Representation Growth in Finite Quasisimple Groups

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Representation growth

- Studying finite-dimensional complex representations
- Let G be a group
- Definition: $r_n(G)$ is the number of (inequivalent) n-dimensional irreducible representations of G
- Example: $r_1(S_3) = 2$, $r_2(S_3) = 1$ and $r_n(S_3) = 0$ for $n \geq 3$
- Idea is to examine the growth rate of $r_n(G)$ when n becomes large

Auxiliary notion: Zeta functions

- A *representation zeta function* of a group G is the following sum taken over irreducible characters of G:

$$\zeta_G(s) \quad = \sum_{\chi \in Irr(G)} \chi(1)^{-s} \quad = \quad \sum_{n=1}^{\infty} \frac{r_n(G)}{n^s}$$

- In general, this is a complex analytic function which may not be defined everywhere
- $\zeta(0)$ is the number of irreducible characters (or conjugacy classes)
- $\zeta(-2) = |G|$ (if G is finite)

Theorem 1 (M. Liebeck, A. Shalev, 2003)

- Let L be any Lie type, with rank r and the number of positive roots u
- Let $L(q)$ denote a finite quasisimple group of type L over $\mathbb{F}(q)$
- Then, as $q \to \infty$,

$$\zeta_{L(q)}(s) \to \begin{cases} 0, & \text{if } s > r/u \\ \infty, & \text{if } s < r/u \end{cases}$$

- This means that for some constant c,

$$r_n(L(q)) < cn^{r/u} \qquad \text{for all } q$$

- Note: The constant depends on the Lie type, but not on q
- Application: Random walks in groups of Lie type

Digression: Quasisimple groups

- $G = G'$ and $G/Z(G)$ is simple
- Finite simple groups are classified
- Interesting families are *alternating groups* and *groups of Lie type*
- Any quasisimple group lies "between" a simple group and its *full covering group*
- Any representation of a quasisimple group (including the simple groups) is also a representation of the full covering group
- F.c. groups for simple groups of Lie type are typically pleasant to work with (e.g. $SL_n(q)$)

Proof of Theorem 1 (part I)

- Relies on Deligne-Lusztig theory of complex characters of algebraic groups
- Apart from finitely many "small" groups, the full covering group of a simple group of Lie type is the finite simply-connected fixed-point subgroup of the algebraic group of the same Lie type
- The characters are partitioned into *Lusztig series* $\mathcal{E}(s)$, parametrized by semisimple conjugacy classes s
- There is a bijective correspondence between characters in $\mathcal{E}(s)$ and unipotent characters of the centralizer of s in the *dual group*

Proof of Theorem 1 (part II)

- Key formula is

$$\chi(1) = |G^* : C_{G^*}(s)|_{p'} \cdot (\psi_s(\chi))(1),$$

  where G^* is the dual group and $\psi_s(\chi)$ is the unipotent character corresponding to χ and s
- Need to estimate the number of semisimple conjugacy classes and the values $|G^* : C_{G^*}(s)|_{p'}$
- The number of positive roots is used to estimate the centralizer index

Theorem 2 (Generalisation, J.H.)

- Liebeck and Shalev had proved similar results for alternating groups and their full covering groups
- For $n > 1$, define

$$s_n = \sum_{H \text{ quasisimple}} r_n(H)$$

- Theorem 2: There is a constant c such that $s_n < cn$
- More precisely, if those groups of Lie type that have $r/u > d$ are excluded, we get $s_n < cn^d$ ($r = u$ only for SL_2)

Proof of Theorem 2 (part I, alternating groups)

- Two cases: alternating groups and groups of Lie type
- Liebeck and Shalev (2003): for any ε there is a d_0 s.t. $r_n(A_d) < n^{\varepsilon}$ for $d \geq d_0$
- Characters of S_n are parametrized by partitions of d
- Considered different ranges for the first member in the partition to get

$$\sum_d r_n(A_d) < cn^{\varepsilon}$$

- Adding the characters of the covering group was not a problem, using results of A. Wagner (1977)

Proof of Theorem 2 (part II, groups of Lie type, large rank)

- The f.c. groups of simple groups of Lie type $H_r(q)$ are classified by their rank r and size of the field q, e.g. $SL_{r+1}(q)$
- Groups with large rank handled separately
- Key result 1: $\chi(1) > kq^r$ for some constant k (Landazuri and Seitz, 1974)
- Key result 2: for any $\varepsilon > 0$ there is r_0 s.t. $r_n(H_r(n)) < cn^\varepsilon$ when $r \geq r_0$ (Liebeck and Shalev, 2003)
- Moreover, the number of quasisimple groups under a f.c. group is bounded by r

Proof of Theorem 2 (part III, groups of Lie type, small rank)

- For groups of small rank, used Theorem 1 and the following
- Theorem 3 (Liebeck & Shalev, 2003): For a fixed Lie type (of fixed rank), there is a set of strictly increasing polynomials $\{f_1, \ldots, f_d\}$ s.t. if $H(q)$ is a f.c. group of this type and χ is an irreducible character of $H(q)$, then $\chi(1) = f_i(q)$ for some $i \leq d$.
- This means that for any n, there are at most d values of q that contribute

Application to Theorem 2

- Consider a classical group G of type SL_n, SO_n or Sp_n over $\mathbb{C}$
- The subgroups of G are partitioned into conjugacy classes
- If two n-dimensional representations are equivalent, the corresponding subgroups belong to the same class
- The conjugating element can be shown to lie in the *conformal* group of the given type
- Finally, conjugacy classes under conformal groups split into at most two classes under G
- Result: The number of conjugacy classes of finite irreducible quasisimple groups of G is less than cn

Further developments

- Generalising to *modular* representations (no Deligne-Lusztig theory available)
- Getting values for the constants (possible from the proofs, but extremely laborious)
- On the other hand, concrete information of the characters can be used to find a constant satisfying $s_n < cn$
- The groups $SL_2(q)$ are the only ones with $r/u = 1$
- For other groups, $r_n(G) < cn^d$ with $d < 1$, so there is some room for d
- So called *gap results* of character degrees can be used to estimate c, also in the modular case (but bound on d will become worse)