Algebra II
Department of Mathematics and Statistics
Problem sheet 13 (final sheet)
Thu 6.5.2010

1. a) Show that the set $\mathbb{A}$ of algebraic numbers is a field. (You can use results proved in lectures and problem sessions.)

   b) Let Ω be the algebraic closure of the prime field $\mathbb{F}_p$. Show that the set of roots of the polynomial $X^{p^n} - X$ in Ω is a field.

2. Prove that every algebraically closed field is infinite.

3. In the following cases, find a splitting field of the polynomial f over the field K, and determine its degree as an extension over K:

   a) $f = X^3 - 2$, $K = \mathbb{Q}$

   b) $f = X^4 - 7$, $K = \mathbb{Q}$

   c) $f = X^4 - 7$, $K = \mathbb{F}_{11}$.

4. Let Ω be the algebraic closure of a field K. An algebraic extension $L \subset \Omega$ of K is called *normal*, if for every K-automorphism $\sigma : \Omega \to \Omega$ it holds that $\sigma(L) \subset L$. Show that the splitting field of any set of polynomials over K is a normal extension of K.

5. Show that $\mathbb{Q}(\sqrt{2}) \ncong \mathbb{Q}(\sqrt{3})$ (as fields).

6. In the following cases, determine the Galois group of the extension L/K. If the extension is a Galois extension, find all intermediate extensions L/M, where $K \subset M \subset L$.

   a) $L = \mathbb{Q}(\sqrt{2})$, $K = \mathbb{Q}$

   b) $L = \mathbb{Q}(\sqrt[3]{2})$, $K = \mathbb{Q}$

   c) $L = \mathbb{Q}(\sqrt{2}, i)$, $K = \mathbb{Q}$.