Algebra II
Department of Mathematics and Statistics
Problem sheet 10 (2 pages)
Thu 22.4.2010

1. Let R be an integral domain. Prove the following claims (Lemma 11.1):

   a) The elements $a, b \in R$ are associates if and only if $a = bc$, where $c \in R$ is a unit.

   b) If $a, b \in R$ are associates, and $a = bc$, then c is a unit.

   c) All units are associates.

2. Show that in the ring $\mathbb{Z}[i\sqrt{5}] = \{a+bi\sqrt{5} \mid a, b \in \mathbb{Z}\}$, 2 is an irreducible number that divides the product $(1 + i\sqrt{5})(1 - i\sqrt{5})$, but does not divide its factors. Deduce that $\mathbb{Z}[i\sqrt{5}]$ is not a unique factorisation domain.

   Hint: If 2 is not irreducible, then $2 = (a+bi\sqrt{5})(c+di\sqrt{5})$ for some $a, b, c, d \in \mathbb{Z}$. Consider the modules (i.e. lengths) of the complex numbers appearing in the equation.

3. a) Let R be a ring, and let $b \in R$. Show that the map $\tau_b : R[X] \to R[X]$, where $\sum_i a_i X^i \mapsto \sum_i a_i (X + b)^i$, is a ring isomorphism. Deduce that $f \in R[X]$ is irreducible if and only if $\tau_b(f)$ is irreducible.

   b) Let p be a prime. Show that $X^p - 1 = (X - 1)g$, where $g \in R[X]$ is irreducible over $\mathbb{Q}$.

   Hint: Consider the polynomial $\tau_1(X^p - 1)$ and use Eisenstein's Criterion.

4. a) Let R be a ring. Show that $R[X, Y] \cong R[X][Y]$ (as R-algebras).

   b) Let L be an extension of the field K, and let a and b be elements of L. Show that $K(a, b) = K(a)(b)$.

5. In the following cases, determine the degree of the subextension $K(A)$ of L/K generated by the set A:

   a) $K = \mathbb{Q}$, $L = \mathbb{R}$, $A = \{\sqrt{2}\}$.

   b) $K = \mathbb{Q}$, $L = \mathbb{C}$, $A = \{\sqrt{2}, i\}$.

   c) $K = \mathbb{F}_2$, $L = \mathbb{F}_2[X]/\langle X^5 + X^3 + 1\rangle$, $A = \{\overline{X}^2\}$.

   Hint: $[K(A) : K]$ is a factor of $[L : K]$.

6. Find out which of the following polynomials are irreducible in the ring $\mathbb{Q}[X]$:

(a)	$X^3 + 2X^2 + X - 4$	(d)	$-3X^5 + 6X^3 - 2$
(b)	$X^3 + 2X^2 + X - 5$	(e)	$X^6 - 5X + 10$
(c)	$7X^4 + X^3 - 2X^2 + 6X + 1$	(f)	$2X^3 + 5X^2 + 6X + 24.$

Furthermore, show that the polynomials $3XY^2 - XY + 2$ and $X^2 - Y$ are irreducible in the ring $\mathbb{Q}[X, Y] \cong \mathbb{Q}[X][Y]$. (You can use Eisenstein's Criterion.)