb{R}$ as the other. Consider the multiplication table of the basis vectors. If $b^2 = x + yb$ for some $x, y \in \mathbb{R}$, define $b' = b - y/2$. Thus, you may assume that $b^2 \in \mathbb{R}$.

6. Consider the real group algebra $\mathbb{R}S_3$ over the symmetric group S_3.

   a) Find a one-dimensional subalgebra of $\mathbb{R}S_3$.

   b) The algebra $\mathbb{R}S_3$ is also an $\mathbb{R}S_3$-module, when the scalar multiplication is defined by $x.y = x \cdot y$. From this module, find a one-dimensional $\mathbb{R}S_3$-submodule that is different from the subalgebra found in part a).

   Hint: In part a), find a vector $\sum_{\sigma \in S_3} x_\sigma \sigma$, such that the subspace it generates is closed under the algebra multiplication.