Algebra II
Department of Mathematics and Statistics
Problem sheet 12
Thu 29.4.2010

1. Find the following extension degrees:

   a) $[\mathbb{Q}(e^{2\pi i/5}) : \mathbb{Q}]$
   b) $[\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}]$
   c) $[\mathbb{Q}(i, e^{\pi i/3}) : \mathbb{Q}]$
   d) $[\mathbb{F}_2(\overline{X}^5) : \mathbb{F}_2]$, where $\mathbb{F}_2(\overline{X}^5) \subset \mathbb{F}_2[X]/\langle X^4 + X^3 + 1\rangle$
   e) $[\mathbb{Q}(e^{2\pi i/9}) : \mathbb{Q}]$.

   Hint: (c) You can show that $\mathbb{Q}(i, e^{\pi i/3}) = \mathbb{Q}(i, \sqrt{3})$. (d) Since the multiplicative group K^* is finite, there is a positive integer n, such that $(\overline{X}^5)^n = 1$.

2. Prove Theorem 13.7: A finite extension of a field K is finitely generated and algebraic over K.

3. Let $A = \{2^{1/n} \mid n \in \mathbb{N}\}$. Show that $\mathbb{Q}(A)$ is an infinite algebraic extension of $\mathbb{Q}$. Does the extension $\mathbb{Q}(A)/\mathbb{Q}$ have a finite set of generators?

4. Assume known the transcendentality of π over the rational numbers. Show that both squaring (or quadrature) of the circle and doubling the cube are impossible as compass and straightedge constructions.

5. Suppose L is an extension of K, and let $A \subset L$ be an arbitrary set of elements algebraic over K. Show that $K(A)/K$ is an algebraic extension. (Remember Theorem 12.7.)

6. Let $\mathbb{A} \subset \mathbb{C}$ be the set of all numbers algebraic over $\mathbb{Q}$. Prove that $\mathbb{A}$ is *algebraically closed* in $\mathbb{C}$, that is, if $\alpha \in \mathbb{C}$ is a root of some polynomial with coefficients in $\mathbb{A}$, then $\alpha \in \mathbb{A}$.