Representation Growth of Finite Quasisimple Groups

by

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"He wants profit from learning. Give him a penny!"

— Euclid, circa 300 B.C.

Abstract

In this thesis, we establish concrete numerical upper bounds for the representation growth of various families of finite quasisimple groups. Let G be a finite quasisimple group and let $r_n(G)$ denote the number of inequivalent irreducible n-dimensional linear representations of G. We describe certain infinite collections $\mathcal{C}$ of finite quasisimple groups and derive upper bounds to the growth of $r_n(G)$ as a function of n; the bounds hold for any G in $\mathcal{C}$. We also bound the total number $s_n(\mathcal{C})$ of inequivalent faithful irreducible n-dimensional representations of groups in $\mathcal{C}$.

Three cases are examined: the complex representation growth of alternating groups and their Schur covers, the complex representation growth of groups of Lie type, and the cross-characteristic modular representation growth of groups of Lie type. In all the cases, it is necessary to find lower bounds for the minimal dimensions of irreducible representations, and also to classify the representations of some of the smallest possible dimensions.

The main results are in all cases upper bounds to the growth of $r_n(G)$ or $s_n(\mathcal{C})$ for a given collection $\mathcal{C}$. All bounds have the form cn^s, where c and s are some constants that depend on the collection under study, with s being always at most 1.

The results are applied to a known problem concerning the number of conjugacy classes of maximal subgroups in classical groups. By Aschbacher's Theorem, the maximal subgroups of finite classical groups can be classified into so-called geometrical types, but there are some additional almost simple subgroups that do not fit into this classification. However, these almost simple subgroups are obtained from representations of quasisimple groups, and the number of conjugacy classes of such subgroups can be estimated by counting the number of irreducible representations.

Declaration of Originality

I hereby declare that this thesis and the work reported herein was composed by and originated entirely from me. Information derived from the published and unpublished work of others has been acknowledged in the text and references are given in the list of sources.

Copyright Declaration

Preface

Groups are algebraic systems that offer a mathematical description of the notion of symmetry. Whether the symmetry exists in the physical nature or in a mathematical construction, there is always a specific group that corresponds to it. It is often said that group theory was born around 1830, when the French mathematician Évariste Galois examined symmetries among roots of polynomials. By extracting general properties from these symmetries he could decide which polynomials admitted a formula for solving the roots and which did not.

When symmetries are examined abstractly as groups, the specific origin of the individual symmetries is lost. Then properties common to different symmetries can be classified, and one obtains new knowledge that was not apparent from the original context. However, sometimes one can get lost dealing with abstractions alone, and having a concrete realisation of the symmetry in view might bring the investigations back on track. Representations of groups can provide these realisations.

Linear representations of groups were first studied in the last years of the 19th century, and they have since been found extremely useful. They are interpretations of groups as symmetries of vector spaces: of lines, planes and multidimensional spaces. Linear algebra, the theory of vectors, provides an extensive number of tools and machinery for working with vector spaces, and a linear representation of a group enables one to use this machinery in study of abstract symmetry.

In this work, I have obtained numerical results related to the number of linear representations of certain classes of groups. The results bound the number of different representations that can exist in a vector space, by a number related to the dimension of that space. I hope the bounds will be useful in the study of the groups and of the symmetries they describe. On the other hand, it is also possible to look at the situation from the opposite end. Attesting to this, I have myself applied the results to obtain knowledge, not of the groups being represented, but of the spaces containing the representations. This application I have included in the final chapter of the thesis.

I have intended the text to be approachable for anyone familiar with the basics of modern algebra, especially group theory. I use quite freely such concepts as quotient groups, homomorphisms, centres and automorphisms, as well as fields, algebraic closures and characteristics. Most of what goes beyond these basic notions is explained in Chapter 2, but some theory that is specific to the individual results is left to the corresponding chapters. I have not proved any results that were known before, but instead offer references to the corresponding work in the literature. An exception is Proposition 3.13, which was indeed previously known, but for which I have given another kind of proof, using my own results.

* * *

I wish to thank my supervisor, Professor Martin Liebeck, for his never-faltering patience and support. He has always understood how difficult the work may be at times, and even more importantly, how necessary it is also to have other interests in life besides one's studies. I would also like to thank my wife, Kaisa Nuolioja, for supporting my venture without any hesitation and for providing many of those interests mentioned above. In addition to her, all my dear friends in Finland, London and abroad have made life more bearable during this long project. Moreover, I shall not forget my parents, who have always supported me and stayed close ever since I left home. To them I may add my aunt Raili, my grandparents, and also my godfather Pertti Rantiala, who first presented me with delightful glimpses of the world of logic and science. On a more pragmatic note, I am grateful to Jenny and Antti Wihuri Foundation as well as the Engineering and Physical Sciences Research Council for their generous financial support. I am also grateful for the Imperial College London for arranging such a stimulating study environment. Finally, special thanks go to Johanna Rämö, who as my faithful friend and colleague perhaps best understands what I have been through during these four years.

Helsinki, 17 June 2013
Jokke Häsä

Esipuhe

Ryhmät ovat algebrallisia järjestelmiä, joiden avulla erilaisisa symmetrioita voidaan käsitellä matemaattisesti. Symmetriaa voi tulla vastaan luonnossa tai se voi ilmetä jossakin matemaattisessa konstruktiossa, mutta yhtä kaikki jokaiseen havaittuun symmetriaan liittyy aina jokin tietty ryhmä. On sanottu, että ryhmäteoria sai alkunsa vuoden 1830 tietämillä, kun ranskalainen matemaatikko Évariste Galois tutki polynomien juurten välisiä symmetrioita. Eristämällä näistä symmetrioista yhteisiä piirteitä hän saattoi selvittää, millä polynomeista on ratkaisukaava ja millä ei.

Tutkimalla symmetrioita abstrakteina ryhminä voidaan jättää huomiotta niiden erilaiset alkuperät. Tällöin päästään luokittelemaan eri symmetrioille yhteisiä ominaisuuksia, ja sen myötä on mahdollista löytää säännönmukaisuuksia, jotka eivät alun perin olleet ilmeisiä. Toisinaan kuitenkin pelkkien abstraktioiden kanssa toimiminen saattaa eksyttää tutkijan, ja tällöin jonkin konkreettisen symmetrian ilmentymän tarkasteleminen voi auttaa palaamaan oikealle polulle. Ryhmien esitykset tarjoavat tällaisia ilmentymiä.

Ryhmien lineaaristen esitysten tutkimus alkoi 1800-luvun viimeisinä vuosina, ja siitä pitäen ne on havaittu erittäin hyödyllisiksi työkaluiksi. Ne mahdollistavat ryhmien tulkitsemisen vektoriavaruuksien kuten suorien, tasojen tai useampiulotteisten avaruuksien symmetrioina. Lineaarialgebra, joka tutkii vektorien teoriaa, tarjoaa suunnattoman määrän välineitä vektoriavaruuksien parissa työskentelyyn, ja ryhmän lineaarinen esitys mahdollistaa näiden välineiden käytön myös abstraktin symmetrian tutkimuksessa.

Tässä työssä kuvailen saavuttamiani tuloksia, jotka liittyvät tietyntyyppisten ryhmien lineaaristen esitysten lukumäärään. Tulokset rajoittavat kussakin vektoriavaruudessa tavattavien esitysten määrää luvulla, joka liittyy avaruuden ulottuvuuslukuun. Toivon, että havaintoni osoittautuvat hyödyllisiksi tarkastelemieni ryhmien ja niihin liittyvien symmetrioiden tutkijoille. Tilannetta voi kuitenkin tarkastella myös toisesta suunnasta, ja tähän liittyen olen itsekin käyttänyt tuloksiani apuna selvittäessäni eräitä ominaisuuksia, jotka kuuluvat – eivät ryhmille, joiden esityksiä tutkin – vaan avaruuksille, joihin esitykset sisältyvät. Kyseisen sovelluksen olen esittänyt kirjan viimeisessä luvussa.

Olen yrittänyt kirjoittaa tekstini siten, että sitä voisi ymmärtää jokainen modernin algebran, erityisesti ryhmäteorian, perusteet tunteva. Käytän vapaasti sellaisia termejä kuin tekijäryhmä, homomorfismi, ryhmän keskus ja automorfismi, samaten sellaisia kuin kunta, algebrallinen sulkeuma ja karakteristika. Näiden ulkopuolelle jäävät käsitteet on suurimmaksi osaksi selitetty luvussa 2, mutta sellainen teoria, joka liittyy läheisesti kuhunkin lopputulokseen, on esitetty samassa luvussa tuloksen kanssa. En ole todistanut uudelleen ennestään tunnettuja tuloksia, vaan viittaan sen sijaan vastaavaan teokseen alan kirjallisuudessa. Poikkeuksena tästä on lause 3.13, joka on tosin tunnettu jo aiemmin, mutta jolle olen esittänyt uudenlaisen todistuksen omien tulosteni pohjalta.

* * *

Haluan kiittää ohjaajaani professori Martin Liebeckiä hänen ehtymättömästä tuestaan ja kärsivällisyydestään. Hän on aina ollut ymmärtäväinen kohtaamieni haasteiden suhteen, ja mikä erityisen tärkeää, hän on jaksanut muistuttaa siitä, että elämässä on tärkeämpiäkin asioita kuin väitöskirjan kirjoittaminen. Haluan lisäksi kiittää vaimoani Kaisa Nuoliojaa siitä, että hän tarjosi epäröimättä tukensa pyrkimyksilleni. Hän myös liittyy suureen osaan noista mainituista tärkeämmistä asioista. Hän ja kaikki ystäväni Suomessa, Lontoossa ja ulkomailla ovat tehneet elämästä siedettävämpää tämän pitkän projektin aikana. En saata myöskään unohtaa vanhempiani, jotka ovat aina olleet tukenani ja pysyneet läheisinä siitä asti, kun lähdin kotoa. Heihin voisin lisätä myös Railitätini ja isovanhempani, sekä myös kummisetäni Pertti Rantialan, joka ensimmäisenä tutustutti minut tieteen ja loogisen ajattelun ihmeellisyyksiin. Käytännöllisemmällä tasolla olen kiitollinen Jenny ja Antti Wihurin säätiölle ja EPSRC-tiedeneuvostolle heidän avokätisestä taloudellisesta tuestaan. Yhtä lailla olen kiitollinen Imperial Collegelle inspiroivasta opiskeluympäristöstä. Lopuksi aivan erityinen kiitos kuuluu Johanna Rämölle, joka ystävänä ja kollegana ehkä parhaiten ymmärtää, mitä olen saanut kokea näiden neljän tapahtumarikkaan vuoden aikana.

Helsingissä, 17. kesäkuuta 2013
Jokke Häsä

Praefatio

Gruppi sunt compositiones algebraicae, quibus notio symmetriae mathematice describi ac tractari potest. Nihil refert, num symmetria in natura exstet an in aliqua constructione mathematica: ad quamvis symmetriam gruppus quidam semper pertinet. Dicunt theoriam grupporum circa annum 1830 ortam esse, cum mathematicus Francogallicus Evaristus Galois symmetrias inter radices expressionum polynominalium investigaret. Ex his symmetriis regulas communes exhibendo contigit ei, ut discerneret, quae aequationes polynomiales radicis adhibendis solvi possint quaeque non.

Cum symmetria per gruppum suum investigatur illis condicionibus ignoratis, quibus symmetria primum exorta est, et proprietates communes symmetriarum diversarum originum indicantur, novae cognoscitur regulae, quae primo visu non clare apparuerunt. Saepe autem accidit investigatori, ut cum rebus solum abstractis incumbat, via, quam sequeretur, obscuretur. Illis temporibus utile sit concretum habere exemplum symmetriae, quod rursus ad viam rectam ducere possit. Repraesentationes haec exempla parare possunt.

Repraesentationes lineares grupporum primum investigatae sunt annis ultimis saeculi XIX, inde a quo tempore mathematici eas maxime utiles putaverunt. Repraesentationes faciunt, ut gruppi sicut symmetriae spatiorum vectorum – linearum, planarum aut spatiorum plurium dimensionum – videantur. Algebra linearis, ut vocatur theoria vectorum, maximam copiam instrumentorum profert, quibus spatia vectorum investigentur, et repraesentatio linearis alicuius gruppi permittit, ut iisdem instrumentis etiam symmetria abstracta pertractetur.

In hoc opere inventiones novas propono, quae ad multitudinem repraesentationum certi generis grupporum attinent. Numeros repperi, qui multitudinem repraesentationum in aliquo spatio vectorum exsistentium terminant, pendentes a dimensione eiusdem spatii. Spero excogitationes meas investigatoribus grupporum symmetriarumque usui fore. Fieri autem potest, ut res a parte omnino opposita aspiciatur. Itaque ipse inventiones meas adhibui ad regulas novas reperiendas – non de gruppis repraesentatis – sed de spatiis, ubi repraesentationes continentur. Hanc accommodationem in ultimo capitulo huius libri inclusi.

Textum eomodo scribere conatus sum, ut quilibet elementorum algebrae modernae, praecipue theoriae grupporum, peritus eum inspicere valeat. Notionibus elementariis illius artis libenter usus sum, et quod eas transiit, maximam partem in capitulo 2 explicabitur. Eam autem theoriam, quae firmius cum inventiones ipsas coniuncta est, in capitula iisdem inventionibus parata remisi. Nihil prius cognitum comprobavi, sed de illis ad aptum opus e litteris electum refero. Exceptio quidem est propositio 3.13, antea vero nota, cui novam demonstrationem dedi inventionibus meis propriis utendo.

* * *

Gratias agere velim custodi mei professori Martino Liebeck, pro eius subsidio atque patientia quasi infinita. Semper intellexit, quam difficile hoc munus nonnullis temporibus se praestare possit. Maiore verum momenti, saepe mihi admonuit in vita etiam alias res exstare praeter studia. Gratias maximas ago etiam uxori meae, Kaisae Nuoliojae, quippe quae sine ulla haesitatione conato meo subsidio fuerit atque mihi multas ex illis aliis rebus protulerit. Una cum ea, omnes amici mei in Finnia, Londinii, peregre versantes munere longo durante mihi vitam plus iucundam fecerunt. Neque parentium meorum oblivisci possum, qui semper mihi auxilio erant intimique manebant inde ab eo tempore, quo domo profectus sum. Ad eos etiam addere sinant materteram meam Raili atque avos meos, nec non compatrem meum Pertti Rantialam, qui primus me adducit ad res miras logicas atque scientificas. Quod ad rem oeconomicam pertinet, gratiam habeo Fundationi Jenny et Antti Wihuri atque consilio scientifico EPSRC pro eorum subsidium pecuniarium generosissimum. Similiter gratiam habeo Collegio Imperiali Londiniense, quod condiciones studendi praeparaverit tam animum incitantes. Postremo, gratias singulares ago doctrici Johannae Rämö, quae collega mea amicaque fidelissima fortasse optime omnium intellegit, qualia mihi sustinenda erant his quattuor annis.

Helsinkii, a.d. XV Kal. Iul. anno MMXIII
Jokke Häsä

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1. Introduction

1.1. Representation growth

The main subject of study in this thesis is *representation growth.* By representation growth we mean the growth of the number of existing representations of groups, as a function of the dimension of the representation space. However, we set some restrictions on the representations.

When G is a group, we define

$$r_n(G)$$

as the number of inequivalent irreducible n-dimensional complex representations of G.[1] In general, $r_n(G)$ can attain infinite values if the group G is infinite. The main purpose here is to find bounds for the growth of $r_n(G)$ as a function of n.

A closely related concept is the *representation zeta function*, which is defined as follows:

$$\zeta_G(s) = \sum_{n \geq 1} \frac{r_n(G)}{n^s}\,.$$

Recalling that if χ is the character corresponding to a representation, then $\chi(1)$ equals the dimension of the representation, we can also write the zeta function as

$$\zeta_G(s) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^{-s},$$

where the sum is taken over all irreducible characters of G. The values of ζ_G are not necessarily finite. However, if for some $s > 0$ we have $\zeta_G(s) = c$, then we see that $r_n(G)$ is bounded from above by cn^s.

When the group G is infinite, it may be interesting to look at the growth of $r_n(G)$ for an individual group. However, a finite group has only finitely many irreducible representations, so for a finite group G, the values of $r_n(G)$ would become zero for sufficiently large n. Therefore, for finite groups one might try bounding the values of $r_n(G)$ over a class of groups, saying for example that $r_n(G)$ is less than $\sqrt{n}$ for any group G in some infinite class. This is what Martin Liebeck and Aner Shalev did when they first introduced the concept. In the next section, we look more closely at their results.

Another kind of bound one might endeavour to establish would be a bound to the number of all n-dimensional irreducible representations of groups in some infinite class.

[1] For definitions of these or any unfamiliar concepts, the reader is referred to Chapter 2.

Call this number $s_n(\mathcal{C})$, where $\mathcal{C}$ is the class of groups under study. As the class $\mathcal{C}$ is infinite, we cannot simply add up all the bounds obtained for $r_n(G)$ for individual G in the class to obtain a bound for $s_n(\mathcal{C})$. Instead, we must study how many of the groups in $\mathcal{C}$ actually have n-dimensional representations and use this information together with the bounds for the numbers $r_n(G)$ to bound $s_n(\mathcal{C})$. Often, it is so that we can bound the number of those groups in $\mathcal{C}$ that have representations of dimension less than any given N. This knowledge can then be used to bound $s_n(\mathcal{C})$.

1.2. Background

Representation growth was inspired by the study of subgroup growth, the growth of the number of subgroups as a function of their index. Subgroup growth is by now a well-established field, and one of its first major applications was to determine when an arithmetic group has the so-called congruence subgroup property.

Among the first results in representation growth were the results of Martin Liebeck and Aner Shalev related to finite groups. In [39], they prove an asymptotic upper bound for the representation zeta function for alternating groups. The result can be stated as follows (see [39, Theorem 1.1]).

Theorem A (Liebeck and Shalev)**.** *Consider the alternating group A_d. Given any $s > 0$, there exists a constant c_s, such that for any integer d, we have*

$$r_n(A_d) < c_s n^s \qquad \textit{for all } n > 1.$$

Regarding other simple groups, in [40] the same authors use Deligne–Lusztig classification of irreducible characters of finite groups of Lie type to obtain a similar result for these groups. Writing $\mathcal{L}(q)$ for a quasisimple group of Lie type $\mathcal{L}$ defined over $\mathbb{F}(q)$, the result reads as follows ([40, Corollary 1.4]).

Theorem B (Liebeck and Shalev)**.** *For a fixed Lie type $\mathcal{L}$, with Coxeter number h, there is a constant $c = c(\mathcal{L})$, such that*

$$r_n\big(\mathcal{L}(q)\big) < cn^{2/h} \qquad \textit{for all } q.$$

Moreover, the exponent $2/h$ is smallest possible, so that the bound is asymptotically tight.

Liebeck and Shalev have also given applications regarding subgroup growth ([39]), random walks on groups ([39, 40]), random generation of groups ([41]), and the dimensions of representation varieties ([41]). Let us look more closely at how results on representation growth can be applied to some of these questions.

Application 1. In [41], the authors study spaces $\mathrm{Hom}(\Gamma, G)$ of homomorphisms from a Fuchsian group Γ to a finite simple group G. A Fuchsian group is a group of isometries of the hyperbolic plane having a certain finite presentation in terms of generators and relations. Examples of Fuchsian groups include the fundamental groups of orientable surfaces with genus at least 2. The following is taken from Sections 1 and 3 of the aforementioned paper.

Let Γ be a Fuchsian group. If G is a group of Lie type over a field $\mathbb{F}_q$, the space $\mathrm{Hom}(\Gamma, G)$ can be regarded as the set of q-rational points in the *representation variety* $\mathrm{Hom}(\Gamma, \bar{G})$, where $\bar{G}$ is the simple algebraic group of the same Lie type as G over the algebraic closure of $\mathbb{F}_q$. Estimates of S. Lang and A. Weil explain how the number of q-rational points in an algebraic variety depends on the dimension of the variety. Bounds for the number of points in $\mathrm{Hom}(\Gamma, G)$ can thus be used to obtain the dimension of the representation variety.

Let us take as Γ for example the fundamental group of a compact surface of genus $g \geq 2$, with so-called elliptic generators $x_1, \dots, x_d$. Let $C = (C_1, \dots, C_d)$ be a d-tuple of conjugacy classes C_i of a finite simple group G, with representatives $g_1, \dots, g_d$, and set

$$\mathrm{Hom}_C(\Gamma, G) = \{\varphi \in \mathrm{Hom}(\Gamma, G) \mid \varphi(x_i) \in C_i \text{ for all } i\}.$$

A result of Hurwitz tells us that

$$|\mathrm{Hom}_C(\Gamma, G)| = |G|^{2g-1}|C_1| \cdots |C_d| \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(g_1) \cdots \chi(g_d)}{\chi(1)^{d-2+2g}},$$

where the χ are characters of irreducible representations of G. Noting that $|\chi(g_i)| \leq \chi(1)$ for all i, we see that

$$\frac{\chi(g_1) \cdots \chi(g_d)}{\chi(1)^{d-2+2g}} \leq \chi(1)^{-(2g-2)}.$$

Hence, we can bound the sum in the above expression by $\zeta_G(2g-2)$. The asymptotic results on representation growth proven in the paper give $\zeta_G(s) \to 1$ as $|G| \to \infty$ for any $s > 1$. In this way, an asymptotic bound for the size of $|\mathrm{Hom}(\Gamma, G)|$ is obtained.

Application 2. Another application is mentioned in [40], concerning random walks on groups. The following is taken from Section 6 of that paper.

Let S be a generating set of a finite group G, closed under inverses. At each step of the walk, we move from an element g in the group to gs, where s is an element from S chosen uniformly at random. Let $P^k(g)$ denote the probability of arriving at g after k steps, starting from the identity. If S is the union of a conjugacy class x^G with its inverse, a formula of P. Diaconis and M. Shashahani shows that

$$\|P^k - U\|^2 \leq \sum_{\chi \in \mathrm{Irr}(G)} \left| \frac{\chi(x)}{\chi(1)} \right|^{2k} \chi(1)^2,$$

where U is the uniform distribution and $\| \cdot \|$ is the l_1-norm. Under certain assumptions on x, information on zeta functions can again be used to determine the *mixing time* of the walk; this is the smallest integer k, such that $\|P^k - U\| < 1/e$.

There are also more recent developments in the study of representation growth. In [46], A. Lubotzky and B. Martin connect the representation growth of certain infinite groups to the congruence subgroup property, and in [36], Lubotzky and M. Larsen derive more asymptotic results for the same groups. For finite groups, David Craven showed in [7] that there is no constant upper bound for $r_n(S_d)$ which would hold for every symmetric group S_d.

The view can also be changed to modular representations. A linear representation is called modular when the characteristic of the scalar field of the representation space is non-zero. In [16], R. Guralnick, M. Larsen and P. H. Tiep prove asymptotic results for modular representation growth and use the results to bound the number of conjugacy classes of maximal subgroups in finite groups of Lie type. Following the basic idea in their proof, we establish another version of the maximal subgroups result in Chapter 6 of this thesis.

1.3. Our contribution

In this work, we have built upon the results of Liebeck and Shalev, with a slightly different approach. Firstly, in addition to finding bounds for $r_n(G)$ that hold for all groups G in a given class $\mathcal{C}$, we have tried to bound the number $s_n(\mathcal{C})$ of all representations of these groups simultaneously. Secondly, we are not content with asymptotic results, but instead go for concrete numerical bounds. For example, we show that if the class $\mathcal{C}$ consists of all finite non-abelian simple alternating groups, then $s_n(\mathcal{C})$ is bounded from above by $0.667n$, and also by $2.52\sqrt{n}$.

On the whole, we have concentrated on classes of finite quasisimple groups. A quasisimple group is a perfect group whose quotient over the centre is simple. In particular, all non-abelian simple groups are quasisimple. The importance of quasisimple groups lies mainly in the fact that any projective representation of a non-abelian finite simple group is obtained as a linear representation of a finite quasisimple group. This connection between projective representations and quasisimple groups will be more thoroughly explained in Section 2.2.

When counting representations of quasisimple groups, we want to avoid counting the same representation twice as a representation of a group and its quotient. For example, all representations of a simple linear group $\mathrm{PSL}_n(q)$ are recovered as representations of $\mathrm{SL}_n(q)$, which is likewise quasisimple. For this reason, we let $r_n^f(G)$ denote the number of inequivalent faithful irreducible n-dimensional complex representations, and define

$$s_n(\mathcal{C}) = \sum_{G\in\mathcal{C}} r_n^f(G).$$

We also try to choose the classes $\mathcal{C}$ so that we would avoid taking isomorphic copies of the same group in the same $\mathcal{C}$, but we have not spent too much effort trying to recognise all possible isomorphisms.

The main results of this work regarding complex representations could be cited as follows.

Theorem C (Chapter 3)**.** *Let A_d denote the alternating group of degree d. For any $d \geq 5$, we have*

$$r_n(A_d) \leq 0.667n \qquad \textit{and} \qquad r_n(A_d) \leq 1.16\sqrt{n}$$

for all $n > 1$.

Theorem D (Chapter 3)**.** *Let $\mathcal{C}$ be the class of all finite simple non-abelian alternating groups. We have*

$$s_n(\mathcal{C}) < 0.000118n + 9 \qquad \textit{and} \qquad s_n(\mathcal{C}) < 2.50\sqrt{n} + 9$$

for all $n > 1$.

Theorem E (Chapter 3)**.** *Let $\mathcal{C}$ denote the class of universal covering groups of the alternating groups of degree greater than 7. We have*

$$s_n(\mathcal{C}) < 0.000711n + 6 \qquad \textit{and} \qquad s_n(\mathcal{C}) < 4.92\sqrt{n} + 6,$$

for all $n > 1$.

Theorem F (Chapter 4)**.** *Let $\mathcal{C}$ denote the class of finite quasisimple groups with simple quotient a group of Lie type. We have*

$$s_n(\mathcal{C}) \leq 1.67n$$

for all $n \geq 13$.

There are also variations of these results, where we consider different subclasses of groups and different types of bounds. The complete set of results can be read from the first sections of Chapters 3–4.

Theorem F above has been used by R. Guralnick, M. Larsen and C. Manack in [14] to obtain the following asymptotic bound for the number of conjugacy classes of maximal subgroups in classical Lie groups. We will explain how this application is related to our work in Section 6.4 of Chapter 6.

Theorem G (Guralnick, Larsen and Manack)**.** *Let G be a simple algebraic group of rank r over an algebraically closed field of characteristic zero. Then the number of conjugacy classes of maximal closed subgroups of G is $O(r)$.*

We have also considered modular representations in the case of groups of Lie type. Let $\mathcal{C}$ be the class of all finite quasisimple groups with simple quotient a group of Lie type. For any prime ℓ and any group G in $\mathcal{C}$, we define $r_n^f(G)$ as the number of inequivalent faithful irreducible n-dimensional representations of G over a field of characteristic ℓ not dividing the size of defining field of G. Then we let $s_n(\mathcal{C}, \ell)$ be the sum of $r_n^f(G)$ over all G in $\mathcal{C}$, and prove the following.

Theorem H (Chapter 5)**.** *For any prime ℓ not dividing the size of the defining field, we have*

$$s_n(\mathcal{C}, \ell) < 15.6n$$

for all $n > 1$.

There are also related results for different subclasses of finite quasisimple groups of Lie type; the results are listed in the beginning of Chapter 5.

Theorem H can be used to bound the number of conjugacy classes of maximal subgroups in finite classical groups. For that, we obtain the following result.

Theorem I (Chapter 6)**.** *Let G_0 be a finite simple classical group of dimension n over the field $\mathbb{F}_q$. Assume that G is a finite group such that $G_0 \leq G \leq \mathrm{Aut}(G_0)$. Let $m(G)$ denote the number of conjugacy classes of maximal subgroups of G not containing G_0. Then we have*

$$m(G) < 2n^{5.2} + n \log_2 \log_2 q.$$

In obtaining the results, we have made heavy use of a computer, and in particular, the Maple computing software. For example, we have written code sequences to compute representation dimensions from known formulae and to classify certain polynomials according to their asymptotic behaviour. These programs are relatively straightforward, and we have not found it necessary to include any actual code in the text.

1.4. Structure of the thesis

We shall give full proofs for all the theorems mentioned in the previous section, apart from Theorem G. Before the proofs, we present a quick overview of necessary theory in Chapter 2, where we also fix the terms and notations that will be used. In that chapter, the first section concerns representation theory. The second contains a list of finite simple groups and some further information on the groups of Lie type in the list. There is also an account on quasisimple groups and their connection to projective representations. The final section of the chapter defines the classical groups and relates them to the groups of Lie type.

In Chapter 3, we prove the results related to alternating groups and their covering groups. Chapter 4 is dedicated to proving results on complex representations of groups of Lie type. The modular representations of these groups are considered in Chapter 5. Finally, two applications are presented in Chapter 6. First, we use the results from Chapter 5 to prove Theorem I above. Then we explain how our results from Chapter 4 were used by Guralnick et al. to obtain Theorem G.

2. Preliminary theory

2.1. Representations and characters

2.1.1. Linear representations

A *linear representation* of a group G, or just *representation* for short, is a homomorphism from G into the group $\mathrm{GL}(V)$ of invertible linear transformations of a vector space V. If ρ is a representation of G, then G can also be seen as *acting* on V via $g.v = \rho(g)(v)$. Conversely, any linear action of G on V defines a representation.

We will restrict our attention to the case where G is a finite group. We will also take the dimension of the space V to be non-zero and finite, and the scalar field of V to be algebraically closed. As the dimension is finite, we can view the group $\mathrm{GL}(V)$ as the matrix group $\mathrm{GL}_n(K)$, where K is the scalar field of V. The proofs for all the facts mentioned in this section can be found in [27].

The dimension of the space V is called the *dimension* of the representation, or its *degree*. In this work, the representation degrees are the objects we are mostly interested in. Every group has at least one 1-dimensional representation, called the *trivial* representation, which takes every element to the identity in $\mathrm{GL}(V)$.

When the scalar field of V is the field of complex numbers, the representations are called *complex* representations. On the other hand, if the scalar field has positive characteristic, one talks about *modular* representations.

Two representations defined on the same space V are said to be *equivalent* (or *similar*) if they can be obtained from each other by a transformation of the space V. In other words, the representations ρ_1 and ρ_2 are equivalent if there is an invertible linear transformation T of V, such that

$$\rho_2(g)(v) = \big(T \circ \rho_1(g) \circ T^{-1}\big)(v) \qquad \text{for all } g \in G \text{ and } v \in V.$$

A representation ρ of a quotient group G/H produces a representation $\hat{\rho}$ of the group G by the following rule: $\hat{\rho}(g) = \rho(gH)$. This is called the *lift* of ρ. A representation that is not a lift of a representation of any non-trivial quotient group is called *faithful*. A faithful representation ρ of G is necessarily an injective homomorphism, since otherwise it would be obtained as a lift of a representation of $G/\ker\rho$.

A representation is called *irreducible* if V has no proper non-zero subspace that is stable under the representation action. Let V_1 and V_2 be two complementary subspaces of V, with respective bases $(v_1, \ldots, v_r)$ and $(w_1, \ldots, w_s)$. A *direct sum* of representations

$\rho_1 \colon G \to \mathrm{GL}(V_1)$ and $\rho_2 \colon G \to \mathrm{GL}(V_2)$ is the representation $\rho_1 \oplus \rho_2 \colon G \to \mathrm{GL}(V)$ with the action defined by

$$(\rho_1 \oplus \rho_2)(g)\left(\sum_i a_i v_i + \sum_j b_j w_j\right) = \sum_i a_i \rho_1(g)(v_i) + \sum_j b_j \rho_2(g)(w_j).$$

Maschke's Theorem ([27, Theorem 1.9]) states that if the characteristic of the scalar field of V does not divide the order of G, any representation can be described as a direct sum of irreducible representations. It is therefore usually enough to consider the irreducible representations. The set of irreducible representations of G is denoted by $\mathrm{Irr}(G)$.

Example 2.1. Consider the following 2-dimensional representation of the cyclic group C_4 with generator denoted by g:

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(g) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \rho(g^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \rho(g^3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The matrix $\rho(g)$ is a 90° rotation in $\mathbb{R}^2$, so ρ does not stabilise any non-zero subspace if it is considered as a representation over $\mathbb{R}$. However, as a complex representation, ρ is not irreducible. In fact, changing the basis of $\mathbb{C}^2$ to $\{(1,-i),(1,i)\}$, we see that ρ is equivalent to

$$\sigma(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma(g) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \sigma(g^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma(g^3) = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}.$$

Now, the representation σ can be written as a direct sum of 1-dimensional representations

$$\sigma_1(1) = 1, \quad \sigma_1(g) = i, \quad \sigma_1(g^2) = -1, \quad \sigma_1(g^3) = -i,$$

and

$$\sigma_2(1) = 1, \quad \sigma_2(g) = -i, \quad \sigma_2(g^2) = -1, \quad \sigma_2(g^3) = i.$$

Hence we see that Maschke's Theorem holds in this case.

Irreducible representations have also another useful property. Let V and W be two vector spaces, and let $\rho \colon G \to \mathrm{GL}(V)$ and $\sigma \colon G \to \mathrm{GL}(W)$ be two irreducible representations. Schur's Lemma ([27, Lemma 1.5]) states that any linear map $L \colon V \to W$ that commutes with the actions of G (i.e. $L \circ \rho(g) = \sigma(g) \circ L$) is either trivial or an isomorphism. This result is of fundamental importance in representation theory. Among other things, it is used to show that when the group G is finite, there are only finitely many irreducible representations of G, up to equivalence ([27, Corollary 2.7]).

Let K be the scalar field of the representation space V, and let KG denote the group algebra of G. Because a group acts on the space V via its representations, the representations can also be seen as KG-modules. In this interpretation, a representation is irreducible if it is irreducible as a module, that is, it has no proper non-trivial submodules. A direct sum of representations becomes likewise a direct sum of modules.

Sometimes one also needs to consider scalar fields that are not algebraically closed. If L is a field extension of K, it might be that an irreducible K-representation becomes reducible when seen as an L-representation. When an irreducible representation stays irreducible over all algebraic field extensions, it is called *absolutely irreducible.*

2.1.2. Characters

Assume now that the scalar field of V is $\mathbb{C}$. In this case a great deal of information about a finite-dimensional representation is contained in its *character*. The character of a representation ρ is the map $\chi_\rho \colon G \to \mathbb{C}$ taking each g to the matrix trace of $\rho(g)$. Characters of two equivalent representations are equal, as the trace of a matrix is invariant under conjugation by an invertible transformation. Also, the character is a class function (constant on each conjugacy class of G), because

$$\rho(hgh^{-1}) = \rho(h)\rho(g)\rho(h)^{-1},$$

and $\rho(h)$ is an invertible transformation.

The degree of a representation ρ is also called the degree of its character χ_ρ, and a character of an irreducible representation is called an irreducible character. Trivial characters are defined in the same way as characters of trivial representations.

When G is finite, all characters of irreducible representations of G can be presented as a *character table*. In the character table, each row represents an irreducible character and every column represents a conjugacy class of G. It is also customary to let the first column represent the class of the identity element. The values in this column are the degrees of the various characters, as $\rho(1)$ is the identity matrix and its trace is the dimension of V. The number of irreducible characters is always the same as the number of conjugacy classes ([27, Corollary 2.7]), so the character table is a square.

Example 2.2. The cyclic group $C_4 = \langle g \rangle$ has 4 one-element conjugacy classes, so it also has 4 irreducible characters. The character table of C_4 is shown below:

	1	g	g^2	g^3
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	i	-1	$-i$
χ_4	1	$-i$	-1	i

Notice that in this case all irreducible characters have $\chi_i(1) = 1$, so every irreducible character is 1-dimensional. The character χ_1 is the trivial character, and the characters χ_3 and χ_4 are the characters of the two representations presented in Example 2.1.

Many character tables of finite simple groups can be found in the Atlas of Finite Groups ([6]). In the following chapters, the book is often referred to simply as the Atlas.

2.1.3. Restriction and induction of characters

Let H be a subgroup of G. Any complex character χ of G gives rise to a character $\chi \downarrow H$ of H via the formula $(\chi \downarrow H)(h) = \chi(h)$ for all $h \in H$. The character $\chi \downarrow H$ is called the *restriction* of χ to H. An irreducible character does not necessarily remain irreducible when restricted to a subgroup, and it is often of great interest to find out how restricted characters of G decompose into irreducible characters of H.

In the opposite direction we have an operation called *induction.* A character φ of H can be made into a character $\varphi \uparrow G$ of G in the following way. Define $\varphi^\circ(h) = h$ when $h \in H$ and $\varphi^\circ(h) = 0$ otherwise. Then $\varphi \uparrow G$ is given by

$$(\varphi \uparrow G)(g) = \frac{1}{|H|} \sum_{x \in G} \varphi^\circ(xgx^{-1}).$$

The character $\varphi \uparrow G$ is called the character *induced* from φ to G.

2.1.4. Brauer characters

The character theory described above gains somewhat different properties when translated to modular representations. For example, if the characteristic of the field is p, then every p-dimensional character χ has $\chi(1) = 0$, so the dimension cannot be inferred from the character values like in the case of complex characters. However, there is a notion, due to R. Brauer, of complex-valued functions that somehow imitate the complex characters in the realm of modular representations.

Let p be a prime, and let U stand for the multiplicative group of complex roots of unity of order coprime to p. There exists an algebraically closed field K of characteristic p, such that the multiplicative group K^* is isomorphic to U ([27, Lemma 15.1]). Suppose ρ is a K-representation of G, and let P be the set of p-regular elements of G, that is, elements of order not divisible by p. We define a function $\varphi\colon P \to \mathbb{C}$ as follows.

Let g be an element of P. It can be shown that all the eigenvalues of $\rho(g)$ are roots of unity ([27, Lemma 2.15]). Denote the eigenvalues as $u_1, \ldots, u_n$, and let $u_1^*, \ldots, u_n^*$ be their complex images under the above described isomorphism. Now, the map defined by

$$\varphi(g) = \sum_{i=1}^{n} u_i^*$$

is called a *Brauer character* of G. Note that if the degree of ρ is n, then $\rho(1)$ has the n-fold eigenvalue 1. Thus the corresponding Brauer character has $\varphi(1) = n$, so the degree can be recovered from the values of the Brauer character.

A Brauer character is called *irreducible* if it comes from an irreducible K-representation. We denote the set of irreducible Brauer characters of G by $\mathrm{IBr}(G)$. The number of irreducible Brauer characters of G is the same as the number of conjugacy classes of p-regular elements of G ([27, Corollary 15.11]).

It can be shown that the irreducible Brauer characters are all distinct and linearly independent over $\mathbb{C}$. Also, any Brauer character is a non-negative integral linear combination of irreducible Brauer characters. (See [27, Chapter 15, page 264].)

The Brauer characters provide a link between p-modular representations of G and normal irreducible characters of G in the following sense. Let χ be a complex character of G, and let $\hat{\chi}$ denote the restriction of χ to p-regular elements. It can then be shown that $\hat{\chi}$ is a Brauer character of G ([27, Theorem 15.6]).

If p does not divide the order of G, every representation of G is a direct sum of irreducible complex characters and every irreducible Brauer character is also an irreducible complex character of G ([27, Theorem 15.3]). Thus the interest in Brauer characters arises only when p does divide $|G|$.

2.1.5. Decomposition matrices and blocks

Let $\hat{\chi}$ be a restriction of an irreducible complex character χ of G to the p-regular elements of G. As $\hat{\chi}$ is a Brauer character, we have

$$\hat{\chi} = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi\varphi}\, \varphi$$

for some non-negative integers $d_{\chi\varphi}$. These integers are called the *decomposition numbers* of G for the prime p, and they are uniquely defined because of the linear independence of the irreducible Brauer characters. If $d_{\chi\varphi}$ is non-zero, the Brauer character φ is said to be a *constituent* of the complex character χ. Every irreducible Brauer character is a constituent of some irreducible complex character ([27, Corollary 15.12]).

The decomposition numbers form a matrix with $|\operatorname{Irr}(G)|$ rows and $|\operatorname{IBr}(G)|$ columns, called the *decomposition matrix*. It can be shown that the columns of the decomposition matrix are linearly independent ([27, Theorem 15.10]). This means that knowing all irreducible complex characters of G together with the decomposition matrix of G for the prime p is enough for obtaining all the irreducible p-modular Brauer characters, and hence, all the irreducible p-modular representation degrees of G.

The decomposition matrix has a block structure. More precisely, there exist sets $B \subseteq \operatorname{Irr}(G) \cup \operatorname{IBr}(G)$, for which $d_{\chi\varphi} = 0$ whenever the pair $\{\chi, \varphi\}$ is not included in any set B. The minimal such sets are called *p-blocks*. Each element of $\operatorname{IBr}(G)$ belongs to a unique p-block, and the same holds for the elements of $\operatorname{Irr}(G)$. (See pages 270–272 of [27], and especially Theorem 15.19.)

The block structure can also be visualised using *Brauer graphs*. A Brauer graph of G for the prime p has as its vertices the irreducible characters of G, and the edge (χ_1, χ_2) exists if both χ_1 and χ_2 have a common p-Brauer character of G as a constituent. It follows that the connected components of the Brauer graph for the prime p correspond precisely to the p-blocks of G.

Some Brauer character tables are given in the Atlas of Brauer Characters ([30]), although it does not contain as many groups as the Atlas of Finite Groups. In the following, the Atlas of Brauer Characters will be referred to as the "Modular Atlas".

2.2. Simple groups and projective representations

The quasisimple groups are certain extensions of simple groups related to so-called projective representations. We will first look briefly at finite simple groups and especially at the simple groups of Lie type. Then we will define projective representations and quasisimple groups and explain the connection between the two concepts.

2.2.1. Finite simple groups

A group is called *simple* if it has no proper non-trivial normal subgroups, or equivalently, if it has no factor groups that are not isomorphic to itself or the trivial group. After a tremendous amount of work published by many mathematicians in several hundred journal articles, the classification of all finite simple groups has been achieved. Accordingly, every finite simple group belongs to (at least) one of the following classes:

1. cyclic groups of prime order
2. alternating groups A_d for $d \geq 5$
3. a) classical groups of Lie type

   b) exceptional groups of Lie type
4. sporadic groups.

Information on the definitions of these groups can be found in R. Wilson's book [63]. Many of the smaller simple groups are also contained in the Atlas of Finite Groups ([6]), which contains the complex character tables along with a lot of other useful information. The proof of the classification is being collected and revised in the ongoing book series project by D. Gorenstein, R. Lyons and R. Solomon (see [12]).

The cyclic groups are the only abelian finite simple groups. The alternating groups are index 2 subgroups of symmetric groups, consisting of even permutations. The sporadic groups are the only finite family in the classification, consisting of 26 groups. The Atlas contains information on all sporadic groups.

2.2.2. Simple groups of Lie type

In diversity, the groups of Lie type are perhaps the richest family of finite simple groups. They are obtained as subgroups of automorphism groups of simple Lie algebras, as explained in [4].

The simple complex Lie algebras were classified by Wilhelm Killing and Élie Cartan. Each simple complex Lie algebra has its own root system, denoted by its Dynkin letter and rank. (More about root systems can be found in [4, Chapter 2].) The so-called classical root systems are A_r, B_r, C_r and D_r, and they can have any positive rank r (although some systems with small r are equal, for example A_3 and D_3). The exceptional systems come with a specific rank, and they are E_6, E_7, E_8, F_4 and G_2. In Figure 2.1 we present the Dynkin diagrams corresponding to each root system. The meaning of the diagrams is explained e.g. in [4, Chapter 3.4].

All the simple complex Lie algebras can also be realised over finite fields, and their automorphism groups give rise to finite simple groups (see [4, Chapter 4]). These groups are called *Chevalley* groups, by the inventor of the method, although some of the groups were known before him. The Chevalley groups are generally denoted $X_r(q)$, where X_r is the root system and q is the size of the finite field.

However, because of symmetries appearing in some Dynkin diagrams, there is another class of groups discovered by R. Steinberg, and also independently by Tits and Hertzig

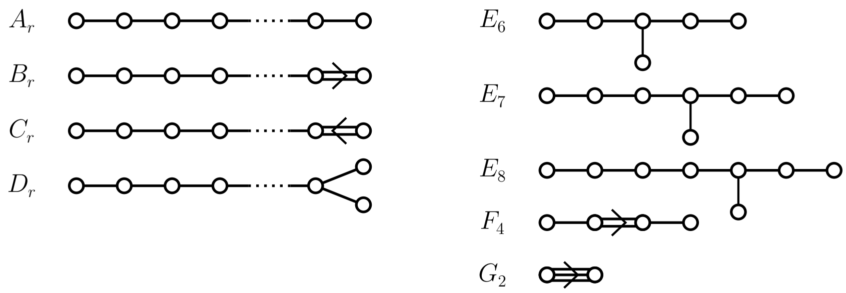

Figure 2.1: The Dynkin diagrams of root systems of simple complex Lie algebras. The nodes represent fundamental roots, and the number of nodes is the rank of the system. The number of edges between nodes represents the angle between the corresponding roots, and an arrow between two nodes points to the node that represents the shorter of the two roots.

(see [4, Chapters 13–14]). When the Dynkin diagram and defining field have automorphisms of the same order, there is a subgroup of the Chevalley group, called the *twisted* group. These groups are denoted by placing the order of the automorphism as a left-hand superscript in the name of the corresponding Chevalley group. Also, instead of the size of the field used in the definition, one replaces the size of the fixed field of the defining automorphism. The Steinberg twisted groups are ${}^2A_r(q)$ and ${}^2D_r(q)$ for any r, and ${}^3D_4(q)$ and ${}^2E_6(q)$ for fixed ranks. For example, ${}^2A_r(2)$ is actually defined over the field of 4 elements.

There is yet another type of groups that only appear for certain fields. When the size of the field is $q = 2^{2k+1}$ for some integer k, one can perform the twist operation to groups of types B_2 and F_4, thereby obtaining the twisted groups ${}^2B_2(q)$ and ${}^2F_4(q)$. Also, when $q = 3^{2k+1}$, we get the twisted group ${}^2G_2(q)$. (See [4, Chapter 14.1].) The groups ${}^2B_2(q)$ were discovered by M. Suzuki and are called *Suzuki* groups, and the groups ${}^2F_4(q)$ and ${}^2G_2(q)$ were discovered by R. Ree and are called *Ree* groups.

Chevalley, Steinberg, Suzuki and Ree groups exhaust all finite simple groups of Lie type. The groups corresponding to root systems with unbounded rank, that is, groups of type A_r, ${}^2A_r$, B_r, C_r, D_r and ${}^2D_r$ are called the *classical* groups of Lie type. They have cropped up in various contexts since the beginning of group theory, and they have descriptions as isometry groups of certain geometries on vector spaces. The classical groups will be discussed further in Section 2.3. The remaining groups are the *exceptional* groups of Lie type.

There is a finite number of groups described above with small rank and small defining field that are in fact not simple. In some cases the group is soluble, so all of its simple subgroups are cyclic. In other cases the group has a simple non-abelian subgroup, and this is then called the simple group of the Lie type in question. Most notable case is the Tits group, which is the commutator subgroup of ${}^2F_4(2)$ and has index 2. This group is often noted separately, as its structure is somewhat different from the other groups of Lie type. For ease of reference, we list all the finite simple groups of Lie type in Table 2.1.

group	field size	other names	when not simple
$A_r(q)$, $r \geq 1$	q	linear	$A_1(2)$, $A_1(3)$: soluble
${}^2A_r(q)$, $r \geq 2$	q^2	unitary	${}^2A_2(2)$: soluble
$B_r(q)$, $r \geq 3$	q	orthogonal	—
$C_r(q)$, $r \geq 2$	q	symplectic	$C_2(2)$: simple subgroup of index 2
$D_r(q)$, $r \geq 4$	q	orthogonal	—
${}^2D_r(q)$, $r \geq 4$	q^2	orthogonal	—
${}^3D_4(q)$	q^3	triality	—
$E_6(q)$	q	—	—
${}^2E_6(q)$	q^2	—	—
$E_7(q)$	q	—	—
$E_8(q)$	q	—	—
$F_4(q)$	q	—	—
$G_2(q)$	q	Dickson	$G_2(2)$: simple subgroup of index 2
${}^2B_2(q)$	$q = 2^{2k+1}$	Suzuki	${}^2B_2(2)$: soluble
${}^2F_4(q)$	$q = 2^{2k+1}$	Ree, Tits ($q = 2$)	${}^2F_4(2)$: simple subgroup of index 2
${}^2G_2(q)$	$q = 3^{2k+1}$	Ree	${}^2G_2(3)$: simple subgroup of index 3

Table 2.1: The finite simple groups of Lie type. Some groups with common rank but different Dynkin letter are isomorphic (because their root systems are equal), so we have set some restrictions for the ranks. (From [4, Chapters 11.1 and 14.4].)

2.2.3. Projective representations

The invertible transformations of a vector space V form the general linear group $\mathrm{GL}(V)$. The centre of this group consists of scalar transformations αI, where α is a non-zero scalar and I is the identity transformation. We will usually identify α with αI. Taking the quotient of the general linear group over scalars, we obtain the *projective* general linear group $\mathrm{PGL}(V)$. The projective general linear group acts on the lines of V (the projective space) in the following manner. Let T be an element of $\mathrm{GL}(V)$ and let $[T]$ denote its coset in $\mathrm{PGL}(V)$. If v is a non-zero vector in V and $\langle v \rangle$ is the line it spans, then $\mathrm{PGL}(V)$ acts on $\langle v \rangle$ by $[T].\langle v \rangle = \langle T(v) \rangle$.

In the previous section, we described linear representations of a group G as homomorphisms into the general linear group $\mathrm{GL}(V)$. Sometimes we are also interested in *projective representations*, which are homomorphism from G into $\mathrm{PGL}(V)$. Clearly, any linear representation ρ defines a unique projective representation through the composition $\pi \circ \rho$, where π is the quotient map from $\mathrm{GL}(V)$ to $\mathrm{PGL}(V)$. However, a group may have more projective representations than linear ones. The theory of projective representations was developed by Issai Schur in his papers [54] and [55]. The parts relevant to this work are covered in [52, Chapter 11] as well as in [27, Chapter 11].

In Chapter 6, we will see an application of projective representations when we look at homomorphisms of finite simple groups into simple classical groups of Lie type. As will be explained in Section 2.3, the simple classical groups are subgroups of $\mathrm{PGL}(V)$, and thus the homomorphisms to be examined are in fact projective representations.

Example 2.3. Consider the Klein four-group $V_4 = \{1, a, b, ab\}$, and the following map R from V_4 into $\mathrm{GL}(\mathbb{C}^2)$:

$$R(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R(a) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad R(b) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad R(ab) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

The map R is not a homomorphism, but the composition $\sigma = \pi \circ R$ into $\mathrm{PGL}(\mathbb{C}^2)$ is, so σ is a projective representation of V_4. In fact, it can be shown that σ does not equal $\pi \circ \rho$ for any linear representation ρ of V_4 (see Example 2.5 below).

2.2.4. Quasisimple groups

A *central extension* of a group G is another group H, such that G is isomorphic to H/M for some subgroup $M \leq H$ contained in the centre $Z(H)$. A central extension of a simple group G that is in addition perfect (i.e. equal to its commutator subgroup) is called a *quasisimple* group. In other words, a group H is quasisimple if it is perfect and the quotient $H/Z(H)$ is simple. The quasisimple groups are related to projective representations in the following way.

When a projective representation σ of a group G is a composition of a linear representation ρ of G and the quotient map π, all information on σ can be inferred from the the linear representation ρ. Unfortunately, not all projective representations are of this form. However, suppose that there is another group H, such that G is quotient of H over a central subgroup, and let π' denote the quotient map. Now, if there is a linear representation ρ of G such that $\sigma \circ \pi' = \pi \circ \rho$, we call ρ a *lift* of σ. In other words, ρ is a lift of σ if the following diagram commutes:

$$\begin{tikzcd} H \arrow[r, dashed, "\rho"] \arrow[d, "\pi'"'] & \mathrm{GL}(V) \arrow[d, "\pi"] \\ G \arrow[r, "\sigma"'] & \mathrm{PGL}(V) \end{tikzcd} \tag{2.1}$$

Such a group H as defined above is sometimes called a *covering group* of G.

Example 2.4. Consider the quaternion group $Q_8 = \{\pm 1, \pm \mathbf{i}, \pm \mathbf{j}, \pm \mathbf{k}\}$. It is well known that V_4 is a quotient of Q_8 over its centre $\{1, -1\}$. Now, there is a complex linear representation ρ of Q_8 defined by

$$\rho(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \rho(\mathbf{i}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \rho(\mathbf{j}) = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \rho(\mathbf{k}) = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}.$$

Let σ denote the projective representation of the Klein four-group V_4 defined in Example 2.3, and let R likewise be the map defined in that example. Moreover, let $\pi' \colon Q_8 \to V_4$ denote the quotient map, taking $\mathbf{i} \mapsto a$ and $\mathbf{j} \to b$. It can be seen that $\pi \circ \rho = \pi \circ R \circ \pi' = \sigma \circ \pi'$, or in other words, that ρ is a lift of σ.

Let now V be a vector space over an algebraically closed field, and let $\rho\colon H \to \mathrm{GL}(V)$ be an irreducible linear representation of a quasisimple group H. If $z \in Z(H)$, we see that $\rho(z)$ commutes with $\rho(h)$ in $\mathrm{GL}(V)$ for all $h \in H$. It then follows from Schur's Lemma that $\rho(z)$ is a scalar ([27, Corollary 1.6]). Writing π for the quotient map $\mathrm{GL}(V) \to \mathrm{PGL}(V)$, we may conclude that $Z(H)$ is in the kernel of $\pi \circ \rho$. Hence, there exists a homomorphism σ from $H/Z(H)$ to $\mathrm{PGL}(V)$ such that $\sigma \circ \pi' = \pi \circ \rho$, where $\pi'\colon H \to H/Z(H)$ is the quotient map. In other words, every irreducible linear representation of a quasisimple group is a lift of a projective representation of the simple quotient.

2.2.5. Schur multipliers and universal covering groups

The lifting problem presented above can be formulated in terms of group cohomology (see e.g. [52, Chapter 11]). Suppose $\rho\colon H \to \mathrm{GL}(V)$ is a lift of a projective representation σ of G, as discussed above. For each $x \in G$, choose a preimage $h_x \in H$ under π', and write L_x for the image $\rho(h_x)$ in $\mathrm{GL}(V)$. Moreover, let us choose L_1 to be the identity. Now, we have $\sigma(x) = \pi(L_x)$ for every $x \in G$, as the diagram in (2.1) commutes. The homomorphism property of ρ then implies that for all $x, y \in G$ there exist scalars $\alpha(x, y)$, such that

$$L_x L_y = \alpha(x, y) L_{xy}.$$

Because of the associativity in $\mathrm{GL}(V)$, the scalars $\alpha(x, y)$ must satisfy

$$\alpha(x, y)\alpha(xy, z) = \alpha(y, z)\alpha(x, yz) \tag{2.2}$$

for any $x, y, z \in G$.

The function $(x, y) \mapsto \alpha(x, y)$ satisfying equation (2.2) is called a *cocycle*. It is not uniquely determined by σ, because it depends on the choice of the preimages h_x. Assume now that another choice has been made, let L'_x denote the resulting elements in $\mathrm{GL}(V)$, and write β for the new cocycle. Then the equality $\pi(L_x) = \sigma(x) = \pi(L'_x)$ ensures that for every $x \in G$ there exists a scalar $\delta(x)$ such that $L'_x = \delta(x)L_x$. Straightforward computation shows that

$$\alpha(x, y)\beta(x, y)^{-1} = \delta(xy)\delta(x)^{-1}\delta(y)^{-1} \tag{2.3}$$

holds for all $x, y \in G$. The above equation, together with the fact that $\delta(1) = 1$, imply that $\alpha\beta^{-1}$ is something called a *coboundary*.

Two cocycles that differ by a coboundary are said to be *cohomologous*. The equivalence classes of cocycles under cohomology form the (second) *cohomology group*. This abelian group is denoted $H^2(G, K^*)$, where K^* is the group of non-zero scalars of V. Hence, we have established that every projective representation σ determines a unique class of cocycles in $H^2(G, K^*)$. Furthermore, it is not hard to see that if σ equals $\pi \circ \rho$ for some linear representation ρ of G, then the resulting cocycle class is the identity class.

Example 2.5. Return once more to Example 2.3. Choosing $L_x = R(x)$ for every $x \in V_4$, we see that the resulting cocycle α has the values shown in the following table:

α	1	a	b	ab
1	1	1	1	1
a	1	-1	1	-1
b	1	-1	-1	1
ab	1	1	-1	-1

It can now be seen that α is not cohomologous to the trivial cocycle. Namely, suppose that δ is such that $\alpha(x, y) = \delta(xy)\delta(x)^{-1}\delta(y)^{-1}$ for all $x, y \in V_4$. For $x \in V_4$, we have $x^2 = 1$, so $\delta(x^2) = 1$. Now we can deduce from the above table that for $x \neq 1$, we have $\delta(x)^2 = -1$, so $\delta(x)$ is i or $-i$. However, this means that if x and y are unequal non-identity elements of V_4, we have

$$\delta(xy)\delta^{-1}(x)\delta^{-1}(y) = \pm i \neq \alpha(x, y),$$

a contradiction. It follows that σ is not $\pi \circ \rho$ for any linear representation ρ of V_4.

The group $H^2(G, \mathbb{C}^*)$ is known as the *Schur multiplier* and denoted simply as $M(G)$. It is well-known that the elements of $M(G)$ are in bijective correspondence with equivalence classes of central extensions of G (see e.g. [52, Theorem 7.59]). In fact, by extending the group G by $M(G)$, Schur managed to show that for any finite group G, there is a group H, such that the following hold:

1) H has a central subgroup M contained in the commutator subgroup of H
2) M is isomorphic to $M(G)$
3) H/M is isomorphic to G
4) all projective representations of G can be lifted to linear representations of H.

The group H is called a *representation group* or a *full covering group* of G. (See [54, § 3] or [52, Theorem 7.66].)

The full covering group of a finite group is not necessarily unique, but all different coverings of the same group have isomorphic commutator subgroups ([54, Satz III]). Also, a full covering group of a perfect group is perfect, so in particular, a full covering group of a non-abelian simple group is perfect and thus unique. Such a unique covering group is called the *universal covering group* of G.

Universal coverings of perfect finite groups are also *universal central extensions*, in the sense that any central extension of a perfect finite group by an abelian group M is isomorphic to a quotient of the universal central extension over a subgroup of M (see [52, Theorem 11.11]). As the quasisimple groups are central extensions of simple groups, this implies that all finite quasisimple groups appear as quotients of the universal covering groups of non-abelian simple groups. Moreover, for any non-abelian simple group G, the different quasisimple extensions of G correspond to different subgroups of the Schur multiplier $M(G)$.

2.3. Classical groups

We have already mentioned classical simple groups as simple groups of Lie type in Section 2.2. As such, they can be defined as groups of automorphisms of Lie algebras. However, the classical groups also have constructions as certain automorphism groups of vector spaces, and in this guise they are perhaps more widely known.

The simple classical groups are not the only groups that are called classical, but the actual definition of a classical group varies with the author. In this section, we shall present some groups that are involved in preserving certain geometries on vector spaces, and all these groups we name classical. The facts presented in this section are found in L. Grove's book [13] and also in [31, Chapter 2], apart from the final two subsections dealing with algebraic groups.

2.3.1. Forms and isometries

For the whole of this section, we will assume that K is a field, V is a finite-dimensional vector space over K, and n is the dimension of V. A *bilinear form* on V is a mapping $\mathbf{f}\colon V \times V \to K$ that is linear in both of its arguments, that is, for all $\alpha \in K$ and $v, w, u \in V$, a bilinear form satisfies

$$\begin{aligned} \mathbf{f}(\alpha v + u, w) &= \alpha \mathbf{f}(v, w) + \mathbf{f}(u, w) \\ \text{and} \qquad \mathbf{f}(v, \alpha w + u) &= \alpha \mathbf{f}(v, w) + \mathbf{f}(v, u). \end{aligned}$$

A bilinear form is called *symmetric* if $\mathbf{f}(v, w) = \mathbf{f}(w, v)$, and *alternating* if $\mathbf{f}(v, v) = 0$, for all $v, w \in V$.

If $\alpha \mapsto \bar{\alpha}$ is an automorphism of K of order two, then a *Hermitian* form on V is a mapping $\mathbf{f}\colon V \times V \to K$ that is not quite bilinear, but instead the following conditions hold for all α in K and v, w, u in V:

$$\begin{aligned} \mathbf{f}(\alpha v + u, w) &= \alpha \mathbf{f}(v, w) + \mathbf{f}(u, w) \qquad \text{and} \\ \mathbf{f}(w, v) &= \overline{\mathbf{f}(v, w)}. \end{aligned}$$

If follows that any Hermitian form satisfies also $\mathbf{f}(v, \alpha w) = \bar{\alpha} \mathbf{f}(v, w)$.

Finally, let $\mathbf{f}$ be a symmetric bilinear form. Then a *quadratic* form is a mapping $Q\colon V \to K$, such that the following conditions hold for all $\alpha \in K$ and $v, w \in V$:

$$\begin{aligned} Q(\alpha v) &= \alpha^2 Q(v) \\ \text{and} \qquad Q(v + w) &= Q(v) + \mathbf{f}(v, w) + Q(w). \end{aligned}$$

The form $\mathbf{f}$ is said to be the bilinear form *associated* with Q. Notice that if the characteristic of K is not two, then the form Q is determined by the choice of $\mathbf{f}$ via $Q(v) = \mathbf{f}(v, v)/2$. In this case, the quadratic form can be seen merely as an auxiliary concept.

From now on, we will use $\mathbf{f}$ to denote a bilinear, hermitian or quadratic form, and whether $\mathbf{f}$ is unary or binary will depend on the context. Also the notation $\mathbf{f}(v)$ should be interpreted as $\mathbf{f}(v, w)$ if $\mathbf{f}$ is a binary form.

An invertible linear transformation g of V is said to *preserve* the form $\mathbf{f}$, if

$$\mathbf{f}\big(g(v)\big) = \mathbf{f}(v) \quad \text{for all } v \in V.$$

Again, $\mathbf{f}(g(v))$ must be interpreted as $\mathbf{f}(g(v), g(w))$ when $\mathbf{f}$ is a binary form. An invertible transformation g that preserves the form $\mathbf{f}$ on V is called an *isometry* of V with respect to $\mathbf{f}$. Given V and $\mathbf{f}$, the isometries form a group which is denoted by $I_{\mathbf{f}}(V)$. The subscript $\mathbf{f}$ may be omitted if the form is clear from context.

Example 2.6. The usual dot product $v \cdot w = v_1 w_1 + \cdots + v_n w_n$ is a symmetric bilinear form on $\mathbb{R}^n$. The corresponding quadratic form is

$$Q(v) = (v \cdot v)/2 = \frac{1}{2}(v_1^2 + \cdots + v_n^2),$$

so in this case, the value of the quadratic from on v is half the squared length of v.

As lengths of vectors and angles between them can be determined from the values of the dot product, a linear transformation that preserves the product must preserve also the said lengths and angles. Such transformations are combinations of rotations and reflections. Thus, in the plane, the isometry group $I(\mathbb{R}^2)$ consists of the following types of matrices:

$$r_\varphi = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \quad \text{and} \quad s_\varphi = \begin{bmatrix} -\cos\varphi & \sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}.$$

The matrices r_φ represent rotations, and the s_φ represent reflections with respect to lines through the origin.

2.3.2. The classical groups

A bilinear or Hermitian form $\mathbf{f}$ is said to be *degenerate* if there is a non-zero vector x, such that $\mathbf{f}(v, x) = 0$ for all $v \in V$ or $\mathbf{f}(x, v) = 0$ for all $v \in V$. A quadratic form is degenerate if its associated bilinear form is. We call the pair $(V, \mathbf{f})$ a *classical space* if either $\mathbf{f}$ is a non-degenerate alternating, Hermitian or quadratic form, or $\mathbf{f}$ is the *trivial* form with $\mathbf{f}(v, w) = 0$ for all $v, w \in V$. These forms will also be called *classical forms*.

From now on, we assume that $(V, \mathbf{f})$ is a classical space. Depending on the type of the form, the group $I_{\mathbf{f}}(V)$ of isometries is called

- *linear*, denoted $\mathrm{GL}(V)$, if $\mathbf{f}$ is trivial
- *symplectic*, denoted $\mathrm{Sp}(V)$, if $\mathbf{f}$ is alternating
- *unitary*, denoted $\mathrm{U}(V)$, if $\mathbf{f}$ is Hermitian
- *orthogonal*, denoted $\mathrm{O}(V)$, if $\mathbf{f}$ is quadratic.

All the isometry groups are contained in $\mathrm{GL}(V)$. As the dimension n of V is finite, the isometries can be realised as matrices in a given basis. If such a basis is chosen, we will use the notation $\mathrm{GL}(V) = \mathrm{GL}_n(K)$, and similar ones, for the isometry groups listed above. Also, if K is the finite field $\mathbb{F}_q$, we shall write $\mathrm{GL}_n(K) = \mathrm{GL}_n(q)$, etc. The unitary group is an exception: since in this case the field size must be a square, we write $\mathrm{U}_n(q)$ when the scalar field is $\mathbb{F}_{q^2}$.

Every isometry group $I(V)$ has a subgroup $S(V)$, called *special*, that consists of those linear transformations whose determinant equals 1. This subgroup will be denoted by placing an S in the name of the isometry group, thus obtaining $\mathrm{SL}(V)$, $\mathrm{SU}(V)$ and $\mathrm{SO}(V)$. For the symplectic groups, the determinant is always 1, so no additional notation is needed ([13, Corollary 3.5]).

Example 2.7. The dot product in Example 2.6 defines the orthogonal group $\mathrm{O}_2(\mathbb{R})$. Any reflection through a line has determinant -1, so the special orthogonal group $\mathrm{SO}_2(\mathbb{R})$ consists only of the matrices r_φ.

Example 2.8. In the complex space $\mathbb{C}^n$, the usual inner product is defined by the formula $\langle x, y\rangle = \sum_i x_i\bar{y}_i$. This is not a bilinear form, but a Hermitian one. Thus, in a complex space $\mathbb{C}^n$ endowed with this inner product, the isometries form the unitary group $\mathrm{U}_n(\mathbb{C})$.

If the linear transformation g does not preserve the form, but allows for scaling by a fixed constant, it is called a *similarity.* More precisely, a transformation g in $\mathrm{GL}(V)$ is an $\mathbf{f}$-similarity if there exists a scalar α in K, such that

$$\mathbf{f}\big(g(v)\big) = \alpha\mathbf{f}(v) \qquad \text{for all } v \in V.$$

The similarities form a subgroup of $\mathrm{GL}(V)$, denoted $\Delta_{\mathbf{f}}(V)$. This subgroup will play an important part in Chapter 6.

Sometimes one may also consider transformations of the space that are not entirely linear. A mapping $g\colon V \to V$ is called *semilinear* if there exists an automorphism σ of K, such that the following hold for all $\alpha \in K$ and $v, w \in V$:

$$\begin{aligned} g(v+w) &= g(v) + g(w) \qquad \text{and} \\ g(\alpha v) &= \alpha^\sigma g(v). \end{aligned}$$

A semilinear transformation is *non-singular* if its kernel is trivial. Notice that for a fully linear transformation (with $\sigma = \mathrm{id}$), a non-singular transformation is invertible. The group of all non-singular semilinear transformations of V is denoted $\Gamma\mathrm{L}(V)$.

A semilinear transformation that preserves a form up to scaling by a fixed scalar and also induces a field automorphism on the form, is called a *semisimilarity.* More precisely, $g \in \Gamma\mathrm{L}(V)$ is an $\mathbf{f}$-semisimilarity if there exist a scalar α and a field automorphism σ, such that

$$\mathbf{f}\big(g(v)\big) = \alpha\mathbf{f}(v)^\sigma \qquad \text{for all } v \in V.$$

The semisimilarities form a group $\Gamma_{\mathbf{f}}(V)$.

Example 2.9. Assume that g is a similarity of the plane $\mathbb{R}^2$ endowed with the dot product. That is, assume there is a real number α, such that

$$g(v)\cdot g(w) = \alpha(v \cdot w) \qquad \text{for all } v, w \in \mathbb{R}^2.$$

Choosing $v = w = (1,0)$, we see that $\alpha = g(v) \cdot g(v)$. It follows that α is uniquely determined by g, and is non-negative. Geometrically, the transformation g consists of an orthogonal transformation and an additional scaling by $\sqrt{\alpha}$. This also explains the term 'similarity'. In geometry, two figures are called similar if one is a scaled copy of the other.

For the dot product, the groups $\Delta(\mathbb{R}^2)$ and $\Gamma(\mathbb{R}^2)$ are the same, since $\mathbb{R}$ has no non-trivial field automorphisms.

2.3.3. Projective groups

All the groups defined above are called classical, but most of them are not simple. The main reason is that the scalar transformations are central in the isometry groups, and thus they form a normal subgroup. To obtain the simple classical groups, we need to factor out the scalars, and in some cases, even go a little further.

If G is any of the classical groups defined above, the corresponding *projective* group is obtained by taking the quotient $G/(G \cap Z)$, where Z denotes the group of non-zero scalar transformations αI with $\alpha \in K$ (we usually identify αI with α). The projective group is denoted by placing a P in front of the name of the group. Thus we obtain, for instance, the projective unitary groups $\mathrm{PU}(V)$ and the projective special orthogonal groups $\mathrm{PSO}(V)$. We can also form projective versions of similarity and semisimilarity groups, and these we denote as $\mathrm{P}\Delta_{\mathbf{f}}(V)$ and $\mathrm{P}\Gamma_{\mathbf{f}}(V)$, respectively.

Let $\mathbf{f}$ be a classical form defined on V. To summarise, we have so far described the following classical groups:

$$S_f(V) \leq I_f(V) \leq \Delta_f(V) \leq \Gamma_f(V).$$

Not all of the above inclusions are proper. For example, all symplectic isometries have determinant one, so they are special. Also, if the scalar field of V has only one non-zero element, we have $I_f(V) = S_f(V)$. Similarly, if the scalar field has no non-trivial automorphisms, then $\Gamma_f(V) = \Delta_f(V)$. The projective variants form a similar chain (of embeddings):

$$\mathrm{P}S_f(V) \leq \mathrm{P}I_f(V) \leq \mathrm{P}\Delta_f(V) \leq \mathrm{P}\Gamma_f(V).$$

Note that sometimes a classical group and its projective variant are isomorphic. For example, suppose that the scalar field is $\mathbb{F}_q$. The non-zero scalars form a cyclic group of order $q-1$ under multiplication. If n and $q-1$ are coprime, we cannot have $\alpha^n = 1$ for any non-zero scalar α, except for $\alpha = 1$. It follows that in this case the only non-zero scalar transformation in $\mathrm{SL}_n(q)$ is the identity, so we have $\mathrm{SL}_n(q) \cong \mathrm{PSL}_n(q)$.

When the scalar field of V is finite and the form is not quadratic, the projective special classical groups are in most cases simple, and they comprise the classical finite simple groups of Lie type mentioned in Section 2.2 (see Theorems 1.13, 3.11 and 11.26 of [13] and the remarks after the proofs). Also, with finite fields the different forms of a non-quadratic classical type lead to isomorphic isometry groups, so we need not specify the form while talking about these groups (see [13, Corollaries 2.12 and 10.4]).

However, the orthogonal groups form an exception in both senses. Firstly, if the dimension n of V is odd, there is only one orthogonal group up to isomorphism, but if the dimension is even, there are two different groups corresponding to different forms (see [13, page 79 and Theorem 12.9], also [31, Proposition 2.5.3]). We follow the convention by denoting these two groups $\mathrm{O}^+(V)$ and $\mathrm{O}^-(V)$. Secondly, if the dimension n is at least two, the special orthogonal group $\mathrm{SO}^{(\pm)}(V)$ possesses a certain subgroup of index two, denoted $\Omega^{(\pm)}(V)$ (see Proposition 2.5.7 and the subsequent discussion in [31]). In this case, the simple orthogonal group is the projective group $\mathrm{P}\Omega^{(\pm)}(V)$.

In Table 2.2 below, we describe which simple classical groups the different groups of classical Lie type correspond to.

group of Lie type	classical group	form	exceptions to simplicity
$A_r(q)$, $r \geq 1$	$\mathrm{PSL}_{r+1}(q)$	trivial	$\mathrm{PSL}_2(2)$, $\mathrm{PSL}_2(3)$
${}^2A_r(q)$, $r \geq 2$	$\mathrm{PSU}_{r+1}(q)$	Hermitian	$\mathrm{PSU}_3(2)$
$B_r(q)$, $r \geq 3$	$\mathrm{P}\Omega_{2r+1}(q)$	quadratic	
$C_r(q)$, $r \geq 2$	$\mathrm{PSp}_{2r}(q)$	alternating	$\mathrm{PSp}_4(2)$
$D_r(q)$, $r \geq 4$	$\mathrm{P}\Omega^+_{2r}(q)$	quadratic $(+)$	
${}^2D_r(q)$, $r \geq 4$	$\mathrm{P}\Omega^-_{2r}(q)$	quadratic $(-)$	

Table 2.2: Classical simple groups as groups of Lie type. (From [31, Table 5.1.A] and Table 2.1 in Section 2.2.)

2.3.4. Classical groups as algebraic groups

In addition to Lie algebras and isometries of vector spaces, there is a third way to define the classical groups. This construction gives a lot of structural information about the groups, and it is especially useful in the study of their representations. However, the subject is so vast that we present here only a brief outline. More details can be found from [5, Chapter 1] or from [31, Section 5.4].

Assume that K is an algebraically closed field. We call a subset of K^m *closed* if it is defined by a finite number of polynomial equations. The closed sets define a topology on K^m, called the *Zariski topology.* A *linear algebraic group* over K is a closed subgroup of $\mathrm{GL}_n(K)$. (An $n \times n$-matrix in interpreted as an element of K^{n^2}.) In other words, a linear algebraic group is any subset of the group of invertible matrices that is defined via a finite number of polynomial equations.

We can see that for example the symplectic groups over an algebraically closed field are linear algebraic groups. Namely, for any bilinear form $\mathbf{f}$ there is a corresponding matrix $B_{\mathbf{f}}$, such that $\mathbf{f}(v, w) = v^{\mathrm{T}} B_{\mathbf{f}} w$ holds for all $v, w \in K^n$. (The matrix elements of $B_{\mathbf{f}}$ can be obtained by considering the values of $\mathbf{f}$ on the basis vectors.) Now, if $\mathbf{f}$ is a non-degenerate alternating form, then g is a symplectic transformation if and only if $g^{\mathrm{T}} B_{\mathbf{f}} g = I$, and this equation can be turned into a system of polynomial equations on the matrix elements.

When the characteristic of K is not two, we can use a symmetric bilinear form to see that orthogonal groups are also linear algebraic groups. Since the determinant of a matrix is a polynomial, the equation $\det(g) = 1$ defines also the special linear, symplectic and orthogonal groups as linear algebraic groups.

Linear algebraic groups are algebraic varieties, and as such they can be studied by the methods of algebraic geometry. A *simple* linear algebraic group is defined to be a linear algebraic group that has no proper closed connected normal subgroups (in the Zariski topology). An example of a simple algebraic group is $\mathrm{SL}_n(K)$. It can be shown that each simple algebraic group has a root system, and the different root systems are the same as the ones of the simple Lie algebras discussed in Section 2.2. (Using the appropriate root system it can be seen that the orthogonal groups are linear algebraic groups also when the characteristic of K is two. For details, see [5, Section 1.11].)

However, the root system does not determine the simple algebraic group uniquely. Instead, for every root system there is a finite number of simple algebraic groups associated with it. The different simple groups sharing a root system are called *isogenous*. Two of these groups have special names, and they are called *simply-connected* and *adjoint* groups, respectively. The first have the largest centre among their isogeny, whereas the centre of any adjoint group is trivial.

Finally, let us explain how the finite classical groups are obtained from the algebraic groups. The same procedure applies also to other groups of Lie type. Let $\bar{\mathbb{F}}_p$ be the algebraic closure of a prime field $\mathbb{F}_p$. Choose a root system and a power of p, denoted q. Let G be a simple linear algebraic group over $\bar{\mathbb{F}}_p$ with that root system.

Let F be the map that takes a matrix in $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$ and raises each element to its q'th power. If G is embedded into $\mathrm{GL}_n(\bar{\mathbb{F}}_p)$ in such a way that F takes G into itself, then F restricted to G is called a *standard Frobenius morphism* of G. A *Frobenius morphism* is then a homomorphism from G to G, some power of which is a standard Frobenius morphism. If F is any Frobenius morphism, the fixed point group

$$G^F = \{g \in G \mid F(g) = g\}$$

is a finite subgroup of G. This subgroup is a group of Lie type. The Chevalley groups arise in this way from standard Frobenius maps, and the twisted groups come from non-standard ones. For example, the standard Frobenius morphism $F_1\colon [a_{ij}] \mapsto [a_{ij}^q]$ defines $\mathrm{SL}_n(q)$ as a subgroup of $\mathrm{SL}_n(\bar{\mathbb{F}}_p)$, whereas the non-standard $F_2\colon [a_{ij}] \mapsto (([a_{ij}^q])^{\mathrm{T}})^{-1}$ defines the group $\mathrm{SU}_n(q)$.

2.3.5. The simply-connected groups

The group G^F is not necessarily simple, but if G is simply-connected, then G^F is quasisimple, apart from a few exceptions where G^F is soluble. (For the exceptions, see Table 2.2.) As a matter of fact, the group G^F is usually the universal covering group of its simple quotient. For the classical groups, this can be seen by comparing the order of the centre of G^F to the Schur multiplier of the simple quotient. For the exceptional types, see e.g. [45].

In Table 2.3, we list the finite group G^F for every simply-connected simple algebraic group G of classical Lie type, together with the corresponding simple quotient G_0 and a certain integer m. This m is such that $\mathbb{Z}_m$ is the generic Schur multiplier of G_0 corresponding to a simply-connected covering group, except when $G_0 = \mathrm{P}\Omega^+_{2r}(q)$ with q odd and r even, for then the multiplier is $\mathbb{Z}_2 \times \mathbb{Z}_2$ ([31, Theorem 5.1.4]). Table 2.4 gives the corresponding information on groups of exceptional Lie type. The exceptional multipliers of all simple groups of Lie type are collected in Table 2.5.

Lie type of G	F	G^F	G_0	m
A_r	standard	$\mathrm{SL}_{r+1}(q)$	$\mathrm{PSL}_{r+1}(q)$	$\gcd(r+1, q-1)$
	non-standard	$\mathrm{SU}_{r+1}(q)$	$\mathrm{PSU}_{r+1}(q)$	$\gcd(r+1, q+1)$
B_r	standard	$\mathrm{Spin}_{2r+1}(q)$	$\mathrm{P}\Omega_{2r+1}(q)$	2
C_r	standard	$\mathrm{Sp}_{2r}(q)$	$\mathrm{PSp}_{2r}(q)$	$\gcd(2, q-1)$
D_r	standard	$\mathrm{Spin}^+_{2r}(q)$	$\mathrm{P}\Omega^+_{2r}(q)$	$\gcd(4, q^m-1)$
	non-standard	$\mathrm{Spin}^-_{2r}(q)$	$\mathrm{P}\Omega^-_{2r}(q)$	$\gcd(4, q^m+1)$

Table 2.3: The finite groups G^F for simply-connected simple algebraic groups G of classical Lie type. The group G_0 is the simple quotient of G^F, except when G^F is soluble or $G^F = \mathrm{Sp}_4(2)$. The integer m is related to the Schur multiplier (see the text for details). From [31, Table 5.1.A].

Lie type of G	F	G_0	m
B_2	non-standard	${}^2B_2(q)$	1
D_4	non-standard	${}^3D_4(q)$	1
E_6	standard	$E_6(q)$	$\gcd(3, q-1)$
	non-standard	${}^2E_6(q)$	$\gcd(3, q+1)$
E_7	standard	$E_7(q)$	$\gcd(2, q-1)$
E_8	standard	$E_8(q)$	1
F_4	standard	$F_4(q)$	1
	non-standard	${}^2F_4(q)$, ${}^2F_4(2)'$	1
G_2	standard	$G_2(q)$, $G_2(2)'$	1
	non-standard	${}^2G_2(q)$, ${}^2G_2(3)'$	1

Table 2.4: The finite simple quotients G_0 of the groups G^F for simply-connected simple algebraic groups G of exceptional Lie type, except when G^F is soluble. The integer m is related to the Schur multiplier (see the text for details). From [31, Table 5.1.B].

G_0	$M(G_0)$
$A_1(4)$, $A_2(2)$, $A_3(2)$, ${}^2A_3(2)$, $C_3(2)$, $G_2(4)$, $F_4(2)$	$\mathbb{Z}_2$
$G_2(3)$	$\mathbb{Z}_3$
$A_1(9)$, $C_2(2)'$, $B_3(3)$	$\mathbb{Z}_6$
$A_2(4)$	$\mathbb{Z}_4 \times \mathbb{Z}_{12}$
${}^2A_3(3)$	$\mathbb{Z}_3 \times \mathbb{Z}_{12}$
${}^2A_5(2)$, ${}^2E_6(2)$	$\mathbb{Z}_2 \times \mathbb{Z}_6$
$D_4(2)$, ${}^2B_2(8)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$G_2(2)'$, ${}^2F_4(2)'$ (not simply-connected)	1

Table 2.5: Exceptional Schur multipliers of finite simple groups of Lie type. From [31, Table 5.1.D] and [45, Section 3].

3. Complex representation growth of the simple alternating groups and their covering groups

3.1. Statement of results

Let A_d denote the alternating group on d symbols. As was mentioned in the Introduction on page 19, Martin Liebeck and Aner Shalev have proved in [39] the following result for complex representation growth of the alternating groups: for any $s > 0$, there is a constant c_s, such that $\zeta_{A_d}(s) < c_s$ for all $d \geq 5$. Hence,

$$r_n(A_d) < c_s n^s \quad \text{for all } n > 1 \text{ and } d \geq 5.$$

In this chapter, we present numerical values for the constants c_s for certain values of s. We also extend our methods to compute similar bounds for the universal covering groups of A_d. Let us first list the results we obtained for the alternating groups.

Theorem 3.1. *(a) For any $d \geq 5$, we have*

$$r_n(A_d) \leq \tfrac{2}{3}n \qquad \textit{for all } n > 1.$$

(b) For any $d \geq 47$, we have

$$r_n(A_d) < 0.0218n \qquad \textit{for all } n > 1.$$

The constant 2/3 in part (a) cannot be made smaller because A_5 has two 3-dimensional representations. The constant in part (b) is an estimate, but we have $r_{46}(A_{47}) = 1$ and $1/46 \approx 0.02174$, so the constant is smallest possible at three significant digits.

Theorem 3.2. *Let $d_0 \in \{5, 12, 39\}$ and $s \in \{5/6,\ 3/4,\ 2/3,\ 1/2\}$, and assume $d \geq d_0$. We have*

$$r_n(A_d) \leq c_{s,d_0} \cdot n^s \qquad \textit{for all } n > 1,$$

where the constants c_{s,d_0} are as given in Table 3.1.

Consider next the sum

$$s_n = \sum_{d \geq 5} r_n(A_d).$$

In the following theorems, we find bounds of the form $s_n < c_s n^s$ for this sum.

$s:$	5/6	3/4	2/3	1/2
$c_{s,5}:$	0.801	0.898	0.962	1.16
$c_{s,12}:$	0.136	0.166	0.203	0.302
$c_{s,39}:$	0.0483	0.0654	<0.0966	<0.293

Table 3.1: Bounding constants for $r_n(A_d)$. Values written with the $<$-sign represent estimates, other constants are known to be smallest possible. All values are rounded up to three significant digits. This notation is used for all tables in this section.

Theorem 3.3. *For all $n > 1$, we have*

(a) $s_n \leq \frac{2}{3}n$
(b) $s_n \leq \frac{2}{21}n + 6$
(c) $s_n < 1.18 \cdot 10^{-4}\, n + 9.$

The multiplicative constants in parts (a) and (b) of the above theorem are smallest possible, since there are two 3-dimensional and eight 21-dimensional representations.

Theorem 3.4. *For all $n > 1$, and for s in $\{5/6, 3/4, 2/3, 1/2\}$, we have*

$$s_n \leq a_s n^s, \quad s_n \leq b_s n^s + 6 \quad \text{and} \quad s_n \leq c_s n^s + 9,$$

where the values of a_s, b_s and c_s are as given in Table 3.2.

$s:$	5/6	3/4	2/3	1/2
$a_s:$	0.801	0.898	1.06	<2.52
$b_s:$	0.159	0.203	0.263	<2.51
$c_s:$	<0.00120	<0.00499	<0.0322	<2.50

Table 3.2: Bounding constants for s_n. For the notation, see Table 3.1.

Furthermore, we prove a general upper bound for the constant c_s as a function of the exponent s.

Theorem 3.5. *There exists an absolute constant C, such that for $0 < s < 1$, we have*

$$s_n \leq C^{1/s} n^s \quad \text{for all } n.$$

Let us then present the results related to the universal covering groups of A_d, which we shall denote by $\tilde{A}_d$. We restrict our attention to faithful representations.

Theorem 3.6. *(a) For any $d \geq 5$, we have*

$$r_n^f(\tilde{A}_d) \leq \tfrac{4}{3}n \qquad \text{for all } n > 1.$$

(b) For any $d \geq 8$, we have

$$r_n^f(\tilde{A}_d) \leq \tfrac{1}{4}n \qquad \text{for all } n > 1.$$

(c) For any $d \geq 38$, we have

$$r_n^f(\tilde{A}_d) < 1.17 \cdot 10^{-5}\, n \qquad \text{for all } n > 1.$$

The constants in parts (a) and (b) are smallest possible as $\tilde{A}_6$ has four 3-dimensional representations and $\tilde{A}_9$ has two 8-dimensional representations. Part (c) is an estimate, but $r_{262144}^f(\tilde{A}_{39}) = 2$ and $2/262144 \approx 0.763 \cdot 10^{-5}$.

Theorem 3.7. *Let $d_0 \in \{5, 8, 30\}$ and $s \in \{5/6,\ 3/4,\ 2/3,\ 1/2\}$, and assume $d \geq d_0$. We have*

$$r_n^f(\tilde{A}_d) \leq c_{s,d_0} \cdot n^s \qquad \text{for all } n > 1,$$

where c_{s,d_0} are as in Table 3.3.

s :	5/6	3/4	2/3	1/2
$c_{s,5}$:	1.61	1.76	1.93	2.45
$c_{s,8}$:	0.354	0.421	0.500	0.708
$c_{s,30}$:	$6.16 \cdot 10^{-4}$	0.00139	< 0.00422	< 0.0804

Table 3.3: Bounding constants for $r_n^f(\tilde{A}_d)$. For the notation, see Table 3.1.

As before, consider the sum

$$\tilde{s}_n(d_0) = \sum_{d \geq d_0} r_n^f(\tilde{A}_d).$$

Because A_6 and A_7 have exceptional covering groups with rather many small-dimensional representations, we give the results on $\tilde{s}_n(d_0)$ separately for $d_0 = 5$ and $d_0 = 8$.

Theorem 3.8. *For all $n > 1$, we have*

(a) $\tilde{s}_n(5) \leq \frac{13}{6}n$ *and* $\tilde{s}_n(8) \leq \frac{3}{8}n$

(b) $\tilde{s}_n(5) \leq \frac{5}{3}n + 3$ *and* $\tilde{s}_n(8) \leq \frac{3}{64}n + 3$

(c) $\tilde{s}_n(5) \leq \frac{7}{6}n + 6$ *and* $\tilde{s}_n(8) < 7.104 \cdot 10^{-4}\, n + 6.$

The multiplicative constants in the above theorem are smallest possible for $d_0 = 5$, since there are thirteen 6-dimensional representations. The constants in parts (a) and (b) are also smallest possible for $d_0 = 8$, since there are three 8-dimensional and six 64-dimensional representations.

Theorem 3.9. *For all $n > 1$, and for s in $\{5/6,\ 3/4,\ 2/3,\ 1/2\}$, we have*

$$\tilde{s}_n(d_0) \leq a_s n^s, \quad \tilde{s}_n(d_0) \leq b_s n^s + 3 \quad \textit{and} \quad \tilde{s}_n(d_0) \leq c_s n^s + 6,$$

where the values for a_s, b_s and c_s are as given in Table 3.4 for $d_0 = 5$ and Table 3.5 for $d_0 = 8$.

s :	5/6	3/4	2/3	1/2
a_s :	2.93	3.40	3.94	5.31
b_s :	2.25	2.61	3.03	< 4.92
c_s :	1.58	1.83	2.12	< 4.92

Table 3.4: Bounding constants for $\tilde{s}_n(5)$.

s :	5/6	3/4	2/3	1/2
a_s :	0.531	0.631	0.750	< 4.92
b_s :	0.0938	0.133	< 0.259	< 4.92
c_s :	< 0.0136	< 0.0591	< 0.259	< 4.92

Table 3.5: Bounding constants for $\tilde{s}_n(8)$.

The results described above are proved in Section 3.4 for the alternating groups (Theorems 3.1–3.5), and in Section 3.6 for the covering groups (Theorems 3.6–3.9). The proofs are based on three types of information:

1. Computed values for all character degrees of every A_d up to $d = 50$ and every $\tilde{A}_d$ up to $d = 60$.
2. Values and multiplicities[1] of the 6 smallest character degrees of any A_d.
3. Lower bounds for other character degrees of A_d and $\tilde{A}_d$ and upper bounds for their multiplicities.

To obtain lower bounds for character degrees, we will state and prove Theorems 3.10 and 3.18 in Sections 3.3 and 3.5, respectively. Theorem 3.10 gives lower bounds to the degrees in certain classes of characters of A_d. Theorem 3.18 is an analogous result for the covering group. The proofs are based on the combinatorial characterisation of characters of the symmetric group and its covering group, using Young diagrams.

[1] Multiplicity is the number of characters with the same degree.

3.2. Representation theory of alternating groups and their covering groups

Let us first recall a few basic results of representation theory of symmetric and alternating groups. These results can be found e.g. in [29]. The irreducible characters of the symmetric group S_d are labelled by partitions of d. Each partition $\lambda = (\lambda_1, \dots, \lambda_k)$ is a decreasing sequence of positive integers, its *parts*, summing up to d. For example, $(6, 4, 4, 2, 1)$ is a partition of 17. The partitions can be represented as *Young diagrams* (also called *Ferrers diagrams*). These consist of left-aligned rows of cells, for which the ith row contains λ_i cells (see Figure 3.1).

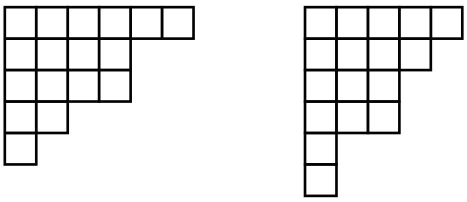

Figure 3.1: The Young diagrams representing the partition $\lambda = (6, 4, 4, 2, 1)$ and its conjugate, $\lambda' = (5, 4, 3, 3, 1, 1)$.

For an irreducible character χ^λ of S_d corresponding to a partition λ, there are two useful formulae for computing the degree. For each cell (i, j) in the Young diagram, let $h_\lambda(i, j)$ denote the number of cells directly to the right and directly below it, adding one for the cell itself. The numbers $h_\lambda(i, j)$ are called *hook lengths* (see Figure 3.2), and the character degree is given by the *Hook Formula* ([29, Theorem 2.3.2])

$$\chi^\lambda(1) = \frac{n!}{\prod_{i,j} h_\lambda(i, j)}. \tag{3.1}$$

The product is taken over all cells in the Young diagram.

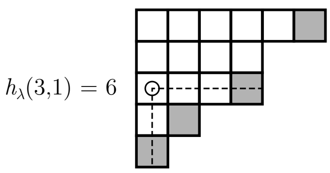

Figure 3.2: A Young diagram showing a hook length and every removable cell.

Another way to compute the degree of χ^λ is recursive. In the Young diagram, there are certain single cells in the right ends of the rows that can be removed so that the resulting diagram still remains a Young diagram corresponding to a partition. (Namely, no row may become shorter than the rows below it; see Figure 3.2.) If λ is partition of d, let Λ^- denote the collection of partitions of $d - 1$ resulting from the removal of single cells in the above manner. Then, we have the *Branching Rule* ([29, Theorem 2.4.3])

$$\chi^\lambda(1) = \sum_{\mu \in \Lambda^-} \chi^\mu(1). \tag{3.2}$$

The Branching Rule gives also a combinatorial characterisation for the degree of χ^λ. A *Young tableau* is a Young diagram obtained by filling in the cells with integers from 1 to d. A Young tableau corresponding to a diagram of a partition λ is called a λ-*tableau*. A λ-tableau is *standard* if the entries increase down along the columns and from left to right along the rows (see Figure 3.3). Now, the cell with the largest entry in a standard λ-tableau can always be removed so that the result is a standard μ-tableau, where μ is a partition of $d-1$. By the Branching Rule and simple induction, we see that the degree of χ^λ is the total number of standard λ-tableaux.

1	3	6	7	10	11
2	5	8	16		
4	12	13	17		
9	14				
15					

1	2	3	7	10	17
4	6	8	11		
5	12	14	16		
9	13				
15					

Figure 3.3: Two standard λ-tableau with $\lambda = (6,4,4,2,1)$. Note that cell 17 is removable in both tableaux.

Let us then turn to alternating groups. For a partition λ, the *conjugate partition* λ' is obtained from by swapping the rows and columns in the corresponding Young diagram (see Figure 3.1). Notice that by the Hook Formula, characters corresponding to conjugate partitions have the same degree. A partition λ is called *self-conjugate*, if $\lambda = \lambda'$. When λ is self-conjugate, the corresponding character χ^λ, when restricted to A_d (see Section 2.1.3), splits into a sum $\chi_1 + \chi_2$ of two irreducible characters. Moreover, the degrees of χ_1 and χ_2 are half of that of the original χ^λ. On the other hand, if the partition λ is not self-conjugate, the characters corresponding to λ and λ' fuse in A_d, leaving only one irreducible character with the original degree. All the irreducible characters of A_d are obtained in this fashion. (See [29, Theorem 2.5.7].) As the degree of a character of A_d can be obtained from one of the two conjugate partitions, we need not consider these partitions separately. Instead, we will concentrate on so called *primary* partitions, where $\lambda_1 \geq \lambda'_1$, that is, there are more columns than rows in the Young diagram.

Lastly, a few words on the faithful characters of the covering groups of S_d and A_d. This theory is thoroughly explained in [26]. When $n \geq 4$, the symmetric group S_d has two non-isomorphic full covering groups, both of which have Schur multiplier 2 ([26, Theorems 2.9 and 2.12]). When $n > 7$, both of these covers have a subgroup of index 2 isomorphic to the universal covering group of the alternating group A_d. This subgroup will henceforth be denoted $\tilde{A}_d$.

One of the covers of S_d, denoted $\tilde{S}_d$, has a combinatorial theory of characters similar to that of S_d. A partition is called *strict* if its parts are strictly decreasing. For each faithful irreducible representation of $\tilde{S}_d$, there is a corresponding *strict partition* of d.

(However, the correspondence is not one-to-one as it is with S_d.) A strict partition can be represented as a *shifted diagram*, where the rows are not left-aligned but the first cell in row i is placed under the second cell in row $i-1$, forming a kind of upside-down "staircase" (see Figure 3.4). Filling in the cells of a shifted diagram with integers from 1 to d gives a *shifted tableau*. As before, a *standard shifted tableau* has the entries in increasing order along its rows and columns.

Figure 3.4: A shifted diagram representing the strict partition $\lambda = (6,4,3,1)$, and a standard shifted λ-tableau.

For a strict partition λ, we shall denote the total number of standard shifted λ-tableaux by g_λ. Let r be the number of parts in the partition λ of d, and let $\varepsilon(\lambda)$ be the *parity* of λ. The parity is 1 (odd) if the number of even parts in λ is odd, and 0 (even) otherwise. Thus $\varepsilon(\lambda)$ corresponds to the parity of a permutation of cycle type λ, and is also the parity of the integer $d+r$. According to [26, Theorem 10.7], the degree of a character of $\tilde{S}_d$ corresponding to λ is given by

$$\chi^\lambda_{\tilde{S}_d}(1) = 2^{(d-r-\varepsilon(\lambda))/2} g_\lambda. \tag{3.3}$$

The Hook Formula and the Branching Rule can also be formulated for shifted diagrams, but they give the value of g_λ instead of the character degree. The Branching Rule is completely analogous to the earlier (see [26, equation (10.5)]), but for the Hook Formula, we need the concept of a *shift-symmetric* diagram of λ. It is a Young diagram obtained from the shifted diagram of λ by adding first a zeroth column of length λ_1, and then, under the first cell of each row i but the last, a column of length λ_{i+1}. The hook length of a cell in the shifted diagram of λ is defined to be the hook length of the shift-symmetric diagram (see Figure 3.5). Now, the Hook Formula (3.1) can be applied to give the value of g_λ ([26, Proposition 10.6]).

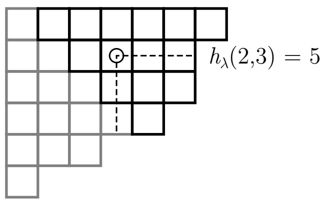

Figure 3.5: A hook length of a cell in a shifted diagram of λ is measured from the corresponding shift-symmetric diagram.

If a strict partition λ is odd, there are two irreducible characters of $\tilde{S}_d$ corresponding to λ. They are called *associates*, and they have the same degree. Restricted to $\tilde{A}_d$, they remain irreducible and fuse to become the same character with the original degree. However, if λ is even, the irreducible character corresponding to λ is *self-associate*. Under restriction to $\tilde{A}_d$, the self-associate character splits into two characters, both irreducible, whose degrees are half of that of the original character. All faithful irreducible characters of $\tilde{A}_d$ are obtained in this way. (See [26, Theorem 8.6].)

3.3. Finding lower bounds to character degrees of S_d

Let us first prove our main tool for obtaining lower bounds for the character degrees. We are going to show that when the length of the first row is restricted, the smallest character degrees of S_n come from two-part partitions. The degree of a character corresponding to a two-part partition $\mu_{d,m} = (d-m, m)$ of d can easily be computed from the Hook Formula (3.1), and it equals

$$\chi^{\mu_{d,m}}(1) = \frac{d!\,(d-2m+1)}{m!\,(d-m+1)!} = \frac{d!}{m!\,(d-m)!} \cdot \frac{d-2m+1}{d-m+1}.$$

The graph of a typical function giving the dependence of the degree of $\chi^{\mu_{d,m}}$ on m is sketched in Figure 3.6. (Instead of factorials, gamma functions are used to make the graph continuous.) By looking at the ratio $\chi^{\mu_{d,m+1}}(1)/\chi^{\mu_{d,m}}(1)$, it is not hard to conclude that the degrees are strictly increasing while $m \leq \lceil \frac{1}{2}(d - \sqrt{d+2}) \rceil$ and strictly decreasing after that.

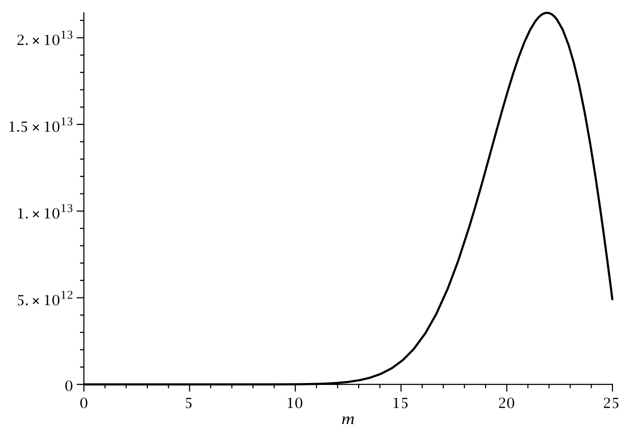

Figure 3.6: The graph of $m \mapsto \chi^{\mu_{d,m}}(1)$ in the case $d = 50$.

For the statement of the theorem, we shall need the following notation. For any integer $m < d$, let $L_{d,m}$ stand for the set of partitions $(\lambda_1, \ldots, \lambda_r)$ of d, where $\lambda_1 \geq \lambda_1'$ and $\lambda_1 \leq d - m$. Notice that $\mu_{d,m} \in L_{d,m}$, and $L_{d,m} \subset L_{d,m'}$ whenever $m \geq m'$.

Theorem 3.10. *Assume d is a positive integer at least 10, and m is an integer such that $0 \leq m \leq d/2$. Then, for all partitions λ in $L_{d,m}$ with at least three parts, we have $\chi^{\lambda}(1) > \chi^{\mu_{d,m}}(1)$.*

Proof. We proceed by induction on d.

Base step. We deal here with some particular partitions. First, we note that if $d = 10$, the claim holds, as can be seen from the following table:

m	$\chi^{\mu_{10,m}}(1)$	$\chi^\lambda(1)$ for $\lambda \in L_{10,m} \setminus L_{10,m+1}$ having ≥ 3 parts
2	35	36 $(8, 1^2)$
3	75	160 $(7, 2, 1)$, 84 $(7, 1^3)$
4	90	315 $(6, 3, 1)$, 225 $(6, 2^2)$, 350 $(6, 2, 1^2)$, 126 $(6, 1^4)$
5	42	288 $(5, 4, 1)$, 450 $(5, 3, 2)$, 567 $(5, 3, 1^2)$, 525 $(5, 2^2, 1)$, 448 $(5, 2, 1^3)$

Next, we look at some small rectangular Young diagrams and notice that the claim also holds for the partitions corresponding to these. Namely,

- $\chi^{(4^3)}(1) = 462$, but $\chi^{\mu_{12,m}}(1) \leq \chi^{(7,5)}(1) = 297$ for all m
- $\chi^{(5^3)}(1) = 6006$, but $\chi^{\mu_{15,m}}(1) \leq \chi^{(9,6)}(1) = 2002$ for all m
- $\chi^{(6^3)}(1) = 87516$, but $\chi^{\mu_{18,m}}(1) \leq \chi^{(11,7)}(1) = 13260$ for all m
- $\chi^{(4^4)}(1) = 24024$, but $\chi^{\mu_{16,m}}(1) \leq \chi^{(10,6)}(1) = 3640$ for all m.

Last, to simplify the induction step, we check here two special forms of partitions where λ has exactly 3 parts, the last of which is 1. We assume that $d \geq 11$.

Case 1. Suppose that $\lambda = (d-2, 1, 1)$. By the Hook Formula,

$$\chi^\lambda(1) = \frac{(d-1)(d-2)}{2} = \frac{d(d-3)}{2} + 1.$$

On the other hand, if $\lambda \in L_{d,m}$, we must have $m \leq 2$. Three cases occur:

$$\chi^{\mu_{d,0}}(1) = 1, \quad \chi^{\mu_{d,1}}(1) = d - 1 \quad \text{and} \quad \chi^{\mu_{d,2}}(1) = \frac{d(d-3)}{2}.$$

In each of these, the inequality $\chi^\lambda(1) > \chi^{\mu_{d,m}}(1)$ holds, as $d \geq 11$.

Case 2. Suppose that $\lambda = (a, a, 1)$, where $a = \frac{d-1}{2}$. (This means that d is odd.) The Hook Formula gives

$$\chi^\lambda(1) = \frac{d!}{(a-1)!\,(a+1)\,a!\,(a+2)} = d! \Big/ \frac{(a+1)!\,(a+2)!}{a(a+1)}.$$

We will focus on the divisor in the last expression (right side of the division symbol '/'), which we call D_1, and which equals

$$D_1 = \frac{4\left(\frac{d+1}{2}\right)!\left(\frac{d+3}{2}\right)!}{d^2 - 1}.$$

We claim that $\chi^\lambda(1)$ is greater than the degree of any character corresponding to a two-part partition $\mu_{d,m}$. Such a character has the degree

$$\chi^{\mu_{d,m}}(1) = \frac{d!\,(d-2m+1)}{m!\,(d-m+1)!} = d! \Big/ \frac{m!\,(d-m+1)!}{d-2m+1}.$$

We focus again on the divisor in the last expression, which we shall call $D_2(m)$, and argue that its smallest possible value is still greater than D_1. This will prove the claim.

As noted before the statement of the theorem, the expression $D_2(m)$ obtains its minimum at $m_0 = \lceil \frac{1}{2}(d - \sqrt{d+2}) \rceil$. Substituting this value into $D_2(m)$ and cleaning up the denominator results in this inequality:

$$D_2(m) \geq D_2(m_0) \geq \frac{\lceil \frac{d-\sqrt{d+2}}{2} \rceil! \lfloor \frac{d+\sqrt{d+2}+2}{2} \rfloor!}{\sqrt{d+2}+1}.$$

Denote the expression on the right-hand side by D_2^*. Direct calculation shows that

$$D_2^*/D_1 = \frac{d^2-1}{4(\sqrt{d+2}+1)} \cdot \frac{\frac{d+5}{2} \cdot \frac{d+7}{2} \cdots \lfloor \frac{d+\sqrt{d+2}+2}{2} \rfloor}{\lceil \frac{d-\sqrt{d+2}+2}{2} \rceil \cdots \frac{d-1}{2} \cdot \frac{d+1}{2}}.$$

The product above the line in the rightmost fraction contains $\lfloor \frac{1}{2}(\sqrt{d+2}-1) \rfloor$ terms, and the product below the line has $\lfloor \frac{1}{2}(\sqrt{d+2}+1) \rfloor$ terms, which is exactly one term more. Since each term below the line is less than each term above the line, we can conclude that

$$D_2^*/D_1 \geq \frac{d^2-1}{4(\sqrt{d+2}+1)} \cdot \frac{1}{\frac{d+1}{2}} = \frac{d-1}{2(\sqrt{d+2}+1)}.$$

With $d \geq 11$, the last expression is greater than 1; hence we get $D_2(m) > D_1$ for all m, and finally $\chi^{\lambda}(1) > \chi^{\mu_{d,m}}(1)$ for all m.

Induction step. Suppose that $d > 10$ and the claim holds for all integers at least 10 and less than d. Assume that $0 \leq m \leq d/2$ and $\lambda \in L_{d,m}$ has at least three parts. Moreover, suppose that λ is not (4^3), (5^3), (6^3) or (4^4), nor either of $(d-2,1,1)$ or $(\frac{d-2}{2}, \frac{d-2}{2}, 1)$. Denote $r = \mathrm{len}(\lambda)$, the number of parts in λ.

Notice first that $\chi^{\mu_{d,0}}(1) = 1$, so that clearly $\chi^{\lambda}(1) > \chi^{\mu_{d,0}}(1)$.

Assume next that $m = d/2$. Now $\mu_{d,m} = (m,m)$, and by the Branching Rule, we have $\chi^{\mu_{d,m}}(1) = \chi^{\mu_{d-1,m-1}}(1)$. If $r > 3$ or $\lambda_3 \geq 2$, let $\lambda(1) = (\lambda_1, \ldots, \lambda_r - 1)$. Now, we have $\chi^{\lambda}(1) \geq \chi^{\lambda(1)}(1)$ by the Branching Rule, and $\lambda(1)_1 = \lambda_1 \leq (d-1)-(m-1)$, so $\lambda(1) \in L_{d-1,m-1}$. Notice that $\lambda(1)$ still has at least 3 parts, so we can apply the induction hypothesis to get $\chi^{\lambda(1)}(1) > \chi^{\mu_{d-1,m-1}}(1)$. Hence, $\chi^{\lambda}(1) > \chi^{\mu_{d,m}}(1)$.

On the other hand, if $\lambda = (\lambda_1, \lambda_2, 1)$, we know by the assumptions made about λ that $\lambda_2 > 1$. Hence we can choose $\lambda(1) = (\lambda_1, \lambda_2 - 1, 1)$, and proceed as above to get $\chi^{\lambda}(1) > \chi^{\mu_{d,m}}(1)$.

We may now assume that $1 \leq m < d/2$. Using the Branching Rule, we get

$$\chi^{\mu_{d,m}}(1) = \chi^{\mu_{d-1,m}}(1) + \chi^{\mu_{d-1,m-1}}(1). \tag{3.4}$$

Here we break the argument into different cases.

Case 1. Suppose that $\lambda_1 = \lambda_2$ but $\lambda_k > \lambda_{k+1}$ for some $2 \leq k < r = \mathrm{len}(\lambda)$. Because of the assumptions made about λ, either $r > 3$ or $\lambda_r \geq 2$. Denoting

$$\lambda(1) = (\lambda_1, \ldots, \lambda_k - 1, \ldots, \lambda_r) \quad \text{and} \quad \lambda(2) = (\lambda_1, \ldots, \lambda_r - 1),$$

the Branching Rule gives

$$\chi^{\lambda}(1) \geq \chi^{\lambda(1)}(1) + \chi^{\lambda(2)}(1). \tag{3.5}$$

Because $\lambda_1 = \lambda_2$ and λ has at least three parts, we have $\lambda_1 \leq \frac{d-1}{2}$. Also, we assumed that $m \leq \frac{d-1}{2}$, so

$$\lambda(1)_1 = \lambda(2)_1 = \lambda_1 \leq \frac{d-1}{2} = d - 1 - \frac{d-1}{2} \leq d - 1 - m.$$

This implies that $\lambda(1)$ and $\lambda(2)$ are in $L_{d-1,m} \subset L_{d-1,m-1}$. Since $m \leq \frac{d-1}{2}$ and $\lambda(1)$ and $\lambda(2)$ have at least three parts, the induction hypothesis yields

$$\chi^{\lambda(1)}(1) > \chi^{\mu_{d-1,m}}(1) \quad \text{and} \quad \chi^{\lambda(2)}(1) > \chi^{\mu_{d-1,m-1}}(1).$$

By these equations and those of (3.4) and (3.5), we get $\chi^{\lambda}(1) > \chi^{\mu_{d,m}}(1)$.

Case 2a. Suppose $\lambda_1 > \lambda_2$, $\lambda_1 > r = \mathrm{len}(\lambda)$, and either $r > 3$ or $\lambda_r \geq 2$. Denote

$$\lambda(1) = (\lambda_1 - 1, \ldots, \lambda_r) \quad \text{and} \quad \lambda(2) = (\lambda_1, \ldots, \lambda_r - 1),$$

so that $\chi^{\lambda}(1) \geq \chi^{\lambda(1)}(1) + \chi^{\lambda(2)}(1)$. Now, because $\lambda(1)_1 = \lambda_1 - 1 \leq d - m - 1$ and $\lambda(1)_1 \geq \mathrm{len}(\lambda(1))$, we have $\lambda(1) \in L_{d-1,m}$. Also, $\lambda(2)_1 = \lambda_1 \leq d - m$, so $\lambda(2) \in L_{d-1,m-1}$. As in the previous case, the induction hypothesis gives

$$\chi^{\lambda(1)}(1) + \chi^{\lambda(2)}(1) > \chi^{\mu_{d-1,m}}(1) + \chi^{\mu_{d-1,m-1}}(1) = \chi^{\mu_{d,m}}(1).$$

Case 2b. Suppose $\lambda = (\lambda_1, \lambda_2, 1)$, where $\lambda_1 > \lambda_2$. We know from the assumptions that $\lambda_1 > 3$ and $\lambda_2 > 1$. Denote

$$\lambda(1) = (\lambda_1 - 1, \lambda_2, 1) \quad \text{and} \quad \lambda(2) = (\lambda_1, \lambda_2 - 1, 1).$$

As above, we have $\lambda(1) \in L_{d,m-1}$ and $\lambda(2) \in L_{d-1,m-1}$, so that by the induction hypothesis,

$$\chi^{\lambda}(1) \geq \chi^{\lambda(1)}(1) + \chi^{\lambda(2)}(1) > \chi^{\mu_{d,m-1}}(1) + \chi^{\mu_{d-1,m-1}}(1) = \chi^{\mu_{d,m}}(1).$$

Case 3a. Suppose $\lambda_1 > \lambda_2$, $\lambda_1 = r$ and $\lambda_r \geq 2$. Here, the conjugate partition λ' is in $L_{d,m}$ and satisfies the assumptions of Case 1 (notice that λ' has more than three parts, since $d > 10$). Because the character degrees corresponding to conjugate partitions are the same, we can refer to Case 1.

Case 3b. Suppose that $\lambda_1 > \lambda_2$ and $\lambda_1 = r$, and that $\lambda_r = 1$ but $\lambda_k > \lambda_{k+1}$ for some $2 \leq k < r$. We know that $r > 3$, because $d > 10$. Denote

$$\lambda(1) = (\lambda_1, \ldots, \lambda_k - 1, \ldots, \lambda_r) \quad \text{and} \quad \lambda(2) = (\lambda_1, \ldots, \lambda_{r-1}).$$

Now, since the first row and column of the Young diagram corresponding to λ are of equal length and do not contain all the cells in the diagram, we can reason as follows. If d is even, we have $\lambda_1 \leq d/2$ and $m \leq d/2 - 1$ (we have assumed $m < d/2$), so that

$$\lambda(1)_1 = \lambda_1 \leq \frac{d}{2} = d - \left(\frac{d}{2} - 1\right) - 1 \leq d - m - 1.$$

However, if d is odd, we have $\lambda_1 \leq \frac{d-1}{2}$ and $m \leq \frac{d-1}{2}$, so that

$$\lambda(1)_1 = \lambda_1 \leq \frac{d-1}{2} = d - \left(\frac{d-1}{2}\right) - 1 \leq d - m - 1.$$

In both cases, $\lambda(1)$ is in $L_{d-1,m}$. On the other hand, we have $\lambda(2)_1 = \lambda_1 \leq d - m$, so $\lambda(2)$ is in $L_{d-1,m-1}$. As above, the induction hypothesis gives $\chi^{\lambda}(1) > \chi^{\mu_{d,m}}(1)$.

Case 3c. Suppose $\lambda_1 > \lambda_2 = \cdots = \lambda_r = 1$ and $\lambda_1 = r$. Denoting $a = \frac{d-1}{2}$ (so that $d = 2a+1$), we can write $\lambda = (a+1, 1^a)$. Let

$$\lambda(1) = \big(a, 1^a\big) \quad \text{and} \quad \lambda(2) = \big(a+1, 1^{a-1}\big).$$

Using the Branching Rule and the fact that $\lambda(1) = \lambda(2)'$, we get

$$\chi^{\lambda}(1) = \chi^{\lambda(1)}(1) + \chi^{\lambda(2)}(1) = 2\chi^{\lambda(2)}(1). \tag{3.6}$$

If $m \leq a-1$, then

$$\lambda(2)_1 = a + 1 = d - 1 - (a-1) \leq d - 1 - m,$$

so that $\lambda(2) \in L_{d-1,m} \subset L_{d-1,m-1}$. As before, by the induction hypothesis and equations (3.4) and (3.6), we get $\chi^{\lambda}(1) > \chi^{\mu_{d,m}}(1)$.

On the other hand, if $m = a$, we have to trace back one step further. If $d = 11$, then $m = 5$, with $\chi^{\lambda}(1) = \chi^{(6,1^5)}(1) = 252$ and $\chi^{\mu_{11,5}}(1) = 132$. Thus we can assume that $d > 11$.

By the Branching Rule, we have

$$\begin{aligned}
\chi^{\mu_{d,m}}(1) &= \chi^{(a+1,a)}(1) \\
&= \chi^{(a,a)}(1) + \chi^{(a+1,a-1)}(1) \\
&= 2\chi^{(a,a-1)}(1) + \chi^{(a+1,a-2)}(1) \\
&= 2\chi^{\mu_{d-2,m-1}}(1) + \chi^{\mu_{d-2,m-2}}(1).
\end{aligned} \tag{3.7}$$

Denote

$$\lambda(3) = \big(a, 1^{a-1}\big) \quad \text{and} \quad \lambda(4) = \big(a+1, 1^{a-2}\big),$$

so that $\chi^{\lambda(2)}(1) = \chi^{\lambda(3)}(1) + \chi^{\lambda(4)}(1)$, and

$$\chi^{\lambda}(1) = 2\left(\chi^{\lambda(3)}(1) + \chi^{\lambda(4)}(1)\right). \tag{3.8}$$

Since $\lambda(3)_1 = a = d - m - 1$, we have $\lambda(3) \in L_{d-2,m-1}$, and since $\lambda(4)_1 = a+1 = d-m$, we have $\lambda(4) \in L_{d-2,m-2}$. By the induction hypothesis and equations (3.7) and (3.8), we get

$$\chi^{\lambda}(1) > 2\big(\chi^{\mu_{d-2,m-1}}(1) + \chi^{\mu_{d-2,m-2}}(1)\big) > \chi^{\mu_{d,m}}(1).$$

Case 4. Suppose $\lambda_1 = \cdots = \lambda_r$. We have already checked the small rectangular diagrams in the Base Step, so we may assume that $d \geq 20$.

For this case, we have $\chi^{\lambda}(1) = \chi^{(\lambda_1,\ldots,\lambda_r-1)}(1)$, so tracing back one step does not get us anywhere. However, it is conceivable that since the two-part partition $\mu_{d,m}$ branches at most like a binary tree, and λ — having at least three rows — branches more wildly, we can prove this case by going back far enough. The only question that arises is how to keep track of the branching.

Imagine that we have taken n cells away from the rectangular diagram of λ so that after the removal of each single cell, the resulting diagram remains a valid Young diagram. Call the resulting diagram A (see Figure 3.7). The removed cells form a cluster, which, when rotated 180 degrees, becomes a valid Young diagram B representing some partition δ of n. Numbering the cells in the order they are removed from the complete rectangle, we can make diagram B into a standard δ-tableau. Thus, the number of ways the particular diagram A can be reached starting from the complete rectangle is the same as the total number of standard δ-tableaux. By the discussion in Section 3.2, this number is $\chi^{\delta}(1)$. We have therefore arrived at the following representation for the degree of χ^{λ}:

$$\chi^{\lambda}(1) = \sum_i \left(\chi^{\delta(i)}(1) \cdot \chi^{\lambda(i)}(1)\right).$$

In the above formula, the summation index i enumerates all Young diagrams with n cells removed from the diagram of λ, each $\lambda(i)$ is a partition corresponding to such a diagram, and each $\delta(i)$ is a partition corresponding to the diagram formed from the removed cells. The number of summands is the same as the number of partitions $\delta(i)$, which is at most the number of irreducible representations of S_n. (Not all partitions of n can be realised as clusters of cells removed from A.)

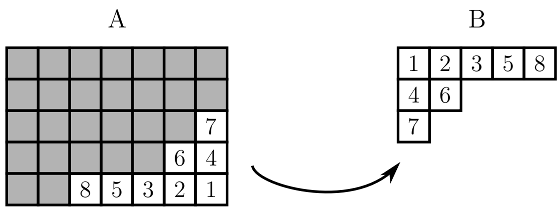

Figure 3.7: Cells removed from a rectangular diagram form a Young tableau.

Relying on the induction hypothesis and the fact that the two-part partition $\mu_{d,m}$ only branches like a binary tree, it suffices to find an n such that

$$\sum_i \chi^{\delta(i)}(1) \geq 2^{n+1}.$$

Moreover, it may happen that λ contains only three rows, and after taking away n cells, we still need to have three rows left. To achieve this, we consider only those partitions δ of n that have at most three parts, first part being at most 5 and the others at most 4. (Here we again use the assumptions made about the shape of λ.)

We will prove that $n = 8$ suffices. The following table shows the suitable partitions $\delta(i)$ and the corresponding character degrees:

i	$\delta(i)$	$\chi^{\delta(i)}(1)$
1	$(5,3)$	28
2	$(5,2,1)$	64
3	$(4,4)$	14
4	$(4,3,1)$	70
5	$(4,2,2)$	56
6	$(3,3,2)$	42

According to the above discussion, we now have

$$\chi^{\lambda}(1) \geq \sum_{i=1}^{6} \left(\chi^{\delta(i)}(1) \cdot \chi^{\lambda(i)}(1)\right) \geq 274 \min_{i} \chi^{\lambda(i)}(1). \tag{3.9}$$

Let $k \leq (d-8)/2$. We claim that each $\lambda(i)$ is in $L_{d-8,k}$. If λ has more than three parts, we have $\lambda(i)_1 = \lambda_1 \leq d/4$. Now, since $d \geq 20$, we get

$$d - 8 - k \geq \frac{d-8}{2} = \frac{d}{4} + \frac{d-16}{4} > \frac{d}{4}.$$

On the other hand, if λ has three parts, then $\lambda(i)_1 \leq \lambda_1 = d/3$. If $\lambda_1 > 7$, we have $d \geq 24$, and

$$d - 8 - k \geq \frac{d-8}{2} = \frac{d}{3} + \frac{d-24}{6} \geq \frac{d}{3}.$$

Finally, if $\lambda = (7^3)$, we have $k \leq 6$, and so

$$d - 8 - k = 13 - k \geq 7 = \lambda_1 \geq \lambda(i)_1.$$

In each case, $\lambda(i)_1 \leq d - 8 - k$, so $\lambda(i) \in L_{d-8,k}$.

Each $\lambda(i)$ has three rows, since we have ruled out the small rectangles for λ. As $d - 8 \geq 10$, we can now apply the induction hypothesis to get $\chi^{\lambda(i)}(1) > \chi^{\mu_{d-8,k}}(1)$ for all possible k. Applying the Branching Rule to μ gives at most two new branches at each step, so after 8 steps we have at most $2^8 = 256$ branches. This combined with (3.9) finally gives

$$\chi^{\lambda}(1) \geq 274 \min_{i} \chi^{\lambda(i)}(1) > 256 \max_{k} \chi^{\mu_{d-8,k}}(1) \geq \chi^{\mu_{d,m}}(1).$$

The above inequality concludes the last case and the whole proof. □

Remark. The result also holds for $d < 10$, with two exceptions: the partition $(3,3,3)$ gives a character of degree 42, while $\chi^{\mu_{9,4}}(1) = 42$ and $\chi^{\mu_{9,3}}(1) = 46$.

The above theorem tells us that to find small characters degrees, we only need to look at two-part partitions. We shall need the following approximation for these degrees.

Lemma 3.11. *Suppose $0 < r < 1/2$, and write $m = \lceil rd \rceil$. Whenever we have $d \geq d_0$, where $d_0 \geq \frac{5-8r}{(1-2r)^2}$, then*

$$\phi_1(r)k_1(r)^d \quad < \quad \chi^{\mu_{d,m}}(1) \quad < \quad \phi_2(r)k_2(r)^d,$$

where

$$\begin{aligned}
\phi_1(r) &= \exp\left(\frac{1}{12d_0+1} - \frac{1}{12r(1-r)d_0}\right)\frac{1-2r}{(1-r)\sqrt{2\pi r(1-r)}}\ , \\
k_1(r) &= \frac{1}{\sqrt[2d_0]{d_0}} \cdot \frac{1}{r^r(1-r)^{1-r}}\ , \\
\phi_2(r) &= \frac{1-2r}{r\sqrt{2\pi r(1-r-1/d_0)}}\ , \quad \textit{and} \\
k_2(r) &= \frac{1}{r^r(1-r-1/d_0)^{1-r}}.
\end{aligned}$$

Proof. For a fixed d, denote $m_0(d) = \frac{1}{2}(d - \sqrt{d+2})$. As noted in the start of this section, the map $g\colon m \mapsto \chi^{d,m}(1)$ is strictly increasing when $m \leq m_0(d)$, and the same holds for its continuous version $\bar{g}$, obtained by using gamma functions instead of factorials. Because $m_0(d)/d \to 1/2$ as $d \to \infty$, we may choose d_0 to be such that $m_0(d) \geq rd+1$ whenever $d \geq d_0$. It turns out that $d_0 = \frac{5-8r}{(1-2r)^2}$ is sufficient.

Now, $\bar{g}(m)$ is increasing up to $m = rd+1$, so $\bar{g}(rd) \leq g(\lceil rd \rceil) \leq \bar{g}(rd+1)$ holds for all $d \geq d_0$. The result follows from substituting Stirling's approximation (see [51])

$$\exp\left(\frac{1}{12n+1}\right) < \frac{n!}{\sqrt{2\pi n}\,(n/e)^n} < \exp\left(\frac{1}{12n}\right)$$

into the expressions for $\bar{g}(rd)$ and $\bar{g}(rd+1)$. (Wherever a factorial of a non-integer is implied, the gamma function is assumed instead.) For the lower bound we get

$$\bar{g}(rd) = \frac{d!}{(rd)!((1-r)d)!} \cdot \frac{(1-2r)d+1}{(1-r)d+1} > ABC,$$

where

$$\begin{aligned}
A &= \exp\left(\frac{1}{12d+1} - \frac{1}{12r(1-r)d}\right), \\
B &= \frac{1}{\sqrt{2\pi r(1-r)d}}\left(\frac{1}{r^r(1-r)^{1-r}}\right)^d, \quad \text{and} \\
C &= \frac{(1-2r)d+1}{(1-r)d+1}.
\end{aligned}$$

We find that A is increasing in d, so we can bound A from below by substituting d_0 for d. Also, it is easy to see that C is decreasing in d, and that $C \to \frac{1-2r}{1-r}$ as $d \to \infty$. Hence

$C > \frac{1-2r}{1-r}$. On the other hand, $\ln d/(2d)$ has its maximum at $d = e$ and is decreasing afterwards, so for all $d \geq d_0$,

$$\ln \sqrt[2d_0]{d_0} = \frac{\ln d_0}{2d_0} \geq \frac{\ln d}{2d},$$

which leads to

$$\left(\sqrt[2d_0]{d_0}\right)^d \geq \sqrt{d},$$

whence

$$B \geq \frac{1}{\sqrt{2\pi r(1-r)}} \left(\frac{1}{\sqrt[2d_0]{d_0}} \cdot \frac{1}{r^r(1-r)^{1-r}}\right)^d.$$

This proves the lower bound.

For the upper bound we use

$$\bar{g}(rd+1) = \frac{d!}{(rd+1)!(d-rd-1)!} \cdot \frac{(1-2r)d-1}{(1-r)d} > DEFG,$$

where

$$\begin{aligned}
D &= \exp\left(\frac{1}{12d} - \frac{1}{12(rd+1)+1} - \frac{1}{12(d-rd-1)+1}\right) \\
E &= \frac{1}{\sqrt{2\pi(r+1/d)(1-r-1/d)}}, \\
F &= \left(\frac{1}{(r+1/d)^r(1-r-1/d)^{1-r}}\right)^d \cdot \frac{1-r-1/d}{r+1/d}, \quad \text{and} \\
G &= \frac{(1-2r)d-1}{(1-r)d}.
\end{aligned}$$

Again, D is increasing and tends to 1 as $d \to \infty$, so we can bound D from above by 1. Also, G is increasing, so $G < \frac{1-2r}{1-r}$. Furthermore, as $0 < 1/d \leq 1/d_0$, we can deduce that

$$E < \frac{1}{\sqrt{2\pi r(1-r-1/d_0)}}$$

and

$$F < \left(\frac{1}{r^r(1-r-1/d_0)^{1-r}}\right)^d \cdot \frac{1-r}{r}.$$

This is enough to confirm the claimed upper bound. □

Next we will show that if $d \geq 51$ and m is small enough, we know the smallest character degree in $L_{d,m}$.

Lemma 3.12. *Assume $d \geq 51$. For any integer $m \leq \lceil 0.32d \rceil$ and for all $\lambda \in L_{d,m}$, we have $\chi^\lambda(1) \geq \chi^{\mu_{d,m}}(1)$.*

Proof. Assume that $\lambda \in L_{d,m}$. According to Theorem 3.10, if λ has at least three parts, the claim holds. Thus we may suppose that $\lambda = \mu_{d,m'}$, where $m' \geq m$. In the discussion preceding Theorem 3.10, it is mentioned that the values of $\chi^{\mu_{d,m}}(1)$ are first increasing in m and then decreasing, finally reaching the value $\chi^{\mu_{d,\lfloor d/2 \rfloor}}(1)$. Denote $m_0(d) = \lceil 0.32d \rceil$. Supposing that $m \leq m_0(d)$, we only need to show that $\chi^{\mu_{d,m_0(d)}}(1) \leq \chi^{\mu_{d,\lfloor d/2 \rfloor}}(1)$, as this would imply two things. First, for all $k \geq m_0(d)$, we would have $\chi^{\mu_{d,k}}(1) \geq \chi^{\mu_{d,m_0(d)}}(1)$, and secondly, $\chi^{\mu_{d,k}}(1)$ would be increasing for $k \leq m_0(d)$. This means that if $m' < m_0(d)$, then $\chi^{\mu_{d,m}}(1) \leq \chi^{\mu_{d,m'}}(1)$ (by monotonicity), and if $m' \geq m_0(d)$, we get

$$\chi^{\mu_{d,m}}(1) \leq \chi^{\mu_{d,m_0(d)}}(1) \leq \chi^{\mu_{d,m'}}(1).$$

Denote $f(d) = \chi^{\mu_{d,\lfloor d/2 \rfloor}}(1)$. As discussed above, it suffices to prove the inequality $f(d) \geq \chi^{\mu_{d,m_0(d)}}(1)$. For values of d from 51 to 99 this can be checked by direct computation. Let us therefore assume that $d \geq 100$. Now, since $\frac{5-8\cdot 0.32}{(1-2\cdot 0.32)^2} < 100$, we can use the upper bound from Lemma 3.11 with $d_0 = 100$ and $r = 0.32$ to get $\chi^{\mu_{d,m_0(d)}}(1) < 0.97\cdot 1.9^d$. We now want to find a lower bound for $f(d)$.

Observe that if d is even, we have $f(d) = f(d-1)$ by the Branching Rule. We may thus assume that d is even. Now, denoting $d = 2k$, we have

$$f(d) = \frac{(2k)!}{k!(k+1)!} = \left(\frac{k}{2}+1\right)\left(\frac{k}{3}+1\right)\cdots\left(\frac{k}{k-1}+1\right)\cdot 2.$$

We take logarithms and bound the resulting sum from below with an integral to get

$$\begin{aligned}
\ln f(d) &= \sum_{i=2}^{k} \ln\left(\frac{k}{i}+1\right) > \int_2^{k+1} \ln\left(\frac{k}{x}+1\right)\,dx \\
&= \ln \frac{(2k+1)^{2k+1}}{(k+2)^{k+2}(k+1)^{k+1}} + 2\ln 2 \\
&> \ln \frac{(2k)^{2k+1}}{(k+2)^{2k+1}(k+1)^2}.
\end{aligned}$$

Let $a = 1/80$. As $d \geq 100$, we have $k \geq 50$, whence $2k(1-a)/(k+2) \geq 1.9$. This gives

$$\begin{aligned}
\ln \frac{(2k)^{2k+1}}{(k+2)^{2k+1}(k+1)^2} &= (2k+1)\ln\frac{2k(1-a)}{k+2} + \underbrace{\ln\frac{(k/40)^{2k+1}}{(k+1)^2}}_{>0} \\
&> (2k+1)\ln 1.9.
\end{aligned}$$

We have proved that for even $d \geq 100$, we have $f(d-1) = f(d) > 1.9^{d+1} > 1.9^d$. Since $\max\{\chi^{\mu_{d-1,m_0(d)}}(1), \chi^{\mu_{d,m_0(d)}}(1)\} < 1.9^d$, this proves the claim. $\square$

To close this section, we use the results obtained above to determine the seven minimal character degrees of A_d. The same result was obtained by R. Rasala in [50], but we prove it here again as an example of how our results can be applied.

	partition	degree	d_0
1.	$(d-1,1)$	$d-1$	4
2.	$(d-2,2)$	$d(d-3)/2$	5
3.	$(d-2,1^2)$	$(d-1)(d-2)/2$	6
4.	$(d-3,3)$	$d(d-1)(d-5)/6$	6
5.	$(d-3,1^3)$	$(d-1)(d-2)(d-3)/6$	8
6.	$(d-3,2,1)$	$d(d-2)(d-4)/3$	7
7.	$(d-4,4)$	$d(d-1)(d-2)(d-7)/24$	8

Table 3.6: Minimal character degrees of A_d.

Proposition 3.13. *Assume $d \geq 51$. Then the seven smallest character degrees of A_d are as given in Table 3.6.*

Proof. Consider characters corresponding to partitions with $\lambda_1 > d-4$. These partitions are listed in Table 3.6, along with $\mu_{d,4}$. By Lemma 3.12, they give the seven minimal non-trivial character degrees of S_d for $d \geq 51$. For each partition, there is a positive integer d_0, such that if $d \geq d_0$, then $\lambda_1 > \lambda_1'$, so the character remains unsplit in A_d. The values of d_0 are given in the table. Also, for any character χ^ν of S_d that splits in A_d, we must have $\nu_1 \leq (d+1)/2$, so that $\nu \in L_{d,m}$, where $m = (d-1)/2$. Hence, using Lemma 3.12 again yields $\chi^\nu(1) > \chi^{\mu_{d,5}}(1)$. Furthermore, it is easy to check that as $d > 15$, we have $\chi^{\mu_{d,5}}(1) > 2\chi^{\mu_{d,4}}(1)$. Thus, the degree of a character of A_d corresponding to a splitting character χ^ν of S_d will be greater than that of $\chi^{\mu_{d,4}}$. This shows that the character degrees listed in Table 3.6 are minimal also in A_d. □

3.4. Proving the main results on alternating groups

In this section we will prove Theorems 3.1–3.4. We assume that $d \geq 51$, and set $r = 0.32$. Let also $\delta = 7$, and divide all partitions of d into three sets: $A = L_{d,\lceil rd \rceil}$, $B = L_{d,\delta} \setminus L_{d,\lceil rd \rceil}$ and $C = \{\lambda \mid \lambda_1 > d - \delta\}$. Notice that for $\lambda \in A$, we have $\lambda_1 \leq d - \lceil rd \rceil$, and for $\lambda \in B$, we have $d - \lceil rd \rceil < \lambda_1 \leq d - \delta$. The number of partitions in C does not depend on d. Lemma 3.12 ensures that the smallest degree of a character corresponding to any partition in sets A and B is always given by $\mu_{d,\lceil rd \rceil}$ and $\mu_{d,\delta}$, respectively.

The idea behind the sets A, B and C is the following. The majority of characters of S_d come from partitions contained in A, and for the degrees of these characters we get an exponential lower bound. For the set B, we get a weaker bound for the degrees, but there are also fewer characters. Set C is there to take care of the few characters with smallest degree, and the characters are chosen as they are in order to get good explicit constants.

For $X \in \{A, B, C\}$, let $r_n^X(A_d)$ denote the number of non-trivial irreducible characters of A_d having degree n and corresponding to a partition in X. Similarly, we write

$$\zeta^X_{A_d}(t) = \sum_{n>1} \frac{r^X_n(A_d)}{n^t}.$$

First, let us exhibit some facts about the character degrees coming from the set A. We will repeatedly make use of the following well-known fact.

Proposition 3.14 (see e.g. [10])**.** *The total number $p(d)$ of partitions of d is bounded from above by $c_p^{\sqrt{d}}$, where $c_p = e^{\pi\sqrt{2/3}}$.*

Lemma 3.15.

(a) For any $\lambda \in A$, we have $\chi^\lambda(1) > 0.450 \cdot 1.8009^d$.

(b) For any $d \geq 51$, we have $\zeta^A_{A_d}(t) < Z^A_t$, where t and Z^A_t are as listed in Table 3.7.

t :	1	5/6	3/4	2/3	1/2
Z^A_t :	$3.020 \cdot 10^{-8}$	$3.801 \cdot 10^{-6}$	$4.264 \cdot 10^{-5}$	$4.784 \cdot 10^{-4}$	0.06022
t :	0.97	0.80	0.72	0.63	0.47
Z^A_t :	$7.210 \cdot 10^{-8}$	$9.996 \cdot 10^{-6}$	$1.019 \cdot 10^{-4}$	0.001386	0.1438

Table 3.7: Bounds for $\zeta^A_{A_d}(t)$.

Remark. The values for t on the first row correspond to the values for s in Theorem 3.1. The other values are chosen to get good explicit results in Theorems 3.3 and 3.4.

Proof. Lemma 3.12 ensures that the partition in A giving the smallest character degree is $\mu_{d,\lceil rd\rceil}$. For part (a), Lemma 3.11 gives the desired lower bound, since $\frac{5-8r}{(1-2r)^2} < 51$.

For part (b), we count separately those character degrees that come from self-conjugate partitions. Let $p(d)$ and $p'(d)$ denote the number of ordinary and self-conjugate partitions of d, respectively. The characters coming from self-conjugate partitions split as discussed in Section 3.2, and the others fuse with their conjugate counterparts. Thus, writing $b(d)$ for $\chi^{\mu_{d,\lceil rd\rceil}}(1)$, we get the following estimate:

$$\zeta^A_{A_d}(t) < \frac{2p'(d)}{(b(d)/2)^t} + \frac{(p(d) - p'(d))/2}{b(d)^t} = \frac{p(d) + (2^{t+2} - 1)p'(d)}{2b(d)^t}.$$

The values of this expression can be computed explicitly up to $d = 120$ for each t listed in Table 3.7, and the maximal values appear as Z^A_t.

On the other hand, we see that $\zeta^A_{A_d}(t) < 2^{t+1}p(d)/b(d)^t$. Using Proposition 3.14 and the result from part (a), we get

$$\zeta^A_{A_d}(t) < \frac{2^{t+1}}{0.450^t} \cdot \frac{c_p^{\sqrt{d}}}{1.8009^{td}},$$

where $c_p = \exp\bigl(\pi\sqrt{2/3}\bigr)$. At $d = 120$, the values of the above expression are below Z_t^A for each t in Table 3.7. Also, some elementary calculus tells us that when $d \geq 120$, the expression is strictly decreasing in d for each t. This proves the claim. □

For the set B, we prove the following.

Lemma 3.16.

(a) For any $\lambda \in B$, we have $\chi^\lambda(1) > (d - 4.187)^7/5040$.

(b) For any $d \geq 51$, we have $\zeta^B_{A_d}(t) < Z_t^B$, where t and Z_t^B are as listed in Table 3.8.

$t:$	1	$5/6$	$3/4$	$2/3$	$1/2$
$Z_t^B:$	$1.077 \cdot 10^{-7}$	$2.674 \cdot 10^{-6}$	$1.372 \cdot 10^{-5}$	$7.248 \cdot 10^{-5}$	0.002311
$t:$	$1 - 1/\delta$	$5/6 - 1/\delta$	$3/4 - 1/\delta$	$2/3 - 1/\delta$	$1/2 - 1/\delta$
$Z_t^B:$	$1.683 \cdot 10^{-6}$	$4.489 \cdot 10^{-5}$	$2.437 \cdot 10^{-4}$	0.001390	0.05553

Table 3.8: Bounds for $\zeta^B_{A_d}(t)$.

Proof. Let $l < \lceil rd \rceil$. First, Lemma 3.12 ensures that for any $\lambda \in B$ that has $\lambda_1 \leq d - l$, we have

$$\chi^\lambda(1) \geq \chi^{\mu_{d,l}}(1) = \frac{1}{l!} \cdot d(d-1)(d-2)\cdots(d-l+2)(d-2l+1). \tag{3.10}$$

For part (a), we set $l = \delta$ and check that for $d \geq 51$, the above polynomial is strictly greater than $(d - 4.187)^7/7!$.

For part (b), notice first that there are no self-conjugate partitions in B. Thus the number of irreducible characters of A_d that come from B and have $\lambda_1 = d - l$ is $p(l)/2$, where $p(l)$ is the number of partitions of l. Writing $b(d,l)$ for the minimal character degree $\chi^{\mu_{d,l}}(1)$, we get

$$\zeta^B_{A_d}(t) < \frac{1}{2} \sum_{l=\delta}^{\lceil rd \rceil - 1} \frac{p(l)}{b(d,l)^t}.$$

Values of this expression can be computed explicitly up to $d = 204$ for each t in Table 3.8, and the maximal ones appear as Z_t^B. On the other hand, using equation (3.10) we can derive a lower bound to $b(d,l)$, so that

$$\zeta^B_{A_d}(t) < \frac{1}{2} \sum_{l=\delta}^{\lceil rd \rceil - 1} c_p^{\sqrt{l}} \left(\frac{l!}{(d-2l)^l} \right)^t,$$

where $c_p = \exp\bigl(\pi\sqrt{2/3}\bigr)$ (see Proposition 3.14). Let us denote the above bound $S_{t,d}$.

Next, we will show that for any t listed in Table 3.8, the values of $S_{t,d}$ are below Z_t^B when $d > 204$. This is true for $202 \leq d \leq 204$, so we need only show that $S_{t,d}$ is decreasing in periods of $\lfloor 1/r \rfloor = 3$, that is, $S_{t,d} \geq S_{t,d+3}$ for all $d \geq 202$. To this end, let us estimate the difference

$$2(S_{t,d} - S_{t,d+3}) \geq \sum_{l=\delta}^{\lceil rd \rceil - 1} c_p^{\sqrt{l}} \left(\frac{l!}{(d-2l)^l} \right)^t - \sum_{l=\delta}^{\lceil rd \rceil} c_p^{\sqrt{l}} \left(\frac{l!}{(d-2l+3)^l} \right)^t .$$

After collecting terms under one summation sign the latter difference of sums becomes $\left(\sum_{l=\delta}^{\lceil rd \rceil - 1} E_l\right) - F$, where

$$E_l = c_p^{\sqrt{l}} \left(\left(\frac{l!}{(d-2l)^l} \right)^t - \left(\frac{l!}{(d-2l+3)^l} \right)^t \right)$$

and

$$F = c_p^{\sqrt{\lceil rd \rceil}} \left(\frac{\lceil rd \rceil !}{(d - 2\lceil rd \rceil + 3)^{\lceil rd \rceil}} \right)^t .$$

To show that this difference is positive, we write k for $\lceil rd \rceil - 1$ and examine the ratio

$$\begin{aligned} \frac{E_k + E_{k-1}}{F} = & \frac{c_p^{\sqrt{k}}}{c_p^{\sqrt{k+1}}} W_0(d)^t \left(X_0(d)^t - Y_0(d)^t \right) \\ & + \frac{c_p^{\sqrt{k-1}}}{c_p^{\sqrt{k+1}}} W_1(d)^t \left(X_1(d)^t - Y_1(d)^t \right), \end{aligned}$$

where

$$\begin{aligned} W_0(d) &= \frac{d-2k+1}{k+1} > \frac{(1-2r)d+1}{rd+1}, \\ W_1(d) &= \frac{(d-2k+1)^2}{k(k+1)} > \frac{((1-2r)d+1)^2}{rd(rd+1)}, \\ X_0(d) &= \left(1 + \frac{1}{d-2k}\right)^k \geq \left(1 + \frac{1}{(1-2r)d+2}\right)^{rd-1}, \\ X_1(d) &= \left(1 - \frac{1}{d-2k+2}\right)^{k-1} > \left(1 - \frac{1}{(1-2r)d+2}\right)^{rd-2}, \\ Y_0(d) &= \left(1 - \frac{2}{d-2k+3}\right)^k < \left(1 - \frac{2}{(1-2r)d+5}\right)^{rd}, \quad \text{and} \\ Y_1(d) &= \left(1 - \frac{4}{d-2k+5}\right)^{k-1} < \left(1 - \frac{4}{(1-2r)d+7}\right)^{rd-1}. \end{aligned}$$

Call the above bounds $\bar{W}_0(d)$, $\bar{W}_1(d)$, $\bar{X}_0(d)$, $\bar{X}_1(d)$, $\bar{Y}_0(d)$ and $\bar{Y}_1(d)$, respectively. Now, $\bar{W}_0(d)$ in increasing in d, but $\bar{W}_1(d)$ is decreasing and tends to $\left(\frac{1-2r}{r}\right)^2$. Also, $\bar{X}_0(d)$ is increasing, and $\bar{Y_0}(d)$ and $\bar{Y_1}(d)$ are decreasing. Moreover, $\bar{X_1}(d)$ is decreasing and tends to $\exp\left(-\frac{r}{1-2r}\right)$. Hence, for all $d \geq 202$, we have

$$\begin{aligned}\frac{E_k + E_{k-1}}{F} &\geq \frac{c_p^{\sqrt{64}}}{c_p^{\sqrt{65}}} \bar{W_0}(202)^t \left(\bar{X_0}(202)^t - \bar{Y_0}(202)^t\right) \\ &\qquad + \frac{c_p^{\sqrt{63}}}{c_p^{\sqrt{65}}} \left(\frac{1-2r}{r}\right)^{2t} \left(\left(e^{-\frac{r}{1-2r}}\right)^t - \bar{Y_1}(202)^t\right) \\ &> 0.852 \cdot 1.123^t \cdot \left(2.330^t - 0.186^t\right) \\ &\qquad + 0.725 \cdot 1.265^t \cdot \left(0.411^t - 0.038^t\right).\end{aligned}$$

The last expression is greater than 1 for each t listed in Table 3.8. Therefore, we have $(E_k + E_{k-1}) > F$, which in turn means that $S_{t,d} > S_{t,d+3}$ for all $d \geq 202$. This concludes the proof. $\square$

Finally, for the set C, we have the following.

Lemma 3.17. *For any $d \geq 51$ and $t > 0$, we have $\zeta^C_{A_d}(t) \leq \zeta^C_{A_{51}}(t)$. Moreover, we have $\zeta^C_{A_{51}}(t) = Z^C_t$ for t and Z^C_t listed in Table 3.9.*

t :	1	5/6	3/4	2/3	1/2
Z^C_t :	0.02178	0.04455	0.06483	0.09604	0.2303

Table 3.9: Values for $\zeta^C_{A_{51}}(t)$.

Proof. All partitions in C are of the form $\lambda_\nu = (d-m, \nu_1, \ldots, \nu_k)$, where $\nu = (\nu_1, \ldots, \nu_k)$ is a partition of $m < \delta$. The number of partitions in C is thus $\sum_{m<\delta} p(m) = 29$. From the Hook Formula, it can be seen that for each subpartition ν, there is a strictly increasing polynomial f_ν that gives the character degree corresponding to λ_ν in terms of d. Now,

$$\zeta^C_{A_d}(t) = \sum_\nu \chi^{\lambda_\nu}(1)^{-t} = \sum_\nu f_\nu(d)^{-t}.$$

Because the polynomials f_ν are increasing, we have that $\zeta^C_{A_d}(t)$ is decreasing in d. Finally, we can compute the values of $\zeta^C_{A_{51}}(t)$ using the polynomials f_ν. These values appear in Table 3.9 as Z^C_t for $t \in \{1, 5/6, 3/4, 2/3, 1/2\}$. $\square$

Now we are finally in a position to prove the main theorems.

Proof of Theorems 3.1 and 3.2. Let s be among 1, 5/6, 3/4, 2/3 and 1/2. We use a computer to calculate all the character degrees of A_d up to $d = 50$. Then we compute the ratios $r_n(A_d)/n^s$ for $d \leq 50$, and check that the bounds given in the statement of the theorems hold when $d \leq 50$. Moreover, the computations show that bounds claimed to be best possible really are so.

For $d > 50$, we use the bound

$$\frac{r_n(A_d)}{n^s} \leq \zeta^A_{A_d}(s) + \zeta^B_{A_d}(s) + \zeta^C_{A_d}(s).$$

Bounds for the zeta functions are obtained in Lemmas 3.15, 3.16 and 3.17. Using these bounds, we find that $r_n(A_d)/n^s < C_s$, where the C_s are listed in the following table:

s :	1	5/6	3/4	2/3	1/2
C_s :	0.02179	0.04456	0.06489	0.09660	0.2929

Comparing these values with the computed bounds for $d \leq 50$ gives the result. $\square$

Proof of Theorems 3.3 and 3.4. Denote $S = \{1, 5/6, 3/4, 2/3, 1/2\}$. We are looking for maximal values over n of the ratios $Q^s(n)_b = (s_n - b)/n^s$, where b assumes values 0, 6 and 9, and s is in S. We have computed the degrees of all characters of alternating groups A_d for $d \leq 50$. By Proposition 3.13, the seventh smallest character degree of any A_d with $d \geq 51$ is at least $n_0 = 229075$. Therefore, up to $n = n_0 - 1$ we can compute all values of $Q^s(n)_b$ by taking into account also the first six character degrees given by Proposition 3.13. The maximal values are given in Table 3.10, together with the value n where the maximum was obtained, as well as the corresponding value of s_n.

s	$Q^s(n)_0$	(n, s_n)	$Q^s(21)_6 \quad (s_n = 8)$	$Q^s(12012)_9 \quad (s_n = 10)$
1	2/3	$(3, 2)$	2/21	1/12012
5/6	0.8007	$(3, 2)$	0.1582	$3.985 \cdot 10^{-4}$
3/4	0.8973	$(8, 5)$	0.2029	$8.716 \cdot 10^{-4}$
2/3	1.052	$(21, 8)$	0.2628	0.001907
1/2	1.746	$(21, 8)$	0.4365	0.009125

Table 3.10: Maximal values of $Q^s(n)_b$ for $n < 229075$. The decimal values are rounded up to four significant digits.

Assume then that $n \geq n_0$. Denote

$$Q^{X,s}(n) = \frac{1}{n^s} \sum_{d \geq 51} r^X_n(A_d), \quad \text{when } X \in \{A, B\},$$

$$\text{and} \qquad Q^{D,s}(n) = \frac{1}{n^s} \sum_{d \leq 50} r_n(A_d).$$

We also define

$$Q^{C,s}(n)_b = \frac{1}{n^s} \left(\sum_{d \geq 51} r^C_n(A_d) - b \right).$$

The values of $Q^{D,s}(n)$ are obtainable from the computed data, and maximal values over $n \geq n_0$ are listed in Table 3.11 below as upper bounds to $Q^{D,s}(n)$. (If $d \leq 50$, there are only finitely many n for which A_d has representations of degree n.) To get estimates for $Q^{X,s}(n)$, where X is A or B, we use Lemmas 3.15 and 3.16.

Let us first concentrate on $Q^{A,s}(n)$. By part (a) of Lemma 3.15, we know that

$$r_n^A(A_d) = 0 \qquad \text{for} \qquad d > \frac{\ln n - \ln 0.450}{\ln 1.8009}.$$

Also, if $\zeta_{A_d}^A(t) < Z_t^A$ holds for some t mentioned in part (b) of the same Lemma, then $r_n^A(A_d) < Z_t^A n^t$ holds for all $d \geq 51$. Hence,

$$Q^{A,s}(n) < \frac{\ln n - \ln 0.450}{\ln 1.8009} \cdot Z_t^A n^{t-s}.$$

Denote this upper bound $\bar{Q}^{A,s,t}(n)$. Differentiating, we find that for $t < s$, the expression $\bar{Q}^{A,s,t}(n)$ attains its maximum at

$$n_1^{s,t} = 0.450 \exp\left(\frac{1}{s-t}\right).$$

Looking at Table 3.7, we set $t = 0.97$ for $s = 1$, $t = 0.80$ for $s = 5/6$, etc. The maximal values $\bar{Q}^{A,s,t}(n_1^{s,t})$ are listed in Table 3.11 as upper bounds to $Q^{A,s}(n)$.

Similarly, using the results of Lemma 3.16, we get

$$Q^{B,s}(n) < \left(5040^{1/7} + 4.187n^{-1/7}\right) \cdot Z_s^B,$$

with Z_s^B listed in Table 3.8. Call these upper bounds $\bar{Q}^{B,s}(n)$. Since $\bar{Q}^{B,s}(n)$ is strictly decreasing in n, we get $Q^{B,s}(n) < \bar{Q}^{B,s}(n_0)$ for all $n > n_0$. The values of $\bar{Q}^{B,s}(n_0)$ are listed in Table 3.11 as upper bounds to $Q^{B,s}(n)$.

Finally, we need to add the characters coming from the set C. As explained in the proof of Theorems 3.1 and 3.2 above, degrees of characters are given by polynomials f_ν, where ν is a partition of $m < \delta$. There are 29 such polynomials altogether, and each of them is strictly increasing. This means that for each n, there are at most 29 characters with degree n, so $\sum_{d \geq 51} r_n^C(A_d) \leq 29$. Thus, for $n \geq n_0$, we get

$$Q^{C,s}(n)_b \leq \frac{29-b}{n_0^s}.$$

The values for $s \in S$ and $b \in \{0,6,9\}$ of the above expression are listed in Table 3.11 as upper bounds to $Q^{C,s}(n)_b$. Now,

$$Q^s(n)_b = \sum_{X \in \{A,B,D\}} Q^{X,s}(n) + Q^{C,s}(n)_b$$

for all $s \in S$, $b \in \{0,6,9\}$ and $n \geq n_0$. Upper bounds for $Q^s(n)_b$ can thus be computed by adding together the bounds for $Q^{X,s}(n)$ and $Q^{C,s}(n)_b$ obtained above. This is done in the three bottom rows of Table 3.11.

The claims of Theorems 3.3 and 3.4 can now be verified by comparing the values in Table 3.11 with the maxima for $n < n_0$ given in Table 3.10. □

s :	1	5/6	3/4	2/3	1/2
$Q^{A,s}(n)$:	$1.113 \cdot 10^{-6}$	$1.497 \cdot 10^{-4}$	0.001591	0.01990	2.246
$Q^{B,s}(n)$:	$6.852 \cdot 10^{-6}$	$1.828 \cdot 10^{-4}$	$9.922 \cdot 10^{-4}$	0.005660	0.2261
$Q^{D,s}(n)$:	$2.204 \cdot 10^{-5}$	$1.775 \cdot 10^{-4}$	$5.034 \cdot 10^{-4}$	0.001429	0.01150
$Q^{C,s}(n)_0$:	$1.266 \cdot 10^{-4}$	$9.903 \cdot 10^{-4}$	0.002770	0.007747	0.06060
$Q^{C,s}(n)_6$:	$1.005 \cdot 10^{-4}$	$7.854 \cdot 10^{-4}$	0.002197	0.006144	0.04806
$Q^{C,s}(n)_9$:	$8.731 \cdot 10^{-5}$	$6.830 \cdot 10^{-4}$	0.001911	0.005343	0.04179
$Q^{s}(n)_0$:	$1.567 \cdot 10^{-4}$	0.001500	0.005844	0.03457	2.518
$Q^{s}(n)_6$:	$1.305 \cdot 10^{-4}$	0.001295	0.005271	0.03296	2.506
$Q^{s}(n)_9$:	$1.174 \cdot 10^{-4}$	0.001193	0.004984	0.03216	2.499

Table 3.11: Upper bounds for the ratios $Q^s(n)_b = (s_n(A_d) - b)/n^s$ with $n \geq 229075$ and for other related expressions (see the text for details).

Proof of Theorem 3.5. We fix s so that $0 < s < 1$, and use the same approach as in the two previous proofs. Without loss of generality, we may assume that $d \geq 51$. We keep also our choice of $r = 0.32$, so that we are able to use Lemma 3.12 and its consequences. However, we shall need $s - 1/\delta > 0$ unless the set B is empty, so we take $\delta = \min\{\lceil 2/s \rceil, \lceil rd \rceil\} \geq 3$. Then the sets A, B and C may be defined as before, but the number of partitions in C now depends on s.

Using the same notation as in the previous proof, we examine

$$Q^s(n) = Q^{A,s}(n) + Q^{B,s}(n) + Q^{C,s}(n),$$

where

$$Q^{X,s}(n) = \frac{1}{n^s} \sum_{d \geq 51} r_n^X(A_d).$$

For $Q^{A,s}(n)$, we use Lemma 3.15. Supposing that Z_t^A is an upper bound for $\zeta_{A_d}^A(t)$, not dependent on d, the lemma yields

$$Q^{A,s}(n) < \frac{\ln n - \ln 0.450}{\ln 1.8009} \cdot Z_t^A n^{t-s},$$

where t is any number satisfying $0 < t < s$ (see the previous proof for details). By simple calculus, this expression has its maximum at $0.449 \exp\left(\frac{1}{s-t}\right)$, and this maximum is

$$\max_n Q^{A,s}(n) < \frac{Z_t^A}{e \ln 1.8009 \cdot 0.450^{s-t}(s-t)}. \tag{3.11}$$

On the other hand, for any λ in B, we find that $\chi^\lambda(1) \geq (d - 2\delta)^\delta/\delta!$ (see the proof of Lemma 3.16). Hence, if Z_s^B is an upper bound for $\zeta_{A_d}^B(s - 1/\delta)$, we get

$$Q^{B,s}(n) < \left((\delta!)^{1/\delta} + \frac{2\delta}{n^{1/\delta}}\right) \cdot Z_s^B.$$

Applying Stirling's approximation to $\delta!$ (see proof of Lemma 3.11) and using $\delta \geq 3$ leads to the estimate

$$Q^{B,s}(n) < \left(e^{-\frac{107}{108}} + 2\right) \delta \cdot Z_s^B. \tag{3.12}$$

Next, we need to find Z_t^A and Z_s^B. To this end, let us first examine the expression $g_{b,t}(l) = c_p^{\sqrt{l}} b^{-lt}$, where $c_p = \exp(\pi\sqrt{2/3})$, $b > 1$ and $0 < t < 1$. Differentiation shows that this expression has its maximum at

$$l = \left(\frac{\ln c_p}{2t \ln b}\right)^2,$$

and this maximum is

$$\max_l g_{b,t}(l) = \exp\left(\frac{(\ln c_p)^2}{4t \ln b}\right). \tag{3.13}$$

Now, from the proof of Lemma 3.15 we see that

$$\zeta_{A_d}^A(t) < \frac{2^{t+1}}{0.450^t} \cdot g_{b_1,t}(d),$$

where $b_1 = 1.8009$, so we can set

$$Z_t^A = \frac{2^{t+1}}{0.450^t} \cdot \exp\left(\frac{(\ln c_p)^2}{4t \ln b_1}\right).$$

Combining this with (3.11) and choosing $t = s/2$ yields

$$\max_n Q^{s,A}(n) < \frac{4(\sqrt{2})^s}{e \ln b_1 \cdot 0.450^s} \cdot \frac{c_1^{1/s}}{s},$$

where $c_1 = \exp\left(\frac{(\ln c_p)^2}{2 \ln b_1}\right)$. This is clearly less than $C_1^{1/s}$ for some constant C_1.

For Z_s^B, we look at $\zeta_{A_d}^B(t)$, where $t = s - 1/\delta$. Recall that $\delta = \lceil 2/s \rceil$, unless B is empty, in which case we can just take $Z_s^B = 0$. Note also that with $\delta = \lceil 2/s \rceil$, we have $2/s \leq \delta < 3/s$ and $s/2 \leq t < 3s/4$.

The proof of Lemma 3.16 tells us that

$$\zeta_{A_d}^B(t) < \frac{1}{2} \sum_{l=\delta}^{\lceil rd \rceil - 1} c_p^{\sqrt{l}} \left(\frac{l!}{(d-2l)^l}\right)^t.$$

Using Stirling's approximation similarly as in the proof of Lemma 3.11, and noting that $\delta \geq 3 > e$ and that $\frac{l}{d-2l} < \frac{r}{1-2r}$ for $l \leq \lceil rd \rceil - 1$, we get a new bound

$$\zeta_{A_d}^B(t) < \frac{a^t}{2} \sum_{l=\delta}^{\lceil rd \rceil - 1} c_p^{\sqrt{l}} \left(\frac{br}{1-2r}\right)^{lt}, \tag{3.14}$$

where $a = \sqrt{2\pi} e^{1/36}$, and $b = \sqrt[6]{3}/e$. Denote $b_2 = \frac{1-2r}{br} > 2$.

We now estimate the sum in (3.14) in two parts. For any integers l_0 and $l \geq l_0$, we have

$$c_p^{\sqrt{l}} b_2^{-lt} < \Big(\underbrace{c_p^{\frac{1}{t\sqrt{l_0}}} \cdot b_2^{-1}}_{\text{call this } q}\Big)^{lt}.$$

With q as noted in the above expression, we want $q \leq q_0$ for some $q_0 < 1$ to be able to use a geometric series approximation. Therefore, choose $q_0 = 0.9$ and set $l_0 = \lceil c_1/t^2 \rceil$, where

$$c_1 = \left(\frac{\ln c_p}{\ln(b_2 q_0)}\right)^2 > 0.$$

Now,

$$\sum_{l=l_0}^{\lceil rd \rceil - 1} c_p^{\sqrt{l}} b_2^{-lt} < \sum_{l=l_0}^{\infty} q_0^{lt} < \sum_{l=0}^{\infty} q_0^{lt} = \frac{1}{1-q_0^t}.$$

By the Mean Value Theorem, $1 - q_0^t > (1-q_0)t$, so

$$\sum_{l=l_0}^{\lceil rd \rceil - 1} c_p^{\sqrt{l}} b_2^{-lt} < \frac{1}{(1-q_0)t}.$$

On the other hand, (3.13) gives

$$\sum_{l=\delta}^{l_0-1} c_p^{\sqrt{l}} b_2^{-lt} < (l_0 - 1)\exp\left(\frac{(\ln c_p)^2}{4t \ln b_2}\right) < \frac{c_1}{t^2} \cdot c_2^{1/(2t)},$$

where $c_2 = \exp\big(\frac{(\ln c_p)^2}{2 \ln b_2}\big) > 1$. Finally, we wee that

$$\zeta_{A_d}^B(t) < \frac{a^t}{2}\left(\frac{1}{1-q_0} \cdot \frac{1}{t} + c_1 \cdot \frac{c_2^{1/(2t)}}{t^2}\right) < \frac{a^s}{2}\left(\frac{2}{1-q_0} \cdot \frac{1}{s} + 4c_1 \cdot \frac{c_2^{1/s}}{s^2}\right).$$

We set Z_s^B to be the bound on the right hand side. Combining this with (3.12) and recalling that $\delta < 3/s$ shows that $Q^{B,s}(n)$ is less than $C_2^{1/s}$ for some constant C_2.

Lastly, set C contains partitions of the form $\lambda_\nu = (d - m, \nu_1, \ldots, \nu_k)$, where m is less than δ and $\nu = (\nu_1, \ldots, \nu_k)$ is a partition of m. For each m there are exactly $p(m)$ such subpartitions, and for each subpartition there is a strictly increasing polynomial expressing the character degree corresponding to λ_ν in terms of d. Hence, the number of partitions in C giving rise to a fixed character degree n is $\sum_{m<\delta} p(m) < \delta p(\delta)$. Thus,

$$Q^{C,s}(n) < \frac{\delta p(\delta)}{n^s} < \delta c_p^{\sqrt{\delta}}.$$

As $\delta < 3/s$, the above expression is less than $C_3^{1/s}$ for some constant C_3. This concludes the proof. □

3.5. Results on faithful character degrees of the covering group

Results presented here mimic those of Section 3.3 related to the symmetric groups. We first show that the number g_λ of standard shifted tableaux corresponding to a strict partition λ is minimal for two-part partitions. The value of g_λ for a two-part partition $\lambda = (d-m, m)$ will be abbreviated as $g_{d,m}$. It can be computed from the Hook Formula, and equals

$$g_{d,m} = \frac{(d-1)!\,(d-2m)}{m!\,(d-m)!} = \frac{d!}{m!\,(d-m)!}\left(1 - \frac{2m}{d}\right).$$

As in the case of the symmetric groups, we find that $g_{d,m}$ is strictly increasing when $m \leq \lceil \frac{1}{2}(d-1-\sqrt{d+1}) \rceil$ and strictly decreasing after that.

For any integer $m < d$, let $M_{d,m}$ stand for the set of strict partitions of d, for which $\lambda_1 \leq d-m$. Now, we have $(d-m, m) \in M_{d,m}$, and $M_{d,m} \subset M_{d,m'}$ whenever $m \geq m'$.

Theorem 3.18. *Assume d is a positive integer at least 22, and m is an integer such that $0 \leq m \leq (d-1)/2$. Then, for all strict partitions $\lambda \in M_{d,m}$ with at least three parts, we have $g_\lambda > g_{d,m}$.*

Proof. The structure of the proof follows the one of Theorem 3.10, using induction on d.

Base step. We deal here with some special cases. First, it is simple to verify that the claim holds for $d = 22$ by going through all $m \leq 10$ and all strict partitions in $M_{d,m}$ with at least three parts. The details are left out.

Next, we check separately that the claim holds for certain "staircase" partitions $(\lambda_1, \ldots, \lambda_r)$, where $\lambda_{i+1} = \lambda_i - 1$ for all $i < r$. The examined cases are listed below, but the details of the computations are omitted.

$$\begin{array}{lllllll}
r = 3: & (9,8,7), & (10,9,8) & & & \\
r = 4: & (7,6,5,4), & (8,7,6,5) & & & \\
r = 5: & (7,\ldots,3), & (8,\ldots,4), & (9,\ldots,5) & & \\
r = 6: & (7,\ldots,2), & (8,\ldots,3), & (9,\ldots,4), & (10,\ldots,5) & \\
r = 7: & (7,\ldots,1), & (8,\ldots,2), & (9,\ldots,3), & (10,\ldots,4), & (11,\ldots,5)
\end{array}$$

Last, we check directly two special cases where λ has exactly 3 parts, the last of which is 1. We assume that $d \geq 23$.

Case 1. Suppose that $\lambda = (d-3, 2, 1)$. Using the Hook Formula for shifted diagrams, we can compute

$$g_\lambda = \frac{d(d-4)(d-5)}{6} = \frac{1}{6}(d^3 - 9d^2 + 20d).$$

Now, if $\lambda \in M_{d,m}$, we must have $m \leq 3$. Therefore, we need to check four cases:

$$g_{d,0} = 1, \quad g_{d,1} = d-2, \quad g_{d,2} = \frac{1}{2}(d^2 - 5d + 4), \quad g_{d,3} = \frac{1}{6}(d^3 - 9d + 20d - 12).$$

Since $d \geq 23$, we see in each case that $g_\lambda > g_{d,m}$.

Case 2. Suppose that d is even and $\lambda = (a, a-1, 1)$, where $a = d/2$. The Hook Formula gives

$$g_\lambda = \frac{d!}{(a+1)!\,(d-1)\,(a-3)!\,a} = d!\Big/\frac{(a+1)!\,a!\,(d-1)}{(a-1)(a-2)}.$$

Let D_1 stand for the divisor in the last expression (right side of the division symbol). It equals

$$D_1 = \frac{4\left(\frac{d}{2}+1\right)!\left(\frac{d}{2}\right)!\,(d-1)}{(d-2)(d-4)}.$$

We claim that g_λ is greater than $g_{d,m}$ for any $m \leq a-2$. We know that

$$g_{d,m} = \frac{d!\,(d-2m)}{m!\,(d-m)!\,d} = d!\Big/\frac{m!\,(d-m)!\,d}{d-2m}.$$

We focus again on the divisor in the last expression, which we shall call $D_2(m)$, and argue that its smallest possible value is still greater than D_1. This will prove the claim.

As noted before the statement of the theorem, the expression $D_2(m)$ obtains its minimum at $m_0 = \lceil \frac{1}{2}(d-1-\sqrt{d+1}) \rceil$. Substituting this into $D_2(m)$ and cleaning up the denominator results in this inequality:

$$D_2(m) \geq D_2(m_0) \geq \frac{\lceil \frac{d-1-\sqrt{d+1}}{2} \rceil! \cdot \lfloor \frac{d+1+\sqrt{d+1}}{2} \rfloor! \cdot d}{\sqrt{d+1}+1}.$$

Rename the lower bound obtained above D_2^*. Direct calculation shows that

$$D_2^*/D_1 = \frac{d(d-2)(d-4)}{4(d-1)(\sqrt{d+1}+1)} \cdot \frac{\left(\frac{d}{2}+2\right)\left(\frac{d}{2}+3\right)\cdots\left(\frac{d}{2}+\lfloor\frac{1+\sqrt{d+1}}{2}\rfloor\right)}{\left(\frac{d}{2}-\lceil\frac{1+\sqrt{d+1}}{2}\rceil\right)\cdots\left(\frac{d}{2}-2\right)\left(\frac{d}{2}-1\right)}.$$

The product above the line in the rightmost fraction contains $\lfloor \frac{1}{2}(\sqrt{d+1}-1) \rfloor$ terms, and the product below the line has $\lfloor \frac{1}{2}(\sqrt{d+1}+1) \rfloor$ terms, which is exactly one term more. Since each term below the line is less than each term above the line, we can conclude that

$$D_2^*/D_1 \geq \frac{d(d-2)(d-4)}{4(d-1)(\sqrt{d+1}+1)} \cdot \frac{1}{\frac{d}{2}-1} = \frac{d(d-4)}{2(d-1)(\sqrt{d+1}+1)} > \frac{d-4}{2(\sqrt{d+1}+1)}.$$

With $d \geq 23$, the last expression is greater than 1. Hence we get $D_2(m) > D_1$ for all m, and finally $g_\lambda > g_{d,m}$ for all m.

Induction step. Suppose that $d > 22$ and the claim holds for all integers at least 22 and less than d. Assume that $1 \leq m \leq (d-1)/2$ and $\lambda \in M_{d,m}$ has at least three parts. Moreover, suppose that λ is not one of the staircase partitions considered in the base step, nor one of the form $(d-3, 2, 1)$ or $(d/2, d/2-1, 1)$. Denote $r = \operatorname{len}(\lambda)$, the number of parts in λ.

Notice first that $g_{d,0} = 1$, so that clearly $g_\lambda > g_{d,0}$.

Assume next that d is odd and $m = (d-1)/2$. For the two-part partition, we have $(d-m, m) = (m+1, m)$, and by the Branching Rule, we get $g_{d,m} = g_{d-1,m-1}$. If $r > 3$ or $\lambda_3 \geq 2$, let $\lambda(1) = (\lambda_1, \ldots, \lambda_r - 1)$. Now, $g_\lambda \geq g_{\lambda(1)}$ by the Branching Rule, and $\lambda(1)_1 = \lambda_1 = (d-1) - (m-1)$, so $\lambda(1)$ is in $M_{d-1,m-1}$. Notice that $\lambda(1)$ still has at least 3 parts, so we can apply the induction hypothesis to get $g_{\lambda(1)} > g_{d-1,m-1}$. Hence $g_\lambda > g_{d,m}$.

On the other hand, suppose $\lambda = (\lambda_1, \lambda_2, 1)$. Because $d > 6$, we then know that either $\lambda_2 > 2$ or $\lambda_2 = 2$ and $\lambda_1 > 2$. This being the case, we can choose either $\lambda(1) = (\lambda_1, \lambda_2 - 1, 1)$ or $\lambda(1) = (\lambda_1 - 1, \lambda_2, 1)$, and proceed as above to get $g_\lambda > g_{d,m}$.

We may now assume that $1 \leq m < (d-1)/2$. The Branching Rule gives

$$g_{d,m} = g_{d-1,m} + g_{d-1,m-1}. \tag{3.15}$$

Here we break the argument into different cases.

Case 1. Suppose that $\lambda_1 = \lambda_2 + 1$, but $\lambda_k > \lambda_{k+1} + 1$ for some $2 \leq k < r$. Because of the assumptions made about λ, either $r > 3$ or $\lambda_r \geq 2$. Denoting

$$\lambda(1) = (\lambda_1, \ldots, \lambda_k - 1, \ldots, \lambda_r) \quad \text{and} \quad \lambda(2) = (\lambda_1, \ldots, \lambda_r - 1),$$

the Branching Rule gives

$$g_\lambda \geq g_{\lambda(1)} + g_{\lambda(2)}. \tag{3.16}$$

Because $\lambda_1 = \lambda_2 + 1$ and λ has at least three parts, we have $\lambda_1 \leq d/2$. Also, we assumed $m \leq d/2 - 1$, so

$$\lambda(1)_1 = \lambda(2)_1 = \lambda_1 \leq \frac{d}{2} = d - \left(\frac{d}{2} - 1\right) - 1 \leq d - m - 1.$$

This implies that $\lambda(1)$ and $\lambda(2)$ are in $M_{d-1,m} \subset M_{d-1,m-1}$. They both also have at least three parts, and moreover, the inequality $m \leq (d-2)/2$ holds, so the induction hypothesis yields

$$g_{\lambda(1)} > g_{d-1,m} \quad \text{and} \quad g_{\lambda(2)} > g_{d-1,m-1}.$$

By these equations and those of (3.15) and (3.16), we get $g_\lambda > g_{d,m}$.

Case 2a. Suppose $\lambda_1 > \lambda_2 + 1$ and either $r > 3$ or $\lambda_r \geq 2$. Denote

$$\lambda(1) = (\lambda_1 - 1, \ldots, \lambda_r) \quad \text{and} \quad \lambda(2) = (\lambda_1, \ldots, \lambda_r - 1),$$

so that $g_\lambda \geq g_{\lambda(1)} + g_{\lambda(2)}$. Because $\lambda(1)_1 = \lambda_1 - 1 \leq d - m - 1$, we have $\lambda(1) \in M_{d-1,m}$. Also, $\lambda(2)_1 = \lambda_1 \leq d - m$, so $\lambda(2) \in M_{d-1,m-1}$. As in the previous case, the induction hypothesis gives

$$g_{\lambda(1)} + g_{\lambda(2)} > g_{d-1,m} + g_{d-1,m-1} = g_{d,m}.$$

Case 2b. Suppose $\lambda = (\lambda_1, \lambda_2, 1)$, where $\lambda_1 > \lambda_2 + 1$. We know from the assumptions that $\lambda_2 > 2$. Denote

$$\lambda(1) = (\lambda_1 - 1, \lambda_2, 1) \quad \text{and} \quad \lambda(2) = (\lambda_1, \lambda_2 - 1, 1).$$

As above, we have $\lambda(1) \in M_{d,m-1}$ and $\lambda(2) \in M_{d-1,m-1}$, so that by the induction hypothesis,

$$g_\lambda \geq g_{\lambda(1)} + g_{\lambda(2)} > g_{d,m-1} + g_{d-1,m-1} = g_{d,m}.$$

Case 3. Suppose $\lambda_i = \lambda_{i+1} + 1$ for $1 \leq i < r$. We proceed similarly as with the case of rectangular diagrams in the proof of Theorem 3.10. However, as we are working with shifted diagrams, we need to deal with different shapes of diagrams in slightly different ways. Put simply, diagrams with a long bottom row we can settle as before, but if the last row is too short, we need to choose the removable cells differently.

Case 3a. Assume that $\lambda_r \geq 6$. Using the same notation as before, we detach n cells from the bottom right corner of the shifted diagram of λ. We will choose the removed cells so that after turning them around 180 degrees, they form an *unshifted* diagram of a partition δ (see Figure 3.8). Numbering the cells in the order they were removed makes the diagram they form a standard Young tableau. The number of ways to remove the cells is thus the same as the number of standard δ-tableaux, which in turn is the degree of the character of S_n corresponding to δ. The Branching Rule then gives the following bound for g_λ:

$$g_\lambda \geq \sum_i \left(\chi_{S_n}^{\delta(i)}(1) \cdot g_{\lambda(i)} \right).$$

Here, the index i enumerates the clusters of n cells chosen for removal, each $\delta(i)$ is a partition corresponding to the unshifted diagram these cells form, and each $\lambda(i)$ is the strict partition corresponding to the remaining shifted diagram. In general, we cannot have an equality in the above formula, since there are blocks of cells that can be removed from a shifted diagram that do not form a diagram of any partition. (Consider, for instance, removing the two bottom rows of lengths λ_r and $\lambda_r + 1$.)

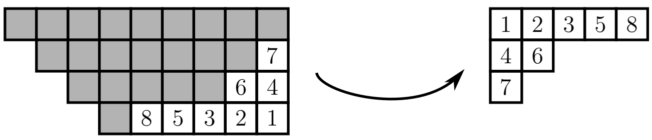

Figure 3.8: If the last row is long, the removed cells form an unshifted Young tableau.

We let $n = 8$ and remove the same six partitions $\delta(i)$ as in the proof of Theorem 3.10, that is, the partitions $(5, 3)$, $(5, 2, 1)$, $(4, 4)$, $(4, 3, 1)$, $(4, 2, 2)$ and $(3, 3, 2)$. We can then refer to the results in the mentioned proof, having first verified that the $\lambda(i)$ satisfy the assumptions in the induction hypothesis.

As $\lambda_r \geq 6$, each $\lambda(i)$ has at least three rows. Also, if $r = 3$, we are assuming that $\lambda_r \geq 9$, as the smaller diagrams were dealt with in the base step. It follows that $d - 8 \geq 22$.

Assume then that $k \leq (d-9)/2$. In order to conclude that $g_{\lambda(i)} > g_{d-8,k}$, we need to show that $\lambda(i)$ is in $M_{d-8,k}$. Firstly, if $r > 3$, we have $\lambda(i)_1 = \lambda_1 \leq d/4 + 3/2$. Since $d \geq 30$, we know that

$$d - 8 - k \geq \frac{d-7}{2} = \frac{d}{4} + \frac{3}{2} + \frac{d-20}{4} > \frac{d}{4} + \frac{3}{2} \geq \lambda(i)_1.$$

On the other hand, if $r = 3$, we have $\lambda(i)_1 \leq \lambda_1 \leq d/3 + 1$. Now, λ_r is at least 9, so $d \geq 30$, and

$$d - 8 - k \geq \frac{d-7}{2} = \frac{d}{3} + 1 + \frac{d-27}{6} > \frac{d}{3} + 1 \geq \lambda(i)_1.$$

In both cases, we get that $\lambda(1) \in M_{d-8,k}$. Hence, the induction hypothesis is satisfied, and the claim holds.

Case 3b. Suppose $\lambda_r = 1$. As $d \geq 23$ and the case $\lambda = (7, 6, 5, 4, 3, 2, 1)$ has been dealt with, we may assume $r \geq 8$.

In this case, we are still removing clusters of n cells from the bottom right corner. This time, however, the removed cells are reflected instead of rotating, so that the bottom right corner cell (the unique one in the last row) becomes the top left corner. The procedure leads to a *shifted* diagram of a strict partition $\zeta(i)$ (see Figure 3.9). The number of ways the cells can be removed is now the same as the number of standard *strict* $\zeta(i)$-tableaux, so again, we get a representation

$$g_\lambda = \sum_i \left(g_{\zeta(i)} \cdot g_{\lambda(i)} \right).$$

The notation is analogous to the previous case.

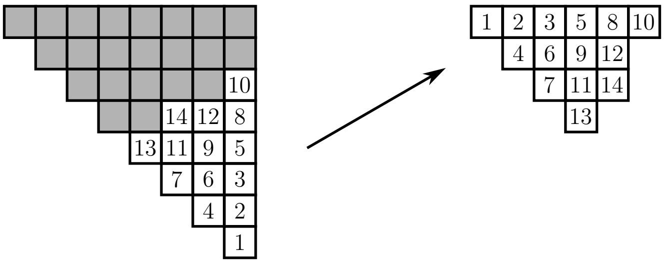

Figure 3.9: When the last row contains only one cell, the removed cells form a shifted tableau.

We let $n = 14$ and list below the necessary $\zeta(i)$ with the corresponding values of $g_{\zeta(i)}$:

i	$\zeta(i)$	$g_{\zeta(i)}$
1	$(8, 4, 2)$	3003
2	$(8, 3, 2, 1)$	1274
3	$(7, 6, 1)$	990
4	$(7, 5, 2)$	2860
5	$(7, 4, 3)$	1872

i	$\zeta(i)$	$g_{\zeta(i)}$
6	$(7, 4, 2, 1)$	2730
7	$(6, 5, 3)$	1274
8	$(6, 5, 2, 1)$	1560
9	$(6, 4, 3, 1)$	1716
10	$(5, 4, 3, 2)$	286

We see that $\sum_i g_{\zeta(i)} = 17565 > 2^{14}$, so we can make the same conclusions in this case as in the previous one, after showing that $\lambda(i) \in M_{d-14,k}$ for all $k \leq (d - 15)/2$.

Supposing that $k \le (d-15)/2$, we see that $\lambda(i)_1 \le \lambda_1 \le d/8 + 7/2$ for all i. Furthermore, d is at least 36, so

$$d - 14 - k \ge d - 14 - \frac{d-15}{2} = \frac{d}{8} + \frac{7}{2} + \frac{3d-80}{8} > \frac{d}{8} + \frac{7}{2} \ge \lambda(i)_1.$$

As before, we have that $\lambda(i) \in M_{d-14,k}$ for all i, which makes the induction work.

Case 3c. Finally, suppose that $2 \le \lambda_r \le 5$. Again, we may assume that $r \ge 8$, as the smaller diagrams were dealt with in the base step. Here, we once more remove blocks of n cells from the shifted diagram of λ, but because $\lambda_r > 1$, the removed cells do not form shifted diagrams. Instead, if we reflect them as in the previous case, they form *semi-shifted* diagrams, where the first few rows might be left-aligned, but the rest form a staircase like in the shifted versions.

Let A now be some semi-shifted diagram of removed cells that corresponds to a strict partition ζ, and let B be the shifted diagram corresponding to the same partition (see Figure 3.10). Enumerate the cells of B in some order, so that B becomes a shifted δ-tableau. Enumerate then the cells of A so that in each row the cells have the same numbers in the same order as in the corresponding row of B. The numbers in A are still in increasing order along rows and columns, so they denote a possible way of removing cells from the diagram of λ. Therefore, the number g_ζ of shifted ζ-tableau is at least the number of ways to remove n cells from the diagram of λ. This leads to the estimate

$$g_\lambda \ge \sum_i \left(g_{\zeta(i)} \cdot g_{\lambda(i)} \right),$$

with notation as before. In general, equality does not hold, as some choices of removed cells cannot be made into a shifted diagram. This happens when some rows in the non-shifted part have equal length.

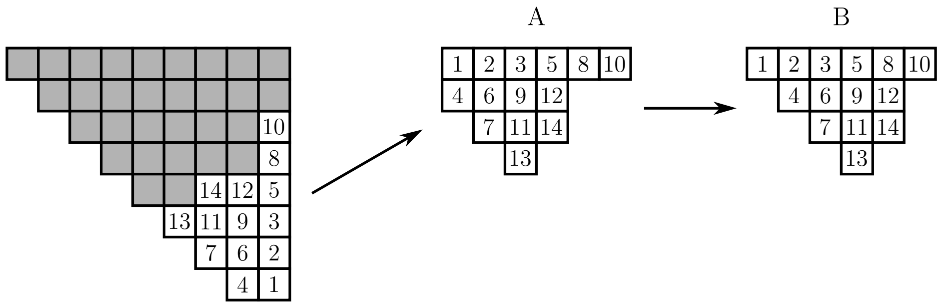

Figure 3.10: If the last row is short, the removed cells form a semi-shifted tableau, which can occasionally be made into a shifted one.

The form of the above estimate implies that we can choose the same partitions for the removable cells as we did in Case 3b. Moreover, we can use the same proof as in the said case to show that the $\lambda(i)$ satisfy the assumptions in the induction hypothesis. Therefore, we can directly apply the conclusion of the previous case.

This settles the final case and the whole proof. $\square$

The next lemma is the analogue of Lemma 3.11. It gives a useful approximation for the value of $g_{d,m}$.

Lemma 3.19. *Suppose* $0 < r < 1/2$*, and write* $m = \lceil rd \rceil$*. Whenever we have* $d \geq d_0$*, where* $d_0 \geq \frac{7-12r}{(1-2r)^2}$*, then*

$$\phi_1(r)k_1(r)^d \;<\; g_{d,m} \;<\; \phi_2(r)k_2(r)^d,$$

where

$$\begin{aligned}
\phi_1(r) &= \exp\left(\frac{1}{12d_0+1} - \frac{1}{12r(1-r)d_0}\right)\frac{1-2r}{\sqrt{2\pi r(1-r)}}\;,\\
k_1(r) &= \frac{1}{\sqrt[2d_0]{d_0}\cdot r^r(1-r)^{1-r}}\;,\\
\phi_2(r) &= \frac{(1-2r)(1-r)}{r\sqrt{2\pi d_0 r(1-r-1/d_0)}}\;,\quad \textit{and}\\
k_2(r) &= \frac{1}{r^r(1-r-1/d_0)^{1-r}}.
\end{aligned}$$

Proof. For a fixed d, denote $m_0(d) = \frac{1}{2}(d-1-\sqrt{d+1})$. As noted before, the map $\gamma\colon m \mapsto g_{d,m}$ is strictly increasing when $m \leq m_0(d)$, and the same holds for its continuous version $\bar{\gamma}$. Because $m_0(d)/d \to 1/2$ as $d \to \infty$, we may choose d_0 to be such that $m_0(d) \geq rd+1$ whenever $d \geq d_0$. It turns out that $d_0 \geq \frac{7-12r}{(1-2r)^2}$ is sufficient.

Now, $\gamma(m)$ is increasing up to $m = rd+1$, so $\bar{\gamma}(rd) \leq \gamma(\lceil rd \rceil) \leq \bar{\gamma}(rd+1)$ holds for all $d \geq d_0$. The result follows from using Stirling's approximation exactly as in the proof of Lemma 3.11. □

We will next show that if $d \geq 61$ and m is small enough, we know the smallest g_λ for all λ in $M_{d,m}$. This lemma imitates Lemma 3.12.

Lemma 3.20. *Assume* $d \geq 61$*. For any integer* $m \leq \lceil 0.33d \rceil$ *and for all* $\lambda \in M_{d,m}$*, we have* $g_\lambda \geq g_{d,m}$*.*

Proof. Denote $f(d) = g_{d,\lfloor (d-1)/2 \rfloor}$ and $m_0(d) = \lceil 0.33d \rceil$. As in the proof of Lemma 3.12, we need to show that $g_{d,m_0(d)} \leq f(d)$. For values of d from 61 to 800 this can be checked by direct computation. Let us therefore assume that $d \geq 801$. Now, since $\frac{7-12\cdot 0.33}{(1-2\cdot 0.33)^2} < 801$, we can use the upper bound from Lemma 3.19 with $d_0 = 801$ and $r = 0.33$ to get $g_{d,m_0(d)} < 0.03 \cdot 1.9^d$. We want to find a lower bound for $f(d)$.

If d is odd, we have $f(d) = f(d-1)$ by the branching formula. We may thus assume that d is odd. Denoting $d = 2k+1$, we have

$$f(d) = \frac{(2k+1)!}{k!(k+1)!}\left(1 - \frac{2k}{2k+1}\right) = \frac{(2k)!}{k!(k+1)!}.$$

As in the proof of Lemma 3.12, we can show that

$$\ln f(d) > (2k+1)\ln 1.9.$$

This means that for odd $d \geq 801$, we have $f(d-1) = f(d) > 1.9^d$. Furthermore, since $\max\{g_{d-1,m_0(d)}, g_{d,m_0(d)}\} < 1.9^d$, the claim is proved. $\square$

3.6. Proving the main results on the covering groups

In this section, we will prove Theorems 3.6–3.9. The arguments imitate closely those of Section 3.4. We assume that $d \geq 61$, and set $r = 0.33$. Divide all strict partitions of d into two sets: $A = M_{d,\lceil rd \rceil}$ and $B = \{\lambda \mid \lambda_1 > d - \lceil rd \rceil\}$.

For $X \in \{A, B\}$, let $r_n^X(\tilde{A}_d)$ denote the number of faithful irreducible characters of $\tilde{A}_d$ having degree n and corresponding to a strict partition in X. Similarly, we write

$$\zeta_{\tilde{A}_d}^X(t) = \sum_{n>1} \frac{r_n^X(\tilde{A}_d)}{n^t}.$$

First, let us exhibit some facts about character degrees corresponding to partitions in A. We shall slightly abuse the notation by writing $\chi_{\tilde{S}_d}^\lambda(1)$ for the degree of the character of $\tilde{S}_d$ corresponding to a strict partition λ, although the character is not uniquely determined by the partition (associate characters share partitions). However, the degree is uniquely determined by λ, so no harm will be done.

Lemma 3.21.

(a) For any $\lambda \in A$, we have $\chi_{\tilde{S}_d}^\lambda(1) > 0.241 \cdot 2.421^d$.

(b) For any $d \geq 61$, we have $\zeta_{\tilde{A}_d}^A(t) < \tilde{Z}_t^A$, where t and $\tilde{Z}_t^A$ are as listed in Table 3.12.

t :	1	5/6	3/4	2/3	1/2
$\tilde{Z}_t^A$:	$4.723 \cdot 10^{-19}$	$2.726 \cdot 10^{-15}$	$2.073 \cdot 10^{-13}$	$1.577 \cdot 10^{-11}$	$9.143 \cdot 10^{-8}$
t :	0.98	0.81	0.73	0.65	0.48
$\tilde{Z}_t^A$:	$1.335 \cdot 10^{-18}$	$9.165 \cdot 10^{-15}$	$5.861 \cdot 10^{-13}$	$3.750 \cdot 10^{-11}$	$2.587 \cdot 10^{-7}$

Table 3.12: Bounds for $\zeta_{\tilde{A}_d}^A(t)$.

Proof. For part (a), recall from Section 3.2 that the character degree of $\tilde{S}_d$ corresponding to a strict partition λ is given by

$$\chi_{\tilde{S}_d}^\lambda(1) = 2^{(d-r-\varepsilon(\lambda))/2} g_\lambda, \tag{3.17}$$

where r is the number of parts in λ, and $\varepsilon(\lambda)$ is the parity of λ (also the parity of the integer $d+r$). Lemma 3.20 ensures that the smallest number g_λ of standard strict tableaux for a partition λ in A is g_{d,m_0}, where $m_0 = \lceil 0.33d \rceil$. As $\frac{7-12\cdot 0.33}{(1-2\cdot 0.33)^2} < 61$, Lemma 3.19 gives the lower bound

$$g_{d,m_0} > 0.2870 \cdot 1.8229^d.$$

It remains to approximate the exponent of 2 in (3.17).

A shifted diagram with r rows must contain at least a triangle-shaped subdiagram of the form $(r, r-1, \dots, 1)$. Hence the number of cells d is at least $r(r+1)/2$. From this we deduce that

$$r \leq -\frac{1}{2} + \frac{\sqrt{1+8d}}{2}.$$

Furthermore,

$$d - r \geq d\left(1 - \sqrt{\frac{1}{4d^2} + \frac{2}{d}}\right) + \frac{1}{2}.$$

By substituting $d = 61$ under the square root, we get the estimate

$$d - r - 1 \geq 0.81874d - \frac{1}{2},$$

which then leads to

$$2^{(d-r-\varepsilon(\lambda))/2} \geq 2^{(d-r-1)/2} \geq 2^{-1/4}\left(2^{0.81874/2}\right)^d.$$

Combining this with the estimate for g_{d,m_0} gives the desired bound.

For part (b), we let $q(d)$ denote the number of strict partitions of d, and $q'(d)$ the number of strict partitions of even parity. The characters corresponding to even partitions split in $\tilde{A}_d$ as discussed in Section 3.2. Thus, writing $b(d)$ for the lower bound given in part (a), we get the following estimate

$$\zeta^A_{\tilde{A}_d}(t) < \frac{q(d) + (2^{t+1}-1)q'(d)}{b(d)^t}.$$

The values of this expression can be computed explicitly up to $d = 110$ for each t listed in Table 3.7, and the maximum values appear in the table as $\tilde{Z}^A_t$.

On the other hand, we see that $\zeta^A_{\tilde{A}_d}(t) < 2^{t+1}q(d)/b(d)^t$. Proposition 3.14 gives a bound for the number of all partitions, so we get

$$\zeta^A_{\tilde{A}_d}(t) < \frac{2^{t+1}}{0.241^t} \cdot \frac{c_p^{\sqrt{d}}}{2.421^{td}},$$

where $c_p = \exp\left(\pi\sqrt{2/3}\right)$. The values of the last expression at $d = 110$ are below $\tilde{Z}^A_t$ for each t in Table 3.7. Also, some elementary calculus tells us that when $d \geq 110$, the expression is strictly decreasing in d for each t. This proves the claim. □

Next, we exhibit bounds related to characters coming from set B. As with $\tilde{S}_d$ above, we write $\chi^\lambda_{\tilde{A}_d}(1)$ for the character degree of $\tilde{A}_d$ corresponding to a strict partition λ, although the character itself is not uniquely determined by the partition.

Lemma 3.22.

(a) For any $\lambda \in B$, we have $\chi^{\lambda}_{\tilde{A}_d}(1) \geq 0.25 \cdot 1.367^d$.

(b) For any $d \geq 61$, we have $\zeta^B_{\tilde{A}_d}(t) < \tilde{Z}^B_t$, where t and $\tilde{Z}^B_t$ are as listed in Table 3.13.

t :	1	5/6	3/4	2/3	1/2
$\tilde{Z}^B_t$:	$1.163 \cdot 10^{-5}$	$2.215 \cdot 10^{-4}$	$9.666 \cdot 10^{-4}$	0.004219	0.08039
t :	0.95	0.78	0.70	0.61	0.45
$\tilde{Z}^B_t$:	$2.814 \cdot 10^{-5}$	$5.687 \cdot 10^{-4}$	0.002340	0.01150	0.1947

Table 3.13: Bounds for $\zeta^B_{\tilde{A}_d}(t)$.

Proof. For part (a), A. Wagner showed in [61] that every faithful character degree of $\tilde{A}_d$ is divisible by $2^{\lfloor (d-s-1)/2 \rfloor}$, where s is the number of ones in the binary representation of d. As s is at most $\log_2 d + 1$, we get

$$2^{\lfloor (d-s-1)/2 \rfloor} \geq 2^{(d-\log_2 d-4)/2} = \frac{(\sqrt{2})^d}{4\sqrt{d}}.$$

Since we have assumed $d \geq 61$, by setting $a = \sqrt[122]{61} \approx 1.0343$ we get $\sqrt{d} < a^d$ (see the proof of Lemma 3.11 for details). This way, we obtain

$$\chi^{\lambda}_{\tilde{A}_d}(1) \geq \frac{1}{4}\left(\frac{\sqrt{2}}{a}\right)^d,$$

which proves part (a).

For part (b), any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ in B has $d - \lambda_1 < \lambda_1$. Thus the number of strict partitions λ in B, that have $\lambda_1 = d - m$ for some m, is the same as the number of strict partitions of m, unless $m = 0$. Write $q(m)$ for the number of strict partitions of m, defining $q(0) = 1$, and write $q'_d(m)$ for the number of those ν among the aforementioned for which the augmented partition $(d-m, \nu_1, \ldots, \nu_r)$ is even. Now, the number of faithful irreducible characters of $\tilde{A}_d$ that correspond to partitions in B becomes

$$S(d) = \sum_{m=0}^{\lceil rd \rceil - 1} q(m) + q'_d(m).$$

Letting $b(d)$ denote the lower bound for the character degrees obtained in part (a), we can write

$$\zeta^B_{\tilde{A}_d}(t) < \frac{S(d)}{b(d)^t}.$$

Values for this expression can be computed explicitly up to $d = 200$ for each t in Table 3.8, and the maximal ones appear as $\tilde{Z}_t^B$.

On the other hand, using Proposition 3.14, we can estimate (very roughly) as follows:

$$\zeta_{\tilde{A}_d}^B(t) < \left(\frac{4}{1.367^d}\right)^t \sum_{m=0}^{\lceil rd \rceil - 1} 2c_p^{\sqrt{m}} < \frac{4^t \cdot 2(rd \cdot c_p^{\sqrt{rd}} + 1)}{1.367^{td}},$$

where $c_p = \exp\bigl(\pi\sqrt{2/3}\bigr)$. The bounding expression on the right is decreasing for $d \geq 200$, and its values at $d = 200$ are smaller than the $\tilde{Z}_t^B$ in Table 3.13. The claim is proved. □

We proceed to proving the main theorems.

Proof of Theorems 3.6 and 3.7. Let s be among 1, 5/6, 3/4, 2/3 and 1/2. We use a computer to calculate all faithful character degrees of $\tilde{A}_d$ up to $d = 60$. Computing the ratios $r_n^f(\tilde{A}_d)/n^s$ for these groups reveals that the bounds given in the theorems hold when $d \leq 60$. Moreover, the computations show that bounds claimed to be best possible really are so.

For $d \geq 61$, we use

$$\frac{r_n^f(\tilde{A}_d)}{n^s} \leq \zeta_{\tilde{A}_d}^A(s) + \zeta_{\tilde{A}_d}^B(s).$$

Bounds for the zeta functions are obtained in Lemmas 3.21 and 3.22. Using these bounds, we find that $r_n^f(\tilde{A}_d)/n^s < C_s$, with C_s given in the following table:

s :	1	5/6	3/4	2/3	1/2
C_s :	$1.164 \cdot 10^{-5}$	$2.216 \cdot 10^{-4}$	$9.667 \cdot 10^{-4}$	0.004220	0.08040

Comparing these values with the computed bounds for $d \leq 60$ gives the result. □

Proof of Theorems 3.8 and 3.9. Denote $S = \{1, 5/6, 3/4, 2/3, 1/2\}$. We are looking for maximal values over n of the ratios $\tilde{Q}_{d_0}^s(n)_b = (\tilde{s}_n(d_0) - b)/n^s$, where b assumes values 0, 3 and 6, s is in S and d_0 is 5 or 8. We have computed the degrees of all faithful characters of the covering groups $\tilde{A}_d$ for d at most 60. From part (a) of Lemma 3.22 we know that the smallest character degree of any $\tilde{A}_d$ with $d \geq 61$ is at least $n_0 = 47843110$. Therefore, for $n < n_0$, we know all values of $\tilde{Q}_{d_0}^s(n)_b$. The maximal values are given in Table 3.14 for $d_0 = 5$ and Table 3.15 for $d_0 = 8$. The tables also show the value n where the maximum was obtained, as well as the corresponding value of $\tilde{s}_n(d_0)$.

Assume then that $n \geq n_0$. For $d < 8$, all representation degrees of $\tilde{A}_d$ are less than n_0, so we need not deal with small d separately. Denote

$$\tilde{Q}^{X,s}(n) = \frac{1}{n^s} \sum_{d \geq 61} r_n^X(\tilde{A}_d), \quad \text{when } X \in \{A, B\},$$

$$\text{and} \qquad \tilde{Q}^{D,s}(n) = \frac{1}{n^s} \sum_{d \leq 60} r_n^f(\tilde{A}_d).$$

s	$\tilde{Q}_5^s(6)_0$	$\tilde{Q}_5^s(6)_3$	$\tilde{Q}_5^s(6)_6$
1	13/6	5/3	7/6
5/6	2.921	2.247	1.573
3/4	3.392	2.609	1.826
2/3	3.938	3.029	2.120
1/2	5.308	4.083	2.858

Table 3.14: Maximal values of $\tilde{Q}_5^s(n)_b$ for $n < 47843110$. All are attained at $n = 6$, and $\tilde{s}_6(5) = 13$. The decimal values are rounded up to four significant digits.

s	$\tilde{Q}_8^s(8)_0$ $(\tilde{s}_n(8) = 3)$	$\tilde{Q}_8^s(64)_3$ $(\tilde{s}_n(8) = 6)$	$\tilde{Q}_8^s(13728)_6$ $(\tilde{s}_n(8) = 7)$
1	3/8	3/64	$7.285 \cdot 10^{-5}$
5/6	0.5304	3/32	$3.565 \cdot 10^{-4}$
3/4	0.6307	0.1326	$7.885 \cdot 10^{-4}$
2/3	3/4	3/16	0.001745
1/2	1.0607	3/8	0.008535

Table 3.15: Maximal values of $\tilde{Q}_8^s(n)_b$ for $n < 47843110$. The decimal values are rounded up to four significant digits.

The values of $\tilde{Q}^{D,s}(n)$ can be inferred from the computed data, and maximal values over $n \geq n_0$ are listed as upper bounds to $\tilde{Q}^{D,s}(n)$ in Table 3.16 below. To get bounds for $\tilde{Q}^{X,s}(n)$ when X is A or B, we use Lemmas 3.21 and 3.22.

Let us first concentrate on $\tilde{Q}^{A,s}(n)$. As in the proof of Theorems 3.3 and 3.4, using part (b) of Lemma 3.21 we get for all $d \geq 61$ that

$$\tilde{Q}^{A,s}(n) < \frac{\ln n - \ln 0.241}{\ln 2.421} \cdot \tilde{Z}_t^A n^{t-s},$$

where t and $\tilde{Z}_t^A$ are as mentioned on the second row of Table 3.12. Denote this upper bound $\bar{Q}^{A,s,t}(n)$. For $t < s$, the expression $\bar{Q}^{A,s,t}(n)$ attains its maximum at

$$n_1^{s,t} = 0.241 \exp\left(\frac{1}{s-t}\right).$$

Looking at Table 3.12, we set $t = 0.98$ for $s = 1$, $t = 0.81$ for $s = 5/6$, etc. The maximal values $\bar{Q}^{A,s,t}(n_1^{s,t})$ are listed in Table 3.11 as upper bounds to $\tilde{Q}^{A,s}(n)$.

Similarly, using the results of Lemma 3.22, we get

$$\tilde{Q}^{B,s}(n) < \frac{\ln n - \ln 0.25}{\ln 1.367} \cdot \tilde{Z}_t^B n^{t-s},$$

with t and $\tilde{Z}_t^B$ as listed on the second row of Table 3.13. For $t < s$, this upper bound, named $\bar{Q}^{B,s}(n)$, attains its maximum at

$$n_2^{s,t} = 0.25 \exp\left(\frac{1}{s-t}\right).$$

Following Table 3.13, we set $t = 0.95$ for $s = 1$, $t = 0.78$ for $s = 5/6$, etc. The maximal values $\bar{Q}^{B,s,t}(n_2^{s,t})$ are listed in Table 3.11 as upper bounds to $\tilde{Q}^{B,s}(n)$.

s :	1	5/6	3/4	2/3	1/2
$\tilde{Q}^{A,s}(n)$:	$2.858 \cdot 10^{-14}$	$1.690 \cdot 10^{-13}$	$1.255 \cdot 10^{-11}$	$9.586 \cdot 10^{-10}$	$5.537 \cdot 10^{-6}$
$\tilde{Q}^{B,s}(n)$:	$7.097 \cdot 10^{-4}$	0.01351	0.05903	0.2582	4.909
$\tilde{Q}^{D,s}(n)$:	$6.304 \cdot 10^{-7}$	$8.578 \cdot 10^{-6}$	$3.164 \cdot 10^{-5}$	$1.168 \cdot 10^{-4}$	0.001588
$\tilde{Q}^{s}_{d_0}(n)_b$:	$7.104 \cdot 10^{-4}$	0.01352	0.05907	0.2584	4.911

Table 3.16: Upper bounds for the ratios $\tilde{Q}^{s}_{d_0}(n)_b = (\tilde{s}_n(d_0) - b)/n^s$ with $n \geq 47843110$ and for other related expressions (see the text for details).

Finally, we note that

$$\tilde{Q}^{s}_{d_0}(n)_b = \sum_{X \in \{A,B,D\}} \tilde{Q}^{X,s}(n)$$

for all $s \in S$, $b \in \{0,3,6\}$, $d_0 \in \{5,8\}$ and $n \geq n_0$. Upper bounds for $\tilde{Q}^{s}_{d_0}(n)_b$ can thus be computed by adding together the bounds obtained for $\tilde{Q}^{X,s}(n)$. This is done in the bottom row of Table 3.16. The claims of Theorems 3.8 and 3.9 can now be verified by comparing these values with the maxima for $n < n_0$ listed in Tables 3.14 and 3.15. $\square$

4. Complex representation growth of finite quasisimple groups of Lie type

4.1. Statement of results

Regarding representation growth of finite groups of Lie type, Martin Liebeck and Aner Shalev have proved in [40] the following upper bound for the growth rate: if $\mathcal{L}$ is a fixed Lie type and $\mathcal{L}(q)$ denotes any finite quasisimple group of type $\mathcal{L}$ over the field $\mathbb{F}_q$, then

$$r_n(\mathcal{L}(q)) < c_{\mathcal{L}}\, n^{2/h_{\mathcal{L}}} \qquad \text{for all } n. \tag{4.1}$$

Here $h_{\mathcal{L}}$ is the Coxeter number of $\mathcal{L}$ (number of roots divided by Lie rank), and $c_{\mathcal{L}}$ is some constant that only depends on the Lie type. The bound is asymptotically tight in the sense that the exponent cannot be smaller. As was explained in the Introduction on page 20, Liebeck and Shalev used their result to find mixing times for random walks on groups of Lie type.

In this section, we shall present numerical upper bounds for the total number of irreducible n-dimensional complex representations of groups that appear in certain families $\mathcal{L}$ of finite quasisimple groups of Lie type. In other words, letting $r_n^f(G)$ denote the number of faithful irreducible n-dimensional representations of G, we will bound the sum

$$s_n(\mathcal{L}) = \sum_{G\in\mathcal{L}} r_n^f(G)$$

for various classes $\mathcal{L}$. In fact, the bounds are of the following type:

$$s_n(\mathcal{L}) < cn^s, \tag{4.2}$$

where c and s are constants that only depend on the family $\mathcal{L}$. The results are presented in the following Theorems 4.1–4.3.

We first deal with the classical groups. We concern ourselves with the following families:

A_1 : linear groups in dimension 2
A' : linear groups in dimension at least 3
${}^2A$: unitary groups in dimensions at least 3
B : orthogonal groups in odd dimension ≥ 7 over a field of odd size
C : symplectic groups in dimension at least 4
D : orthogonal groups of plus type in even dimension ≥ 8
${}^2D$: orthogonal groups of minus type in even dimension ≥ 8.

Regarding that the Liebeck–Shalev bound given by (4.1) is asymptotically tight, we realise that for each $\mathcal{L}$, the smallest possible value for the exponent s in (4.2) is $2/h_{\mathcal{L}}$, where $h_{\mathcal{L}}$ is the largest Coxeter number appearing in the family $\mathcal{L}$. As a matter of fact, we can often push the exponent down to $2/h_{\mathcal{L}}$, but not always. (For details, see the remark on page 105.) Beside these optimal exponents, we list our best efforts as $s_{\mathcal{L}}$ in Table 4.1. Excluding type A_1, we will consider exponents 1, 2/3 and $s_{\mathcal{L}}$ for each classical family $\mathcal{L}$.

$\mathcal{L}:$	A_1	A'	^2A	B	C	D	^2D
$2/h_{\mathcal{L}}:$	1	2/3	2/3	1/3	1/2	1/3	1/3
$s_{\mathcal{L}}:$	1	2/3	2/3	**1/2**	1/2	**1/2**	**1/2**

Table 4.1: Optimal and obtained bounding exponents for the classical families. Where the optimal exponent differs from the obtained one, the latter is written in boldface.

Theorem 4.1. *For all $n > 1$, we have*

$$s_n(A_1) \leq \tfrac{8}{3}n.$$

Also, for $n > 12$, we have

$$s_n(A_1) \leq n + 3.$$

Remark. The constant in the first bound in the previous theorem is smallest possible, since there are eight 3-dimensional representations of quasisimple groups of type A_1. Whether the constant in the second bound is smallest possible depends on a number-theoretical question we could not settle. (See the proof for more details.)

Theorem 4.2. *Let $\mathcal{L}$ denote one of the families of classical groups presented above, and let $s_{\mathcal{L}}$ be as in Table 4.1. Then, for s in $\{1, 2/3, s_{\mathcal{L}}\}$, we have*

$$s_n(\mathcal{L}) \leq c_{\mathcal{L},s}\, n^s \qquad \textit{for all } n > 1,$$

with the constants $c_{\mathcal{L},s}$ given in Table 4.2.

Remark. For the previous theorem, the exact values of $s_n(\mathcal{L})/n^s$ were computed for small values of n. The maximal values of the ratio are given in Table 4.20 on page 110. If an entry in that table equals the corresponding one in Table 4.2, then we know that in that case the discovered constant is smallest possible (at least for small n).

Next, let $\mathcal{E}$ denote the family of finite quasisimple groups of exceptional Lie types $^2B_2$, $^3D_4$, E_6, E_7, E_8, $^2E_6$, F_4, $^2F_4$, G_2 and $^2G_2$.

Theorem 4.3. *For all $n > 1$, we have*

(a) $s_n(\mathcal{E}) \leq n$

(b) $s_n(\mathcal{E}) \leq 1.913n^{2/3}$

(c) $s_n(\mathcal{E}) < 5.057n^{1/2}$.

s	A_1	A'	^2A	B	C	D	^2D
1	8/3	7/8	2/3	2/27	1/2	1/8	1/34
2/3	—	7/4	3.464	2/9	0.8144	0.2804	0.1681
$s_{\mathcal{L}}$	8/3	7/4	3.464	0.3850	1.3417	2.134	2.134

Table 4.2: Bounding constants for classical groups.

Remark. The multiplicative constants in parts (a) and (b) of the above theorem are smallest possible, for there are altogether seven 7-dimensional representations of quasisimple groups of types G_2 and $^2G_2$. Also, the exponent 1/2 is smallest possible, since each group $^2B_2(q)$ has $q/2-1$ representations of dimension q^2+1.

We also get the immediate corollary below.

Corollary 4.4. *Let $\mathcal{H}$ denote the class of all finite quasisimple groups of Lie type. We have*

$$s_n(\mathcal{H}) \leq \tfrac{5}{3}n \qquad \textit{for all } n \geq 13.$$

For $n < 13$, the values of $s_n(\mathcal{H})$ are given in Table 4.3.

$n:$	2	3	4	5	6	7	8	9	10	11	12
$s_n:$	2	10	12	9	25	20	19	13	17	5	23

Table 4.3: Values of $s_n(\mathcal{H})$ for $n < 13$.

Proof. While deriving the main results, we will have computed the multiplicities of all representation degrees less than 2000 for all quasisimple groups of Lie type. The maximum of the ratio $s_n(\mathcal{H})/n$ is reached at $n = 24$, and it equals 5/3, as there are altogether 40 representations of dimension 24. When $n \geq 2000$, we use the bounds in Theorems 4.1, 4.2 and 4.3 to see that the stated upper bound holds. □

We obtain the above results by examining minimal character degrees of groups of Lie type. For all exceptional groups and for classical groups of small rank, the complete information of character degrees is available. For classical groups of large rank, however, we shall have to do with so-called *gap results*, which give the few smallest character degrees. The main bulk of the work is contained in classifying certain polynomials giving the character degrees in such a way that they can then be managed computationally. This classification process is explained in Section 4.3. The actual results are proved in Section 4.4 for the classical groups and in Section 4.5 for the exceptional groups.

4.2. Representation theory of quasisimple groups of Lie type

A universal covering group of a finite simple group of Lie type is completely determined by its simple quotient, which in turn is determined by its Lie family, rank r and the size q of the defining field. We will almost exclusively deal with the universal covering groups, as every irreducible representation of a quasisimple group can be lifted to a representation of the universal covering group. For this reason, we shall denote the universal covering group simply by $H_r(q)$, where H is replaced by the letter of the Lie family in question. For example, $A_2(3)$ is the group $\mathrm{SL}_3(3)$, and $E_6(4)$ is the triple cover of the finite simple group of Lie type E_6 over the field of four elements. Notice that this notation differs from the notation used in Section 2.2, as $E_6(4)$, for example, is here not the simple group but its universal cover. The notation is also different from the notation used in the Atlas.

Apart from 19 exceptions, the universal covering groups of simple groups of Lie type are obtained as follows (see also the end of Section 2.3): Suppose $\mathbf{G}(\bar{\mathbb{F}}_q)$ is a simply-connected simple linear algebraic group over the algebraic closure of $\mathbb{F}_q$, and let F be a Frobenius morphism. In the general case, the finite fixed point group $\mathbf{G}(q) = \mathbf{G}(\bar{\mathbb{F}}_q)^F$ is a universal covering group. We call these covering groups *regular*. The exceptions to this rule were given in Table 2.5, but we list them here again for ease of reference: $A_1(4)$, $A_1(9)$, $A_2(2)$, $A_2(4)$, $A_3(2)$, ${}^2A_3(2)$, ${}^2A_3(3)$, ${}^2A_5(2)$, $B_3(3)$, $C_2(2)$, $C_3(2)$, $D_4(2)$, ${}^2E_6(2)$, $F_4(2)$, $G_2(2)$, $G_2(3)$, $G_2(4)$, ${}^2B_2(8)$ and ${}^2F_4(2)$ (the Tits group). Even in these exceptional cases, the universal covering group is always denoted $H_r(q)$.

Let us now concentrate on the regular universal covering groups and see how their character degrees can be obtained. The behaviour of irreducible characters of finite groups of Lie type can be understood by the means of Deligne–Lusztig theory. That theory uses a generalised induction of characters defined via ℓ-adic cohomology of certain varieties. A brief description of the theory is given in [8, Chapters 11–13]. We are mostly interested in the partitioning of characters into so-called *Lusztig series*, and how the degrees can be calculated from the partition. To describe this, we need the concept of dual algebraic groups.

Let $\mathbf{G}$ and $\mathbf{G}^*$ be simple algebraic groups over an algebraically closed field $\bar{\mathbb{F}}_q$, with maximal tori T and T^*, respectively. The group $\mathbf{G}^*$ is said to be *dual* to $\mathbf{G}$ if there is an isomorphism between the character group $X(T)$ and the cocharacter group $Y(T^*)$ sending roots to coroots. In particular, the dual group to a simply-connected simple algebraic group is always an adjoint group. For example, the groups $\mathrm{SL}_n(\bar{\mathbb{F}}_q)$ and $\mathrm{PGL}_n(\bar{\mathbb{F}}_q)$ are dual to each other. If the groups are also endowed with Frobenius morphisms F and F^*, these morphisms are said to be dual to each other if they are compatible with the above isomorphism. More information about duality can be found in [5, Chapter 4].

Now, by Deligne–Lusztig theory, the characters of $\mathbf{G}^F$, a finite group of Lie type, are partitioned into Lusztig series ([8, Proposition 13.17]). Each Lusztig series $\mathcal{E}(\mathbf{G}^F, (s))$ is labelled by a so-called *geometric conjugacy class* (s) of a semi-simple element s in the dual group $(\mathbf{G}^*)^{F^*}$, and the members of $\mathcal{E}(\mathbf{G}^F, (1))$ are called *unipotent characters*. For any semi-simple $s \in (\mathbf{G}^*)^{F^*}$, there is a bijection ψ_s between $\mathcal{E}(\mathbf{G}^F, (s))$ and $\mathcal{E}(C_{\mathbf{G}^*}(s)^{F^*}, (1))$, the unipotent characters of the centraliser of s in the dual group ([8, Theorem 13.23]). Moreover, there is the following formula for the degree of a character χ in $\mathcal{E}(\mathbf{G}^F,(s))$ (see [8, Remark 13.24]):

$$\chi(1) = \frac{\left|\mathbf{G}^F\right|_{p'}}{\left|C_{\mathbf{G}^*}(s)^{F^*}\right|_{p'}} \cdot \psi_s(\chi)(1). \tag{4.3}$$

Here, p is the characteristic of $\mathbb{F}_q$, and subscript p' denotes the p-prime part.

The unipotent characters of adjoint groups have been discovered by Lusztig, and they are exhibited in [5, 13.8–13.9]. Their degrees are given by polynomials in q (except for the Suzuki and Ree types ${}^2B_2$, ${}^2F_4$ and ${}^2G_2$, for which the polynomials are in terms of $\sqrt{q}$). Let now $H_r(q)$ be a simply-connected finite group of a fixed Lie type H_r. It follows from the formula (4.3), together with the exact form of the degrees of the unipotent characters of adjoint groups, as well as the formulae for the orders of finite groups of Lie type (see e.g. [5, page 75]), that the character degrees of $H_r(q)$ are given by a finite set of polynomials in q (or $\sqrt{q}$ for the Suzuki and Ree groups). These sets of degree polynomials may differ depending on the congruence class of q modulo a fixed number that depends only on the type H_r. The degree polynomials are henceforth referred to as *Lusztig polynomials*.

We shall make use of the Lusztig polynomials in two ways. For Lie types of rank at most 8 (in some cases 7), Frank Lübeck has produced listings of the Lusztig polynomials. He has also computed the multiplicities of these polynomials, that is, how many characters share the same degree polynomial. The lists are available at his website [42]. The multiplicities are also polynomials in q. (A description of how these lists were produced is given in [45].) On the other hand, P. Tiep and A. Zalesskii [60] have found the polynomials giving the few smallest character degrees of classical groups, regardless of rank, and this list has been extended by Guralnick and Tiep [19] and H. N. Nguyen [49] for the symplectic and orthogonal groups.

Looking at equation (4.3) and the resulting Lusztig polynomials more closely, it is evident that the Lusztig polynomials are products of cyclotomic polynomials and a monomial (with a possible positive rational constant). The following lemma gives some simple facts about these kinds of polynomials.

Lemma 4.5. *Consider a polynomial $f(x)$ that is a product of a power of x and a non-zero number of cyclotomic polynomials, not all of them $x+1$ nor all of them $x-1$. Suppose that*

$$f(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0.$$

Let k be the greatest integer smaller than d such that $a_k \neq 0$, and write $a_M = \max_i |a_i|$. Let also $x_0 = a_M/|a_k| + 1$. Then the following are true.

(a) The function $x \mapsto f(x)$ is strictly increasing for all $x > 1$.

(b) Either

$$x^d < f(x) < (x+1)^d \tag{4.4}$$

or

$$(x-1)^d < f(x) < x^d \tag{4.5}$$

holds for all $x \geq x_0$. *Moreover, case* (4.4) *holds if and only if* $a_k > 0$.

Proof. (a) Note that f has the general form

$$f(x) = \prod_{i=1}^{d}(x - \omega_i),$$

where each ω_i is either zero or a root of unity, and not all of them are 0, 1 nor -1. When $x > 1$, the modulus $|x - \omega_i|$ of each factor is positive and increasing in x, and thus their product $|f(x)|$ is strictly increasing in x. Since the value $f(x)$ is real and positive for $x > 1$, it follows that part (a) holds.

(b) Assume $x > 1$. Now, as $|x-1| \leq |x - \omega_i| \leq |x+1|$ is true for every i in the above product, and equality does not hold for all i on either side, we get

$$|x-1|^d < |f(x)| < |x+1|^d.$$

For $x > 1$, all parts of this inequality are real and positive, so we may drop the absolute value signs. It remains to show that $f(x) - x^d$ does not pass over zero when $x \geq x_0$.

Let $x \geq x_0$. Assume first that $a_k > 0$. Then

$$\begin{aligned} f(x) - x^d = \sum_{i=0}^{k} a_i x^i &\geq a_k x^k - a_M \sum_{i=0}^{k-1} x^i = a_k x^k - a_M \frac{x^k - 1}{x-1} \\ &> x^k\bigl(a_k - a_M/(x-1)\bigr) \geq x^k(a_k - |a_k|) = 0. \end{aligned}$$

Similarly, if $a_k < 0$, we get

$$\begin{aligned} x^d - f(x) = -\sum_{i=0}^{k} a_i x^i &\geq -a_k x^k - a_M \sum_{i=0}^{k-1} x^i = |a_k| x^k - a_M \frac{x^k - 1}{x-1} \\ &> x^k\bigl(|a_k| - a_M/(x-1)\bigr) \geq x^k(|a_k| - |a_k|) = 0. \end{aligned}$$

□

4.3. Classifying the degree polynomials

In this section, we explain how to classify and parametrise the Lusztig polynomials in order to perform computations on them. We focus our attention to the universal covering groups of classical groups of rank greater than 1, that is, groups that belong to one of the following families mentioned in the beginning of this section: A', ${}^2A$, B, C, D and ${}^2D$. Moreover, we will only deal with universal covering groups of regular type, as the exceptional covers can be taken care of one by one.

There are systematic isomorphisms between certain small-rank groups in different families (e.g. between B_2 and C_2), and we wish to take each group into account only once. We have therefore excluded some of the smallest ranks in some of the families. For example, for family B we take the smallest applicable rank to be 3, since $B_1(q) \cong A_1(q)$ and $B_2(q) \cong C_2(q)$ for every q. The smallest applicable ranks are denoted as $r_0 = r_0(\mathcal{L})$ and listed in Table 4.4. Also, we will not consider groups of type B in even characteristic, as $B_r(2^k) \cong C_r(2^k)$ for all r.

The complete lists of Lusztig polynomials are available for groups of small rank. We will write $r_1 = r_1(\mathcal{L})$ for the first rank for which Lübeck has not produced complete lists with multiplicities. In other words, lists are available for $r < r_1$. The values of r_1 are also given in Table 4.4.

$\mathcal{L}:$	A'	${}^2A$	B	C	D	${}^2D$
$r_0:$	2	2	3	2	4	4
$r_1:$	9	9	9	9	8	8

Table 4.4: Restrictions for rank in the classical families.

4.3.1. Classifying Lübeck's polynomials

Fix any classical family $\mathcal{L}$ and restrict attention to groups in this family with rank at least $r_0(\mathcal{L})$ and less than $r_1(\mathcal{L})$. Then the Lusztig polynomials of these groups, together with their multiplicities, appear in Lübeck's lists. We also assume that $q \geq q_1 = 49$, for the smaller groups will be dealt with by direct computation.

We shall describe a list of pairs of polynomials (f_i, h_i), labelled by some index set I, where the f_i will run through all Lusztig polynomials pertaining to the family $\mathcal{L}$ (and having suitable rank). Thus, each $f_i(q)$ will be a representation degree of some universal covering group $H_r(q)$ of type $\mathcal{L}$, with $r_0 < r < r_1$. (As $q \geq q_1$, the universal covering group will be of regular type.) On the other hand, the h_i will be chosen so that, for each i, the value $h_i(q)$ will be an upper bound for the multiplicity of the representation degree $f_i(q)$. We achieve this enumeration in the following steps 1–4.

Step 1. Fix a rank r with $r_0 < r < r_1$, and collect into one set F_r all degree polynomials of the groups $H_r(q)$. Enumerate the polynomials as $f_{r,i}$, with i in some index set I_r.

Step 2. For each pair (r, i), with $i \in I_r$, there may exist several multiplicity polynomials $g_1, \ldots, g_k$ for the degree polynomial $f_{r,i}$. (These correspond to different congruence classes of q.) Going through the g_j one by one, construct a polynomial $g_{r,i}$ by the following algorithm:

(a) Let $g_{r,i,1} = g_1$.

(b) For any $j > 1$, if one of $g_j(q)$ and $g_{r,i,j-1}(q)$ dominates the other for all $q \geq q_1$, write $g_{r,i,j}$ for the dominating one. Otherwise, let $g_{r,i,j} = g_j + g_{r,i,j-1}$.

(c) Let $g_{r,i} = g_{r,i,k}$.

This procedure ensures that $g_{r,i}(q)$ is always an upper bound for the multiplicity of $f_{r,i}(q)$.

Step 3. Let $F = \bigcup_r F_r$ and enumerate the new set as $F = \{f_i\}_{i \in I}$. For all $i \in I$, define

$$\eta_i = \sum_{f_{r,i} = f_i} g_{r,i}.$$

In other words, η_i is the sum of all those $g_{r,i}$ that correspond to a given f_i as the latter appears as $f_{r,i}$ for different ranks r.

Step 4. Lastly, to make certain calculations easier, we define h_i to be the polynomial obtained from η_i by disregarding all terms with negative sign.

We now have all the Lusztig polynomials appearing in Lübeck's lists indexed by I. Next, we partition the indices $i \in I$ with regard to the asymptotic behaviour of f_i. More precisely, for any positive rational number a and integer d, we define the following parametrised classes:

$$\begin{aligned} I^+_{a,d} &= \{i \in I \mid aq^d \leq f_i(q) < a(q+1)^d \text{ for } q \geq q_1\} \\ I^-_{a,d} &= \{i \in I \mid a(q-1)^d < f_i(q) < aq^d \text{ for } q \geq q_1\}. \end{aligned}$$

All these sets are clearly disjoint. By going through all Lübeck's lists, we have checked that no degree polynomial has the form $a(q-1)^d$ or $a(q+1)^d$. Now, Lemma 4.5 ensures that for large enough q_1, each $i \in I$ is contained in one of $I^{\pm}_{a,d}$. Again, we have checked computationally that the q_1 we have chosen is adequate. Thus, we know that the sets $I^{\pm}_{a,d}$ form a partition of I.

For the sake of convenience, we shall often write I_ν instead of $I^{\varepsilon}_{a,d}$, abbreviating the parameters as one triple $\nu = (a, d, \varepsilon)$. When ε appears in formulae, we will also write 0 for $+$ and 1 for $-$.

To each triple $\nu = (a, d, \varepsilon)$ we further attach a positive integer N_ν and a polynomial h_ν. Firstly, we let N_ν be the smallest value given by the degree polynomials indexed by I_ν for $q \geq q_1$:

$$N_\nu = \min_{i \in I_\nu} f_i(q_1).$$

Secondly, the polynomial h_ν is defined as follows. For most parameter vectors ν, it can be computationally verified that there is some $i \in I_\nu$ for which h_i dominates the other multiplicities, that is, $h_i(q) \geq \max_{j \in I_\nu} h_j(q)$ for all $q \geq q_1$. In this case, we define $h_\nu = h_i$. Otherwise, we let $h_\nu = \sum_{i \in I_\nu} h_i$.

Example 4.6. The Lusztig polynomials of the groups $A_2(q)$ are given in the two tables below. The degree polynomials are listed as φ_i, and the multiplicity polynomials as ψ_i. The first table is for q congruent to 0 or 2 modulo 3, and the second is for q congruent to 1 modulo 3.

We see that some degree polynomials appear in both tables. We enumerate the degree polynomials as $f_{2,i}$. Of those multiplicity polynomials that correspond to the same degree polynomial in different tables, we choose the bigger one and call it $g_{2,i}$. For example, polynomials φ_3 and φ_{12} are equal, so they will be taken as one polynomial $f_{2,3}$, and the chosen multiplicity will be $g_{2,3}(q) = \frac{1}{3}(q^2 + q)$.

i	φ_i	ψ_i
1	q^2+q	1
2	q^2+q+1	$q-2$
3	q^3-q^2-q+1	$\frac{1}{3}(q^2+q)$
4	q^3-1	$\frac{1}{2}(q^2-q)$
5	q^3	1
6	q^3+q^2+q	$q-2$
7	q^3+2q^2+2q+1	$\frac{1}{6}(q^2-5q+6)$

i	φ_i	ψ_i
8	q^2+q	1
9	q^2+q+1	$q-2$
10	$\frac{1}{3}(q^3-q^2-q+1)$	6
11	$\frac{1}{3}(q^3+2q^2+2q+1)$	3
12	q^3-q^2-q+1	$\frac{1}{3}(q^2+q-2)$
13	q^3-1	$\frac{1}{2}(q^2-q)$
14	q^3	1
15	q^3+q^2+q	$q-2$
16	q^3+2q^2+2q+1	$\frac{1}{6}(q^2-5q+4)$

According to the previous discussion, the next step would be to combine the tables for different ranks. We pretend there is only one rank to be considered, so that $f_i = f_{2,i}$ for all i and the polynomials h_i will be obtained from the $g_{2,i}$ simply by disregarding any negative terms. (In reality, we would take sums of different $g_{r,i}$.) This leads to the following table:

i	$f_{2,i} = f_i$	$g_{2,i}$	h_i	$\nu = (a,d,\varepsilon)$
1	q^2+q	1	1	$(1,2,+)$
2	q^2+q+1	$q-2$	q	
3	$\frac{1}{3}(q^3-q^2-q+1)$	6	6	$(1/3,3,-)$
4	$\frac{1}{3}(q^3+2q^2+2q+1)$	3	3	$(1/3,3,+)$
5	q^3-q^2-q+1	$\frac{1}{3}(q^2+q)$	$\frac{1}{3}(q^2+q)$	$(1,3,-)$
6	q^3-1	$\frac{1}{2}(q^2-q)$	$\frac{1}{2}q^2$	
7	q^3	1	1	$(1,3,+)$
8	q^3+q^2+q	$q-2$	q	
9	q^3+2q^2+2q+1	$\frac{1}{6}(q^2-5q+6)$	$\frac{1}{6}(q^2+6)$	

In the above table, we have also partitioned the rows according to the classes I_ν, whose parameters are displayed in the last column. As an example, for the class with $\nu = (1,3,-)$ we would have $N_\nu = f_5(q_1) = 115200$ and $h_\nu = h_6$.

The significance of the classification described above is revealed in the following lemma.

Lemma 4.7. *Let $n > 1$. Writing $H_r(q)$ for a regular universal covering group of a fixed classical family, we have*

$$\sum_{r<r_1} \sum_{q \geq q_1} r_n\big(H_r(q)\big) \;\leq \sum_{\nu;\, N_\nu \leq n} h_\nu\big((n/a)^{1/d} + \varepsilon\big),$$

where $\nu = (a,d,\varepsilon)$ and $\varepsilon \in \{0,1\}$.

Proof. The idea is to estimate the double sum by adding together all multiplicities of degree polynomials that obtain a particular value n.

Suppose $n > 1$ is a representation degree of some $H_r(q)$ with $r < r_1$ and $q \geq q_1$. Such a degree is given by $f_i(q)$ for some indices i. We argue that in any I_ν there is at most one such i.

Firstly, if $f_i(q) = n$ for some $i \in I^+_{a,d}$, then we have $aq^d \leq n < a(q+1)^d$, and this inequality holds for at most one integer q. The conclusion is also true if $i \in I^-_{a,d}$. On the other hand, we can check computationally that when $q \geq q_1$, we have $f_i(q) \neq f_j(q)$ for any i and j in I_ν, unless $i = j$. Thus, there is at most one $i \in I_\nu$ for which $f_i(q) = n$.

Finally, the polynomial f_i indexed by I_ν cannot have n as a value unless $N_\nu \leq n$. On the other hand, if $n = f_i(q)$ and $i \in I_\nu$, we have $q < (n/a)^{1/d} + \varepsilon$. As h_ν is an upper bound for all the h_i with $i \in I_\nu$, the result holds. □

4.3.2. Minimal degree polynomials and gap results

For ranks at least $r_1(\mathcal{L})$, we do not have complete lists of Lusztig polynomials available. Instead, we will use partial results obtained by Tiep and Zalesskii [60], Nguyen [49] and Guralnick and Tiep [19]. These results give us some of the Lusztig polynomials corresponding to the smallest representation degrees.

The minimal degree results are sometimes called *gap results* because they show that there are only a few characters of the smallest degree n_1, and maybe a few of the degrees $n_1 + 1$ and $n_1 + 2$, whereafter there is a relatively large "gap" before the next degree. Again, after a few degrees, there might be another gap before the next one and so forth.

For example, any group $A_r(q) = \mathrm{SL}_{r+1}(q)$ with $r \geq 2$ has irreducible representations of degrees

$$\begin{aligned} \frac{q^{r+1} - q}{q-1} &= q^r + q^{r-1} + \cdots + q \\ \text{and} \quad \frac{q^{r+1} - 1}{q-1} &= q^r + q^{r-1} + \cdots + q + 1. \end{aligned}$$

The difference between these two degrees is one, and they are both "below the first gap". The next dimension is "above the first gap", given by a polynomial with degree at least $2r - 2$.

For each classical family $\mathcal{L}$, we will now describe a similar classification of the minimal degree polynomials as we did above with Lübeck's polynomials. For $r \geq r_1(\mathcal{L})$, we have listed the necessary polynomials in Tables 4.5 through 4.10. In the tables, φ_i always denotes the polynomial giving the i'th smallest character degree, and ψ_i is the corresponding multiplicity. Notice that the multiplicities are linear polynomials, and do not depend on the rank of the group like the degrees do. In some cases the polynomials depend on the congruence class of q, and this is indicated in a separate column.

Fix a family $\mathcal{L}$. We shall classify the set of minimal degree polynomials of groups of type $\mathcal{L}$ according to their asymptotic behaviour. If φ_i is such a minimal degree polynomial, let a be its leading coefficient and $k(r)$ its degree. Then it is straightforward to check that, for each i, exactly one of the following holds:

$$aq^{k(r)} < \varphi_i(r,q) < a(q+1)^{k(r)} \qquad \text{for all } q \text{ and } r \tag{4.6}$$

$$\text{or} \qquad a(q-1)^{k(r)} < \varphi_i(r,q) < aq^{k(r)} \qquad \text{for all } q \text{ and } r. \tag{4.7}$$

The degree of the polynomial has the form $k(r) = \alpha r + \beta$, where α and β are integers. We classify all the minimal degree polynomials according to properties (4.6) and (4.7) by partitioning the indices i into classes $I^{\varepsilon}_{a,\alpha,\beta}$, where ε is set to $+$ if φ_i satisfies condition (4.6), and to $-$ otherwise. The parameters are shown next to the corresponding polynomials in the tables. We shall often abbreviate the parameters as one vector $\nu = (a, \varepsilon, \alpha, \beta)$, writing I_ν instead of $I^{\varepsilon}_{a,\alpha,\beta}$. In formulae, we shall also write ε as 0 instead of $+$ and 1 instead of $-$.

The last row in the tables is a lower bound to the next character degree, so it indicates the size of the next gap. We call the polynomial appearing in this row the *gap bound*, and denote it by $\Gamma = \Gamma_{\mathcal{L}}$. Degrees given by the actual minimal degree polynomials are said to be *below the gap bound*, and the bigger degrees are *above the gap bound*.

For each gap bound, we can also find a lower bound of the form

$$\Gamma(r,q) \geq a(q-\varepsilon)^{\alpha r+\beta},$$

where a is the leading coefficient of Γ, ε is either 0 or 1, and α and β are integers. Thus, we may attach a parameter vector $(a, \varepsilon, \alpha, \beta)$ also to the gap bound, although this polynomial is not considered to be indexed by any I_ν. The parameters are shown in the tables.

	$\varphi_i(r,q)$	$\psi_i(r,q)$	$(a,\varepsilon,\alpha,\beta)$
	$\frac{q^{r+1}-q}{q-1}$	1	$(1,+,1,0)$
	$\frac{q^{r+1}-1}{q-1}$	$q-2$	
Γ:	$\frac{(q^{r+1}-1)(q^r-q^2)}{(q-1)(q^2-1)}$	—	$(1,+,2,-2)$

Table 4.5: Minimal degree polynomials of $A_r(q)$, $r \geq 9$. (From [60].)

	$\varphi_i(r,q)$	$\psi_i(r,q)$	q	$(a,\varepsilon,\alpha,\beta)$
	$\frac{q^{r+1}-q+\kappa(q-1)}{q+1}$	$1+\kappa(q-1)$		$(1,-,1,0)$
	$\frac{q^{r+1}+1+\kappa(q-1)}{q+1}$	$q+\kappa(1-q)$		
Γ:	$\frac{(q^{r+1}+1-2\kappa)(q^r-q^2+\kappa(q^2-q))}{(q+1)(q^2-1)}$	—	2	$(1,-,2,-2)$
	$\frac{(q^{r+1}+1-2\kappa)(q^r-q^2+\kappa(q^2+1))}{(q+1)(q^2-1)}$	—	>2	

Table 4.6: Minimal degree polynomials of ${}^2A_r(q)$, $r \geq 9$. Here, $\kappa = 0$ if r is even, and $\kappa = 1$ if r is odd. (From [60].)

	$\varphi_i(r,q)$	$\psi_i(r,q)$	q	$(a,\varepsilon,\alpha,\beta)$
	$\frac{q^{2r}-1}{q^2-1}$	1	odd	$(1,+,2,-2)$
	$\frac{(q^r-1)(q^r-q)}{2(q+1)}$	1		$(1/2,-,2,-1)$
	$\frac{(q^r+1)(q^r+q)}{2(q+1)}$	1		
	$\frac{(q^r+1)(q^r-q)}{2(q-1)}$	1		$(1/2,+,2,-1)$
	$\frac{(q^r-1)(q^r+q)}{2(q-1)}$	1		
	$\frac{q^{2r}-1}{q+1}$	$\frac{1}{2}q$	even	$(1,-,2,-1)$
	$\frac{q^{2r}-1}{q+1}$	$\frac{1}{2}(q-1)$	odd	
	$\frac{q^{2r}-1}{q-1}$	$\frac{1}{2}(q-2)$	even	$(1,+,2,-1)$
	$\frac{q^{2r}-1}{q-1}$	$\frac{1}{2}(q-3)$	odd	
	$\frac{q^{2r+1}-q}{q^2-1}$	1	odd	
Γ:	q^{4r-8}	—		$(1,+,4,-8)$

Table 4.7: Minimal degree polynomials of $B_r(q)$, $r \geq 9$. (From [49] for odd q and [19] for even q.)

$\varphi_i(r,q)$	$\psi_i(r,q)$	q	$(a,\varepsilon,\alpha,\beta)$
$\frac{1}{2}(q^r-1)$	2	odd	$(1/2,-,1,0)$
$\frac{1}{2}(q^r+1)$	2	odd	$(1/2,+,1,0)$
$\frac{(q^r-1)(q^r-q)}{2(q+1)}$	1		$(1/2,-,2,-1)$
$\frac{q^{2r}-1}{2(q+1)}$	2	odd	
$\frac{(q^r+1)(q^r+q)}{2(q+1)}$	1		
$\frac{(q^r+1)(q^r-q)}{2(q-1)}$	1		$(1/2,+,2,-1)$
$\frac{q^{2r}-1}{2(q-1)}$	2	odd	
$\frac{(q^r-1)(q^r+q)}{2(q-1)}$	1		
$\frac{q^{2r}-1}{q+1}$	$\frac{1}{2}q$	even	$(1,-,2,-1)$
$\frac{q^{2r}-1}{q+1}$	$\frac{1}{2}(q-1)$	odd	
$\frac{q^{2r}-1}{q-1}$	$\frac{1}{2}(q-2)$	even	$(1,+,2,-1)$
$\frac{q^{2r}-1}{q-1}$	$\frac{1}{2}(q-3)$	odd	
$\frac{(q^{2r}-1)(q^{r-1}-q)}{2(q^2-1)}$	2	odd	$(1/2,+,3,-3)$
$\frac{(q^{2r}-1)(q^{r-1}-1)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-1)(q^{r-1}+1)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-1)(q^{r-1}+q)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-1)(q^{r-1}-1)}{2(q+1)}$	$q-1$	odd	$(1/2,-,3,-2)$
$\frac{(q^{2r}-1)(q^{r-1}+1)}{2(q+1)}$	$q-1$	odd	
$\frac{(q^{2r}-1)(q^r-q)}{2(q^2-1)}$	2	odd	$(1/2,+,3,-2)$
$\frac{(q^{2r}-q^2)(q^r-1)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-q^2)(q^r+1)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-1)(q^r+q)}{2(q^2-1)}$	2	odd	
$\frac{(q^{2r}-1)(q^{r-1}-1)}{2(q-1)}$	$q-3$	odd	
$\frac{(q^{2r}+1)(q^{r-1}-1)}{2(q-1)}$	$q-3$	odd	
$\Gamma\colon \left(\frac{(q^{r-1}+1)(q^{r-2}-q)}{q^2-1}-1\right)\frac{q^{r-1}(q^{r-1}-1)(q-1)}{2}$	—	even	$(1/2,-,4,-6)$
$\frac{(q^{2r-1}-1)(q^{r-1}-1)(q^{r-1}-q^2)}{2(q^4-1)}$	—	odd	

Table 4.8: Minimal degree polynomials of $C_r(q)$, $r \geq r_0$. (From [49] for odd q and [19] for even q.)

	$\varphi_i(r,q)$	$\psi_i(r,q)$	q	$(a,\varepsilon,\alpha,\beta)$
	$\frac{(q^r-1)(q^{r-1}+q)}{q^2-1}$	1		$(1,+,2,-3)$
	$\frac{(q^r-1)(q^{r-1}-1)}{2(q+1)}$	2	odd	$(1/2,-,2,-2)$
	$\frac{(q^r-1)(q^{r-1}+1)}{q-1}$	2	odd	$(1/2,+,2,-2)$
	$\frac{(q^r-1)(q^{r-1}-1)}{q+1}$	$\frac{1}{2}q$	even	$(1,-,2,-2)$
	$\frac{(q^r-1)(q^{r-1}-1)}{q+1}$	$\frac{1}{2}(q-1)$	odd	
	$\frac{q^{2r}-q^2}{q^2-1}$	1		$(1,+,2,-2)$
	$\frac{(q^r-1)(q^{r-1}+1)}{q-1}$	$\frac{1}{2}(q-2)$	even	
	$\frac{(q^r-1)(q^{r-1}+1)}{q-1}$	$\frac{1}{2}(q-3)$	odd	
Γ:	$q^{4r-10}+1$	—		$(1,+,4,-10)$

Table 4.9: Minimal degree polynomials of $D_r(q)$, $r \geq 8$. (From [49].)

	$\varphi_i(r,q)$	$\psi_i(r,q)$	q	$(a,\varepsilon,\alpha,\beta)$
	$\frac{(q^r+1)(q^{r-1}-q)}{q^2-1}$	1		$(1,+,2,-3)$
	$\frac{(q^r+1)(q^{r-1}+1)}{2(q+1)}$	2	odd	$(1/2,-,2,-2)$
	$\frac{(q^r+1)(q^{r-1}-1)}{q-1}$	2	odd	$(1/2,+,2,-2)$
	$\frac{(q^r+1)(q^{r-1}+1)}{q+1}$	$\frac{1}{2}q$	even	$(1,-,2,-2)$
	$\frac{(q^r+1)(q^{r-1}+1)}{q+1}$	$\frac{1}{2}(q-1)$	odd	
	$\frac{q^{2r}-q^2}{q^2-1}$	1		$(1,+,2,-2)$
	$\frac{(q^r+1)(q^{r-1}-1)}{q-1}$	$\frac{1}{2}(q-2)$	even	
	$\frac{(q^r+1)(q^{r-1}-1)}{q-1}$	$\frac{1}{2}(q-3)$	odd	
Γ:	$q^{4r-10}+1$	—		$(1,+,4,-10)$

Table 4.10: Minimal degree polynomials of ${}^2D_r(q)$, $r \geq 8$. (From [49].)

We add still some more properties to the classes of minimal degree polynomials constructed above. To each parameter vector $\nu = (a, \varepsilon, \alpha, \beta)$ we attach (as was done earlier with Lübeck's polynomials) certain bounding numbers and polynomials. Firstly, for some ν as indicated in Tables 4.5–4.10, there is a limitation for those field sizes to which the polynomials indexed by I_ν can be applied. We write $q_0(\nu)$ for the smallest such field. Note that $q_0(\nu)$ is either 2 or 3, depending on ν, and that for type B we have $q_0(\nu) = 3$ for all ν.

Secondly, define

$$f_\nu(r, q) = a(q - \varepsilon)q^{\alpha r + \beta - 1}. \tag{4.8}$$

(In the formula, we write $\varepsilon \in \{0, 1\}$, though as a parameter we usually have $\varepsilon \in \{+, -\}$.) For each ν, it can be checked that f_ν is a lower bound for the degree polynomials indexed by I_ν. More precisely, for all $i \in I_\nu$ and each $r \geq r_1(\mathcal{L})$ and $q \geq q_0(\nu)$, we have $\varphi_i(r, q) \geq f_\nu(r, q)$. Note that if $\varepsilon = +$, then $f_\nu(r, q) = aq^{\alpha r + \beta}$.

For each ν, there is a multiplicity polynomial ψ_i with $i \in I_\nu$ that dominates all other multiplicity polynomials in the same class. We let g_ν denote this largest multiplicity polynomial, in addition dropping any terms with negative sign. Thus, $g_\nu(q) \geq \psi_i(q)$ for all $i \in I_\nu$. Note that all the g_ν are first degree polynomials.

Lastly, we let $N_\nu(r)$ be the smallest value attained by the minimal degree polynomials indexed by I_ν and corresponding to a group of rank $r \geq r_1(\mathcal{L})$. The value of $N_\nu(r)$ is obtained by substituting $q_0(\nu)$ for q in the smallest minimal degree polynomial indexed by I_ν.

The values of $q_0(\nu)$, g_ν and $N_\nu(r)$ are gathered together in Tables 4.11 through 4.16. In these tables, the last row gives bounds related to the gap bound. For a parameter vector μ corresponding to a gap bound Γ, we let f_μ denote a polynomial lower bound to Γ with the same form as in (4.8). The value of $q_0(\mu)$ is also defined in the same way as $q_0(\nu)$ above.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1, +, 1, 0)$	2	q	$2^{r+1} - 2$
gap:	$(1, +, 2, -2)$	2	—	—

Table 4.11: Bounding parameters for minimal degree polynomials of $A_r(q)$.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1, -, 1, 0)$	2	q	$\big(2^{r+1} - \gcd(r, 2)\big)/3$
gap:	$(1, -, 2, -2)$	2	—	—

Table 4.12: Bounding parameters for minimal degree polynomials of ${}^2A_r(q)$.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1, +, 2, -2)$	3	1	$(3^{2r} - 1)/8$
	$(1/2, -, 2, -1)$	3	1	$(3^{2r} - 4 \cdot 3^r + 3)/8$
	$(1/2, +, 2, -1)$	3	1	$(3^{2r} - 2 \cdot 3^r - 3)/4$
	$(1, -, 2, -1)$	3	$q/2$	$(3^{2r} - 1)/2$
	$(1, +, 2, -1)$	3	$q/2$	$(3^{2r} - 1)/2$
gap:	$(1, +, 4, -8)$	3	—	—

Table 4.13: Bounding parameters for minimal degree polynomials of $B_r(q)$.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1/2, -, 1, 0)$	3	2	$(3^r - 1)/2$
	$(1/2, +, 1, 0)$	3	2	$(3^r + 1)/2$
	$(1/2, -, 2, -1)$	2	2	$(2^{2r-1} - 3 \cdot 2^{r-1} + 1)/3$
	$(1/2, +, 2, -1)$	2	2	$2^{2r-1} - 2^{r-1} - 1$
	$(1, -, 2, -1)$	2	$q/2$	$(2^{2r} - 1)/3$
	$(1, +, 2, -1)$	2	$q/2$	$2^{2r} - 1$
	$(1/2, +, 3, -3)$	3	2	$(3^{3r-1} - 3^{2r+1} - 3^{r-1} + 3)/16$
	$(1/2, -, 3, -2)$	3	q	$(3^{3r-1} - 3^{2r} - 3^{r-1} + 1)/8$
	$(1/2, +, 3, -2)$	3	q	$(3^{3r} - 3^{2r+1} - 3^r + 3)/16$
gap:	$(1/2, -, 4, -6)$	2	—	—

Table 4.14: Bounding parameters for minimal degree polynomials of $C_r(q)$.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1, +, 2, -3)$	2	1	$(2^{2r-1} + 3 \cdot 2^{r-1} - 1)/3$
	$(1/2, -, 2, -2)$	3	2	$(3^{2r-1} - 4 \cdot 3^{r-1} + 1)/8$
	$(1/2, +, 2, -2)$	3	2	$(3^{2r-1} + 2 \cdot 3^{r-1} - 1)/2$
	$(1, -, 2, -2)$	2	$q/2$	$(2^{2r-1} - 3 \cdot 2^{r-1} + 1)/3$
	$(1, +, 2, -2)$	2	$q/2$	$(2^{2r} - 4)/3$
gap:	$(1, +, 4, -10)$	2	—	—

Table 4.15: Bounding parameters for minimal degree polynomials of $D_r(q)$.

	$\nu = (a, \varepsilon, \alpha, \beta)$	$q_0(\nu)$	$g_\nu(q)$	$N_\nu(r)$
	$(1, +, 2, -3)$	2	1	$(2^{2r-1} - 3 \cdot 2^{r-1} - 2)/3$
	$(1/2, -, 2, -2)$	3	2	$(3^{2r-1} + 4 \cdot 3^{r-1} + 1)/8$
	$(1/2, +, 2, -2)$	3	2	$(3^{2r-1} - 2 \cdot 3^{r-1} - 1)/2$
	$(1, -, 2, -2)$	2	$q/2$	$(2^{2r-1} + 3 \cdot 2^{r-1} + 1)/3$
	$(1, +, 2, -2)$	2	$q/2$	$(2^{2r} - 4)/3$
gap:	$(1, +, 4, -10)$	2	—	—

Table 4.16: Bounding parameters for minimal degree polynomials of ${}^2D_r(q)$.

From the parameter vectors ν, we can always reconstruct the bounding polynomials f_ν and f_μ defined by (4.8). For example, for the family D, these lower bounds read as follows:

ν	f_ν
$(1,+,2,-3)$	q^{2r-3}
$(1/2,-,2,-2)$	$\frac{1}{2}(q-1)q^{2r-3}$
$(1/2,+,2,-2)$	$\frac{1}{2}q^{2r-2}$
$(1,-,2,-2)$	$(q-1)q^{2r-3}$
$(1,+,2,-2)$	q^{2r-2}
$\mu=(1,+,4,-10)$	q^{4r-10}

Let now $H_r(q)$ be a universal covering group belonging to a classical family $\mathcal{L}$. Consider a parameter vector ν. For any fixed positive integer n, the lower bound given in (4.8) yields upper bounds to those field sizes q and ranks r, for which n can appear as one of the irreducible representation degrees of $H_r(q)$ indexed by I_ν. Namely, we see that if $\varphi_i(r,q)=n$ for some $i\in I_\nu$, then

$$r \leq \frac{1}{\alpha}\left(\frac{\log n - \log a - \varepsilon\log(q_0-1)}{\log q_0} - \beta + \varepsilon\right), \tag{4.9}$$

where $q_0=q_0(\nu)$. We write $r_\nu(n)$ for this upper bound. Note especially that $N_\nu(r)>n$ for $r>r_\nu(n)$.

Also, for each r we have

$$q \leq \left(\frac{n}{a(q_0-1)^\varepsilon}\right)^{\frac{1}{\alpha r+\beta-\varepsilon}}. \tag{4.10}$$

Call this upper bound $q_\nu(r,n)$. Similar bounds can also be obtained if the degree n is above the gap bound. These will be denoted r_μ and q_μ for the gap parameter vector μ.

As an example, we give in Tables 4.17 and 4.18 the expressions for r_ν, q_ν, r_μ and q_μ in the case $\mathcal{L}=D$, and also the expressions for q_μ for all classical families.

These bounds can be now used in the same way as the corresponding bounds for Lübeck's polynomials.

Lemma 4.8. *Write $H(q)$ for a universal covering group of a fixed classical type $\mathcal{L}$ with fixed rank $r\geq r_1(\mathcal{L})$, and let $r_n^<(H(q))$ be the number of irreducible representations of degree n below the gap bound. Then we have*

$$\sum_q r_n^<\big(H(q)\big) \;\leq \sum_{\nu;\,N_\nu(r)\leq n} g_\nu\big(q_\nu(r,n)\big).$$

Proof. The proof is similar to the one of Lemma 4.7. Suppose $n>1$ is a representation degree of some $H(q)$. Such a degree is given by a minimal degree polynomial φ_i. We argue that in any I_ν there is at most one such i.

ν	$q_0(\nu)$	$r_\nu(n)$	$q_\nu(r,n)$
$(1,+,2,-3)$	2	$\frac{1}{2}(\log_2 n+3)$	$n^{\frac{1}{2r-3}}$
$(1/2,-,2,-2)$	3	$\frac{1}{2}(\log_3 n+3)$	$n^{\frac{1}{2r-3}}$
$(1/2,+,2,-2)$	3	$\frac{1}{2}(\log_3 n+\log_3 2+2)$	$(2n)^{\frac{1}{2r-2}}$
$(1,-,2,-2)$	2	$\frac{1}{2}(\log_2 n+3)$	$n^{\frac{1}{2r-3}}$
$(1,+,2,-2)$	2	$\frac{1}{2}(\log_2 n+2)$	$n^{\frac{1}{2r-2}}$
$\mu=(1,+,4,-10)$	2	$\frac{1}{4}(\log_2 n+10)$	$n^{\frac{1}{4r-10}}$

Table 4.17: Expressions for r_ν, q_ν, r_μ and q_μ for the Lie family D.

$\mathcal{L}:$	A'	${}^2A$	B	C	D	${}^2D$
$q_\mu(r,n):$	$n^{\frac{1}{2r-2}}$	$n^{\frac{1}{2r-3}}$	$n^{\frac{1}{4r-8}}$	$(2n)^{\frac{1}{4r-7}}$	$n^{\frac{1}{4r-10}}$	$n^{\frac{1}{4r-10}}$

Table 4.18: Expressions for q_μ for all classical families.

As in the proof of Lemma 4.7, it follows from the definition of the classes I_ν that there is only one possible q for which $\varphi_i(q)=n$ can hold. On the other hand, by going through all minimal degree polynomials of type $\mathcal{L}$, we can check that they all have distinct values at any q (when $r\geq r_1(\mathcal{L})$). Hence, there can exist at most one i such that $\varphi_i(q)=n$.

Now, a minimal degree polynomial giving the degree n cannot be indexed by I_ν unless $N_\nu(r)\leq n$. In this case, $q_\nu(r,n)$ is an upper bound to the possible q for which $H(q)$ can have n as a representation degree, and g_ν dominates all multiplicity polynomials related to I_ν. This justifies the upper bound. □

4.3.3. Above the gap bound

Assume that $H_r(q)$ is a universal covering group of a classical type $\mathcal{L}$, and $r\geq r_1(\mathcal{L})$, as before. If a representation degree of $H_r(q)$ is above the gap bound, we do not know its exact multiplicity. In this case, we shall simply bound the multiplicity by the number of conjugacy classes of $H_r(q)$. Upper bounds to these conjugacy class numbers have been obtained by J. Fulman and R. Guralnick in [11]. The bounds have the form $q^r+B_{\mathcal{L}}q^{r-1}$, where each $B_{\mathcal{L}}$ is a constant depending on the classical family $\mathcal{L}$. The constants are listed in Table 4.19.

The following two simple lemmata allow us to bound the multiplicities of the representation degrees in a fairly effective way. The first one is a simple estimate relieving us from having to sum over all integers when we cannot determine which of them are prime powers.

Lemma 4.9. *Suppose that K is a strictly increasing function on the integers and Q is a positive integer. For summing the values of $K(q)$ over prime powers q, we have the*

$\mathcal{L}:$	A'	${}^2A$	B	C	D	${}^2D$
$B_{\mathcal{L}}:$	3	15	22	30	32	32

Table 4.19: Constants in the Fulman–Guralnick bounds for conjugacy class numbers of classical groups.

following estimate:

$$\sum_{q\leq Q} K(q) \;<\; K(Q)+\frac{1}{2}\sum_{i=3}^{Q-1}K(i)+\sum_{i=1}^{[\log_2 Q]}K(2^i).$$

(Here $[x]$ denotes the integral part of x.)

Proof. Firstly, each q can be either odd or a power of two. The binary powers are handled by the second sum on the right hand side.

For odd values of q, we have two possibilities. If Q is even, the integers from 3 to Q can be partitioned into pairs $(2l-1, 2l)$, and we have $K(2l) > K(2l-1)$ for all l, since K is strictly increasing. Thus,

$$\sum_{\substack{q\leq Q\\ q \text{ odd}}} K(q) \leq \sum_{l=2}^{Q/2}K(2l-1) < \frac{1}{2}\sum_{l=2}^{Q/2}\big(K(2l-1)+K(2l)\big) = \frac{1}{2}\sum_{i=3}^{Q}K(i).$$

On the other hand, if Q is odd, we have by the same argument

$$\sum_{\substack{q\leq Q\\ q \text{ odd}}} K(q) < K(Q)+\frac{1}{2}\sum_{i=3}^{Q-1}K(i).$$

In both cases, we see that the desired inequality holds. □

Lemma 4.10. *Keep the notation of the previous lemma. Additionally, assume that $K(q)$ has the form $q^r + Bq^{r-1}$ for some $B > 0$ and $r \geq 2$. For the summation of $K(q)$ over prime powers q, we obtain*

$$\sum_{q\leq Q} K(q) < \frac{1}{2(r+1)}Q^{r+1}+\left(\frac{2^{r+1}-1}{2^r-1}+\frac{B}{2r}\right)Q^r+\frac{(2^r-1)B}{2^{r-1}-1}Q^{r-1}.$$

Proof. The result follows directly from the lemma above. Notice first that

$$\frac{1}{2}\sum_{i=3}^{Q-1}K(i) < \frac{1}{2}\int_0^Q K(x)\,\mathrm{d}x = \frac{Q^{r+1}}{2(r+1)}+\frac{BQ^r}{2r}. \tag{4.11}$$

Also, using the formula for geometric sum, we get

$$\begin{aligned}\sum_{i=1}^{[\log_2 Q]} K(2^i) &= \sum_{i=1}^{[\log_2 Q]} \left(2^{ri} + B \cdot 2^{(r-1)i}\right) \\ &= \frac{2^r\left(2^{[\log_2 Q]r} - 1\right)}{2^r - 1} + \frac{B \cdot 2^{r-1}\left(2^{[\log_2 Q](r-1)} - 1\right)}{2^{r-1} - 1} \\ &< \frac{2^r Q^r}{2^r - 1} + \frac{2^{r-1} B Q^{r-1}}{2^{r-1} - 1}. \end{aligned} \tag{4.12}$$

Putting together (4.11) and (4.12), and adding $K(Q)$ yields the result. □

4.4. Proving the main results on classical groups

Let $\mathcal{L}$ stand for one of the classical families A', ${}^2A$, B, C, D or ${}^2D$, and let $H_r(q)$ denote a universal covering group of type $\mathcal{L}$, having rank r and being defined over a field of size q. We shall use the parametrisations of Lusztig polynomials described in the previous section to compute upper bounds to

$$s_n(\mathcal{L}) = \sum_r \sum_q r_n\big(H_r(q)\big).^1$$

These upper bounds will then furnish the results stated in Theorems 4.1 and 4.2. We start with groups of rank greater than 1, postponing the case $\mathcal{L} = A_1$.

Proof of Theorem 4.2, excluding the case $\mathcal{L} = A_1$. Fix the value of s among $\{1, 2/3, s_{\mathcal{L}}\}$. (The values for $s_{\mathcal{L}}$ were given in Table 4.1.) We strive to find uniform bounds for the ratios

$$Q_n(\mathcal{L}, s) = \frac{1}{n^s} \sum_r \sum_q r_n\big(H_r(q)\big).$$

We begin by giving a description of different kinds of bounds we shall need, and after this deal with each family $\mathcal{L}$ separately. For the general discussion, assume that the universal covering group $H_r(q)$ is of regular type.

Let us first consider the small rank case, where $r < r_1 = r_1(\mathcal{L})$. Denote

$$Q_n^\circ(\mathcal{L}, s) = \frac{1}{n^s} \sum_{r<r_1} \sum_q r_n\big(H_r(q)\big). \tag{4.13}$$

We can use Lübeck's data ([42]) to compute the exact value of $Q_n^\circ(\mathcal{L}, s)$ for $n \le 10^7$.

When $n > 10^7$, we deal with small and large values of q separately. Letting $q_1 = 49$, we can compute the exact values of

$$R_n^1(\mathcal{L}, s) = \frac{1}{n^s} \sum_{r<r_1} \sum_{q<q_1} r_n\big(H_r(q)\big) \tag{4.14}$$

[1] In this section and the next, sums over r and q are to be taken over positive integers and prime powers, respectively.

for all n (they become zero for large enough n). On the other hand, by Lemma 4.7, we know that

$$\frac{1}{n^s}\sum_{r<r_1}\sum_{q\geq q_1} r_n\big(H_r(q)\big) \leq R_n^2(\mathcal{L}, s),$$

where

$$R_n^2(\mathcal{L}, s) = \frac{1}{n^s}\sum_{\nu;\, N_\nu\leq n} h_\nu\big((n/a)^{1/d}+\varepsilon\big) \tag{4.15}$$

with $\nu = (a, d, \varepsilon)$ parametrising classes of Lusztig polynomials.

Now,[1] Q_n° is bounded from above by $R_n^1 + R_n^2$, and for large enough n, by R_n^2 alone. To resolve the ultimate behaviour of Q_n°, we use still another upper bound for R_n^2:

$$\bar{R}_n^2(\mathcal{L}, s) = \frac{1}{n^s}\sum_{\nu} h_\nu\big((n/a)^{1/d}+\varepsilon\big). \tag{4.16}$$

Consider a fixed $\nu = (a, d, \varepsilon)$. Writing m for the degree of the polynomial h_ν, we see that in each summand above, the greatest power of n is m/d. From Lübeck's data, it is quickly verified that $m/d \leq s_{\mathcal{L}} \leq s$. As every h_ν has only non-negative coefficients, this implies that $\bar{R}_n^2$ is decreasing in n. Therefore, $\bar{R}_n^2$ has a limit, and this limit is 0 if $s > s_{\mathcal{L}}$.

For large ranks, we use minimal degree polynomials. We shall deal separately with degrees below and above the gap bound. First, for a universal covering group H of type $\mathcal{L}$, let $r_n^<(H)$ denote the number of irreducible representations of degree n below the gap bound. Then define

$$Q_n^<(\mathcal{L}, s) = \frac{1}{n^s}\sum_{r\geq r_1}\sum_{q} r_n^<\big(H_r(q)\big). \tag{4.17}$$

For $r \geq r_1$, the smallest value of the gap bound is $\Gamma(r_1, q_0(\mu))$. Call this integer n_0. Using the minimal degree polynomials, we can compute the exact values of $Q_n^<$ for $n < n_0$.

When $n \geq n_0$, Lemma 4.8 gives the following upper bound to $Q_n^<$:

$$R_n^<(\mathcal{L}, s) = \frac{1}{n^s}\sum_{\nu}\sum_{\substack{r\geq r_1\\ N_\nu(r)\leq n}} g_\nu\big(q_\nu(r, n)\big). \tag{4.18}$$

Here, $\nu = (a, \varepsilon, \alpha, \beta)$ parametrises classes of minimal degree polynomials. The values of $R_n^<$ can be computed exactly for arbitrarily large n. To bound the eventual behaviour of $R_n^<$, we use the fact that $N_\nu(r) > n$ when $r > r_\nu(n)$. This leads to the following upper bound to $R_n^<(\mathcal{L}, s)$:

$$\bar{R}_n^<(\mathcal{L}, s) = \sum_{\nu}\frac{1}{n^s}(r_\nu(n) - r_1 + 1)\cdot g_\nu\big(q_\nu(r_1, n)\big). \tag{4.19}$$

Note that since the g_ν are linear polynomials, each summand in the above expression has the form

$$\frac{1}{n^s}(\log_b n - A)\left(Bn^{\frac{1}{\alpha r_1+\gamma}} + C\right)$$

[1] Henceforth, we shall often simplify the notation by suppressing the parameters $\mathcal{L}$ and s.

with B and C non-negative and $\alpha r_1 + \gamma > 1/s \geq 1/s_{\mathcal{L}}$. This expression becomes positive and decreasing when $n \geq b^A \exp\big((s - \frac{1}{\alpha r_1+\gamma})^{-1}\big)$, and tends to 0. This means that $\bar{R}_n^<$ is eventually strictly decreasing.

Lastly, let $r_n^>(H)$ stand for the number of irreducible representations of H with degree n above the gap bound. We mean to bound

$$Q_n^>(\mathcal{L}, s) = \frac{1}{n^s} \sum_{r \geq r_1} \sum_q r_n^>\big(H_r(q)\big). \tag{4.20}$$

This is achieved by using the Fulman–Guralnick bounds for conjugacy class numbers. According to these bounds, the number of classes of $H_r(q)$ is at most $q^r + B_{\mathcal{L}} q^{r-1}$, with $B_{\mathcal{L}}$ given in Table 4.19. We write

$$R_n^>(\mathcal{L}, s) = \frac{1}{n^s} \sum_{r \geq r_1} \left(\sum_{q;\, \Gamma(r,q) \leq n} q^r + B_{\mathcal{L}} q^{r-1} \right), \tag{4.21}$$

where Γ is the gap bound. This is clearly an upper bound to $Q_n^>$.

The exact values of $R_n^>$ can be computed for arbitrarily large n. By Lemma 4.10, we have $\sum_q \big(q^r + B_{\mathcal{L}} q^{r-1}\big) < \Lambda_r(n)$, where

$$\Lambda_r(n) = \frac{1}{2(r+1)}\, q_\mu(r,n)^{r+1} + \left(\frac{2^{r+1}-1}{2^r-1} + \frac{B_{\mathcal{L}}}{2r} \right) q_\mu(r,n)^r + \frac{(2^r-1)B_{\mathcal{L}}}{2^{r-1}-1}\, q_\mu(r,n)^{r-1}.$$

Here, $q_\mu(r,n)$ is the upper bound for q presented in (4.10). It has the form $(an)^{1/(br-c)}$ for some positive integers b and c, with $a = 2$ if $\mathcal{L} = C$ and $a = 1$ otherwise (see Table 4.18). Since $b \leq c$ for all classical families, we find every term in $\Lambda_r(n)$ decreasing in r (for any fixed n). This enables us to bound $R_n^>$ from above simply by

$$\bar{R}_n^>(\mathcal{L}, s) = \frac{r_\mu(n) - r_1 + 1}{n^s}\, \Lambda_{r_1}(n). \tag{4.22}$$

Here we also used the fact that $\Gamma_\mu(r,q) > n$ when $r > r_\mu(n)$ and $q > q_\mu(r,n)$.

Note that $r_\mu(n)$ has the form $a \log_b n + c$, and the highest power of n in $\Lambda_{r_1}(n)$ is less than s, except when $\mathcal{L} = {}^2A$ and $s = 2/3$. In all the other situations, we conclude that $\bar{R}_n^>$ will eventually become decreasing. For the one exceptional case, we note that

$$f_\mu(r,q) = (q-1)q^{2r-3} \geq 2^{-1/2} q^{2r-5/2}.$$

From this inequality, we obtain $q_\mu^*(r,n) = (\sqrt{2}n)^{1/(2r-5/2)}$ to be used in place of $q_\mu(r,n)$ as an upper bound to q. This makes $\bar{R}_n^>$ eventually decreasing.

Remark. For $\bar{R}_n^>$ to become eventually decreasing, we need the highest power of n in $\Lambda_{r_1}(n)$ to be less than s. This is why we cannot always choose $s_{\mathcal{L}} = 2/h_{\mathcal{L}}$, which would otherwise be optimal. If we would have complete lists of Lusztig polynomials available for larger ranks, we could take $r_1(\mathcal{L})$ to be bigger, and this would let us make $s_{\mathcal{L}}$ smaller.

We are now in a position to prove the claims of the theorem case by case for each classical family.

Case $\mathcal{L} = A'$. Here, s is either 1 or 2/3, and $r_1 = 9$. Set $n_0 = \Gamma(r_1, 2) = 173228$. For $n < n_0$, we compute the exact values of $Q_n^\circ + Q_n^<$. Also, the exceptional covers $A_2(2)$, $A_2(4)$ and $A_3(2)$ have all representation degrees below n_0. Using the data given in the Atlas of Finite Groups ([6]), we can compute $Q_n^e(s) = \sum_H r_n(H)/n^s$ for the exceptional covers H. Now, we have $Q_n = Q_n^\circ + Q_n^< + Q_n^e$ when $n < n_0$, and the maximum of $Q_n(s)$ for $n < n_0$ equals 7/8 for $s = 1$, and 7/4 for $s = 2/3$ (the maximum is obtained at $n = 8$ in both cases). For ease of reference, these maxima are listed below the proof in Table 4.20 on page 110.

When $n \geq n_0$, we use the upper bounds derived above. Values of R_n^1 and R_n^2 are handled by the computer. There is only one class of minimal degree polynomials, so

$$R_n^<(s) = \sum_{r=9}^{\infty}{}' \frac{1}{n^{s-1/r}} \quad \text{and} \quad \bar{R}_n^<(s) = (\log_2 n - 8)\frac{1}{n^{s-1/9}}.$$

In the dashed ($'$) sum we add the restriction $n \geq N_\nu(r) = 2^{r+1} - 2$.

On the other hand, above the gap bound we have

$$R_n^>(s) = \frac{1}{n^s}\sum_{r=9}^{\infty}\sum_{q}{}'(q^r + Bq^{r-1}),$$

where $B = 3$, and where we add the restriction $n \geq \Gamma(r,q) = \frac{(q^{r+1}-1)(q^r-q^2)}{(q-1)(q^2-1)}$. Moreover,

$$\bar{R}_n^>(s) = \left(\frac{1}{2}\log_2 n - 7\right)\left(\frac{1}{20n^{s-5/8}} + \frac{6649}{3066n^{s-9/16}} + \frac{511}{85n^{s-1/2}}\right).$$

Set $n_1 = 10^7$ and $n_2 = 2 \cdot 10^{60}$. We define the function F_s as follows:

$$F_s(n) = \begin{cases} Q_n^\circ(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \leq n \leq n_1 \\ R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_1 < n < n_2 \\ \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \geq n_2. \end{cases}$$

From the earlier discussion, it is clear that $F_s(n)$ is an upper bound to $Q_n(s)$ for all $n \geq n_0$. (For $n \geq n_2$, we have $R_n^1 = 0$.) Direct computation shows that the maximum of $F_s(n)$ for $n_0 \leq n < n_2$ is less than 0.007517 when $s = 1$ and 1.17409 when $s = 2/3$. On the other hand, $F_1(n_2) < 10^{-21}$ and $F_{2/3}(n_2) < 0.1$, and when $n \geq n_2$, some simple calculus tells us that $F_s(n)$ is descending for both s.

Let $c_{A',s}$ be the maximal value of $Q_n(s)$ for $n < n_0$ obtained above. As $c_{A',s}$ is greater than the maximum of $F_s(n)$ for $n \geq n_0$ for both values of s, we can conclude that it bounds $Q_n(s)$ from above for all n. This means that $s_n(A') \leq c_{A',s}n^s$, which proves the claim of the theorem in the case $\mathcal{L} = A'$.

Case $\mathcal{L} = {}^2A$. Again, s is either 1 or 2/3, and $r_1 = 9$. Using the same notation as in the previous case, we set $n_0 = \Gamma(r_1, 2) = 57970$, $n_1 = 10^7$ and $n_2 = 2 \cdot 10^{60}$.

For $n < n_0$, we compute the exact values of $Q_n^\circ + Q_n^<$. The exceptional covers ${}^2A_3(2)$, ${}^2A_3(3)$ and ${}^2A_5(2)$ have all their representation degrees below n_0. With the help of the Atlas, we can compute $Q_n^e(s) = \sum_H r_n(H)/n^s$ for the exceptional covers H. Now, $Q_n = Q_n^\circ + Q_n^< + Q_n^e$ when $n < n_0$, and the maximum of $Q_n(s)$ in this range equals 2/3 for $s = 1$ (at $n = 6$), and $13/21^{2/3} \approx 1.7080$ for $s = 2/3$ (at $n = 21$). These maxima are listed in Table 4.20 below the proof.

Suppose then that $n \geq n_0$. Values of R_n^1 and R_n^2 are handled by the computer. We have

$$R_n^<(s) = {\sum_{r=9}^{\infty}}' \frac{1}{n^{s-1/(r-1)}} \quad \text{and} \quad \bar{R}_n^<(s) = (\log_2 n - 7)\frac{1}{n^{s-1/8}}.$$

In the dashed ($'$) sum we add the restriction $n \geq N_\nu(r) = (2^{r+1} - \gcd(r,2))/3$.

On the other hand, above the gap bound we have $R_n^>$ similar to the one in the previous case, with $B = 15$, and if $s = 1$, we get

$$\bar{R}_n^>(s) = \frac{1}{2}(\log_2 n - 13)\left(\frac{1}{20n^{s-2/3}} + \frac{8693}{3066n^{s-3/5}} + \frac{511}{17n^{s-8/15}}\right).$$

For $s = 2/3$, this expression is not eventually decreasing, so we need to make a modification to $q_\mu(r,n)$ according to the discussion below equation (4.22). This leads to

$$\bar{R}_n^>(2/3) = \frac{1}{2}(\log_2 n - 13)\left(\frac{2^{10/31}}{20n^{2/93}} + \frac{8693 \cdot 2^{9/31}}{3066n^{8/93}} + \frac{511 \cdot 2^{8/31}}{17n^{14/93}}\right).$$

The function F_s is defined as follows:

$$F_s(n) = \begin{cases} Q_n^\circ(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \leq n \leq n_1 \\ R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_1 < n < n_2 \\ \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \geq n_2. \end{cases}$$

Now, $F_s(n)$ is an upper bound to $Q_n(s)$ for all $n \geq n_0$. (For $n \geq n_2$, we have $R_n^1 = 0$.) Computation shows that the maximum of $F_s(n)$ for $n_0 \leq n < n_2$ is less than 0.075455 when $s = 1$ and 3.46356 when $s = 2/3$. On the other hand, $F_1(n_2) < 10^{-19}$ and $F_{2/3}(n_2) < 0.3$, and when $n \geq n_2$, it can be seen that $F_s(n)$ is descending for both s.

Let $c_{^2A,1}$ be the maximum of $Q_n(1)$ for $n < n_0$ obtained above, and let $c_{^2A,2/3}$ be the maximum of $F_{2/3}(n)$ for $n_0 \leq n < n_2$. This way we see that, for both s, the value of $c_{^2A,s}$ is an upper bound to $Q_n(s)$ for all n. This proves the claim in the case $\mathcal{L} = {}^2A$.

Case $\mathcal{L} = B$. Here, s is one of 1, 2/3 and 1/2, and $r_1 = 9$. Minimal rank is 3 and only odd q are taken into account. We set $n_0 = \Gamma(r_1, 3) = 22876792454961$ and $n_2 = 10^{20}$. For $n < n_0$, we compute the exact values of $Q_n^\circ + Q_n^<$. The exceptional cover $B_3(3)$ has all its representation degrees below n_0. We compute the values of $Q_n^e(B,s) = r_n(B_3(3))/n^s$. For $n < n_0$, the maximum of $Q_n(s) = Q_n^\circ(s) + Q_n^<(s) + Q_n^e(s)$ equals 2/27 for $s = 1$ (at $n = 27$), 2/9 for $s = 2/3$ (at $n = 27$), and $2/(3\sqrt{3}) \approx 0.38491$ for $s = 1/2$ (at $n = 27$). These maxima are listed in Table 4.20 below the proof.

Suppose then that $n \geq n_0$. Values of R_n^1 and R_n^2 are handled by the computer. There are 5 classes of minimal degree polynomials, parametrised by ν. Thus, $R_n^<(s)$ is a sum of five terms of the form $\sum_r' \Psi_\nu(s,n,r)$, where each sum is taken over those r for which $n \geq N_\nu(r)$. The $N_\nu(r)$ can be found in Table 4.13, and the $\Psi_\nu(s,n,r)$ are listed below:

$$\frac{1}{n^s}, \quad \frac{1}{n^s}, \quad \frac{1}{n^s}, \quad \frac{1}{2(n/2)^{s-1/(2r-2)}}, \quad \frac{1}{2n^{s-1/(2r-1)}}.$$

Furthermore, $\bar{R}_n^<(s)$ is the following sum:

$$\begin{aligned}(\log_3 n - 14)\frac{1}{2n^s} + (\log_3 n - 14)\frac{1}{2n^s} + (\log_3(2n) - 15)\frac{1}{2n^s} \\ + (\log_3(n/2) - 14)\frac{1}{4(n/2)^{s-1/16}} + (\log_3 n - 15)\frac{1}{4n^{s-1/17}}.\end{aligned}$$

Above the gap bound, we have $B = 22$ and

$$\bar{R}_n^>(s) = \left(\frac{1}{4}\log_3 n - 6\right)\left(\frac{1}{20n^{s-5/14}} + \frac{14828}{4599n^{s-9/28}} + \frac{11242}{255n^{s-2/7}}\right).$$

The function F_s is defined as follows:

$$F_s(n) = \begin{cases} R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \leq n < n_2 \\ \bar{R}^1(s) + \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \geq n_2. \end{cases}$$

Here, $\bar{R}^1(s)$ is the maximum of $R_n^1(s)$ for $n > n_2$. (Note that $R_n^1(s)$ becomes zero eventually.) For $n \geq n_0$, $F_s(n)$ is an upper bound to $Q_n(s)$. Computation shows that the maximum of $F_s(n)$ for $n_0 \leq n < n_2$ is less than $7.2102 \cdot 10^{-9}$ when $s = 1$, less than $2.0469 \cdot 10^{-4}$ when $s = 2/3$ and less than 0.034486 when $s = 1/2$. On the other hand, $F_1(n_2) < 10^{-11}$, $F_{2/3}(n_2) < 10^{-5}$ and $F_{1/2}(n_2) < 0.02$, and when $n \geq n_2$, we see that $F_s(n)$ is descending for each s.

Let $c_{B,s}$ be the maximum of $Q_n(s)$ for $n < n_0$ obtained above. For each s, the value of $c_{B,s}$ is greater than the maximum of $F_s(n)$ for $n_0 \leq n < n_2$. Thus $c_{B,s}$ is an upper bound to $Q_n(s)$ for all n and s. This proves the claim in the case $\mathcal{L} = B$.

Case $\mathcal{L} = C$. Here, $s \in \{1, 2/3, 1/2\}$ and $r_1 = 9$. We set $n_0 = \Gamma(r_1, 2) = 352283520$ and $n_2 = 10^{20}$. For $n < n_0$, we compute the exact values of $Q_n^\circ + Q_n^<$. The exceptional covers $C_2(2)$ and $C_3(2)$ have all their representation degrees below n_0. We compute $Q_n^e(s) = \sum_H r_n(H)/n^s$ for the exceptional covers H. For $n < n_0$, the maximum of $Q_n(s) = Q_n^\circ(s) + Q_n^<(s) + Q_n^e(s)$ equals $1/2$ for $s = 1$ (at $n = 4$), $6/20^{2/3} \approx 0.81433$ for $s = 2/3$ (at $n = 20$), and $3/\sqrt{5} \approx 1.34165$ for $s = 1/2$ (at $n = 20$). These maxima are listed in Table 4.20 below the proof.

Suppose then that $n \geq n_0$. Values of R_n^1 and R_n^2 are handled by the computer. There are 9 classes of minimal degree polynomials, and $R_n^<$ and $\bar{R}_n^<$ are obtained as in the previous case. Above the gap bound we have $B = 30$, and

$$\bar{R}_n^>(s) = \left(\frac{1}{4}\log_2 n - 6\right)\left(\frac{2^{10/29}}{20n^{s-10/29}} + \frac{5624 \cdot 2^{9/29}}{1533n^{s-9/29}} + \frac{1022 \cdot 2^{8/29}}{17n^{s-8/29}}\right).$$

The function F_s is defined as follows:

$$F_s(n) = \begin{cases} R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \le n < n_2 \\ \bar{R}^1(s) + \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \ge n_2. \end{cases}$$

As in the previous case, $\bar{R}^1(s)$ is just the maximum of $R_n^1(s)$ for $n \ge n_2$. For all $n \ge n_0$, $F_s(n)$ is an upper bound to $Q_n(s)$. Computation shows that the maximum of $F_s(n)$ for $n_0 \le n < n_2$ is less than $6.2765 \cdot 10^{-5}$ when $s = 1$, less than 0.044328 when $s = 2/3$ and less than 1.17804 when $s = 1/2$. On the other hand, $F_1(n_2) < 10^{-11}$, $F_{2/3}(n_2) < 10^{-4}$ and $F_{1/2}(n_2) < 0.1$, and when $n \ge n_2$, we see that $F_s(n)$ is descending for each s.

Let $c_{C,s}$ be the maximum of $Q_n(s)$ for $n \le n_0$ obtained above. For each s, the value of $c_{C,s}$ is greater than the maximum of $F_s(n)$ for $n_0 \le n < n_2$. Thus $c_{C,s}$ is an upper bound to $Q_n(s)$ for all n and s. This proves the claim in the case $\mathcal{L} = C$.

Case $\mathcal{L} = D$. Here, s is one of 1, 2/3 and 1/2, and $r_1 = 8$. Minimal rank is 4. We set $n_0 = \Gamma(r_1, 2) = 4194305$, $n_1 = 10^7$ and $n_2 = 10^{20}$. For $n < n_0$, we compute the exact values of $Q_n^\circ + Q_n^<$. The exceptional cover $D_4(2)$ has all representation degrees below n_0. We compute the values of $Q_n^e(s) = r_n(D_4(2))/n^s$. For $n < n_0$, the maximum of $Q_n(s) = Q_n^\circ(s) + Q_n^<(s) + Q_n^e(s)$ equals $1/8$ for $s = 1$ (at $n = 8$), $3/35^{2/3} \approx 0.28038$ for $s = 2/3$ (at $n = 35$), and $3/\sqrt{35} \approx 0.5071$ for $s = 1/2$ (at $n = 35$). These maxima are listed in Table 4.20 below the proof.

Suppose then that $n \ge n_0$. Values of R_n^1 and R_n^2 are handled by the computer. There are 5 classes of minimal degree polynomials, and $R_n^<$ and $\bar{R}_n^<$ are obtained as before. Above the gap bound we have $B = 32$, and

$$\bar{R}_n^>(s) = \frac{1}{4}(\log_2 n - 18)\left(\frac{1}{18n^{s-9/22}} + \frac{1021}{255n^{s-4/11}} + \frac{8160}{127n^{s-7/22}}\right).$$

The function F_s is defined as follows:

$$F_s(n) = \begin{cases} Q_n^\circ(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \le n \le n_1 \\ R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_1 < n < n_2 \\ \bar{R}^1(s) + \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \ge n_2. \end{cases}$$

As before, $\bar{R}^1(s)$ is the maximum of $R_n^1(s)$ for $n \ge n_2$. Now, $F_s(n)$ is an upper bound to $Q_n(s)$ for $n \ge n_0$. Computation shows that the maximum of $F_s(n)$ for $n_0 \le n < n_2$ is less than 0.0010420 when $s = 1$, less than 0.16805 when $s = 2/3$ and less than 2.13399 when $s = 1/2$. On the other hand, $F_1(n_2) < 10^{-10}$, $F_{2/3}(n_2) < 0.001$ and $F_{1/2}(n_2) < 1$, and when $n \ge n_2$, we see that $F_s(n)$ is descending for each s.

For $s \in \{1, 2/3\}$, let $c_{D,s}$ be the maximum of $Q_n(s)$ for $n < n_0$ obtained above, and let $c_{D,1/2}$ be the maximum of $F_{1/2}(n)$ for $n_0 \le n < n_2$. This way we see that, for each s, the value of $c_{D,s}$ is an upper bound to $Q_n(s)$ for all n. This proves the claim in the case $\mathcal{L} = D$.

Case $\mathcal{L} = {}^2D$. Here, s is one of 1, 2/3 and 1/2, and $r_1 = 8$. Minimal rank is 4. We set $n_0 = \Gamma(r_1, 2) = 4194305$, $n_1 = 10^7$ and $n_2 = 2 \cdot 10^{70}$. For $n < n_0$, we compute the exact values of $Q_n^\circ + Q_n^<$. There are no exceptional covers. For $n < n_0$, the maximum of $Q_n(s) = Q_n^\circ(s) + Q_n^<(s)$ equals $1/34$ for $s = 1$, $1/34^{2/3} \approx 0.095283$ for $s = 2/3$, and $1/\sqrt{34} \approx 0.1715$ for $s = 1/2$ (all reached at $n = 34$). There maxima are listed in Table 4.20 below the proof.

Suppose then that $n \geq n_0$. Values of R_n^1 and R_n^2 are handled by the computer. There are 5 classes of minimal degree polynomials, and $R_n^<$ and $\bar{R}_n^<$ are obtained as before. Above the gap bound we have $B = 32$, and $\bar{R}_n^>$ is the same as in the previous case.

The function F_s is defined as follows:

$$F_s(n) = \begin{cases} Q_n^\circ(s) + R_n^<(s) + R_n^>(s) & \text{when } n_0 \leq n \leq n_1 \\ R_n^1(s) + R_n^2(s) + R_n^<(s) + R_n^>(s) & \text{when } n_1 < n < n_2 \\ \bar{R}^1(s) + \bar{R}_n^2(s) + \bar{R}_n^<(s) + \bar{R}_n^>(s) & \text{when } n \geq n_2. \end{cases}$$

As before, $\bar{R}^1(s)$ is the maximum of $R_n^1(s)$ for $n \geq n_2$. Now, $F_s(n)$ is an upper bound to $Q_n(s)$ for $n \geq n_0$. Computation shows that the maximum of $F_s(n)$ for $n_0 \leq n < n_2$ is less than 0.0010420 when $s = 1$, less than 0.16805 when $s = 2/3$ and less than 2.13399 when $s = 1/2$. On the other hand, $F_1(n_2) < 10^{-10}$, $F_{2/3}(n_2) < 0.001$ and $F_{1/2}(n_2) < 1$, and when $n \geq n_2$, we see that $F_s(n)$ is descending for each s.

Let $c_{^2D,1}$ be the maximum of $Q_n(1)$ for $n < n_0$ obtained above, and for $s \in \{2/3, 1/2\}$, let $c_{^2D,s}$ be the maximum of $F_s(n)$ for $n_0 \leq n < n_2$. This way we see that, for each s, the value of $c_{^2D,s}$ is an upper bound to $Q_n(s)$ for all n. This proves the claim in the final case $\mathcal{L} = {}^2D$. □

s	A'	^2A	B	C	D	^2D
1	7/8	2/3	2/27	1/2	1/8	1/34
2/3	7/4	1.7080	2/9	0.8144	0.2804	0.09529
$s_\mathcal{L}$	7/4	1.7080	0.3850	1.3417	0.5071	0.1715

Table 4.20: Maximal values of $Q_n(\mathcal{L}, s)$ for small n. (For details, see the case-by-case analysis in the second half of the proof of Theorem 4.2 above.)

It still remains to check the case of linear groups of rank one.

Proof of Theorem 4.1. For fields of size 2 and 3, there are no simple groups of type A_1. Simple groups $\mathrm{SL}_2(4)$ and $\mathrm{PSL}_2(9)$ have exceptional covering groups, but we will not consider $\mathrm{SL}_2(4)$ separately, as it is isomorphic to $\mathrm{PSL}_2(5)$. The character degrees and multiplicities of $A_1(9)$ are listed in the Atlas, and reproduced below in Table 4.21.

For all other finite fields q, the universal covering group is $\mathrm{SL}_2(q)$. For these groups, the generic formulae for complex character degrees and their multiplicities are known, and we give them in Table 4.22.

With this information, it is easy to compute $s_n(A_1)$ for n as large as we wish. We find that for $n \leq 12$, the largest value of $s_n(A_1)/n$ is $8/3$, and this is obtained only at $n = 3$

degree	1	3	4	5	6	8	9	10	12	15
multiplicity	1	4	2	2	6	4	3	3	4	2

Table 4.21: Character degrees of the exceptional covering group $A_1(9)$.

q even		q odd	
degree	multiplicity	degree	multiplicity
$q-1$	$q/2$	$q-1$	$(q-1)/2$
q	1	q	3
$q+1$	$(q-2)/2$	$q+1$	$(q-3)/2$
		$(q-1)/2$	2
		$(q+1)/2$	2

Table 4.22: Non-trivial character degrees of $\mathrm{SL}_2(q)$.

(the characters come from $A_1(5)$, $A_1(7)$ and $A_1(9)$). Similarly, the second largest value $7/3$ is obtained only at $n = 6$, and the third largest $3/2$ only at $n = 4$ and $n = 12$.

The largest character degree of $A_1(9)$ is 15, so after this, all degrees come from the characters of $\mathrm{SL}_2(q)$. We can infer from the character table that for any n, there are at most $n + 3$ characters of degree n. (The maximal number is reached only if $n - 1$, n, $n+1$, $2n-1$ and $2n+1$ are all prime powers.) We also see that $s_{13}(A_1) = 5$, $s_{14}(A_1) = 9$ and $s_{15}(A_1) = 14$, so that $s_n(A_1) < n + 3$ for $n \in \{13, 14, 15\}$. Finally, when $n > 15$, we have $(n + 3)/n < 8/3$, so the upper bound for $n \leq 12$ holds globally. This proves Theorem 4.1, and also completes the proof of Theorem 4.2. □

4.5. Proving the main result on exceptional groups

In this section, we will prove Theorem 4.3. Let $\mathcal{E}$ denote the class of finite quasisimple groups of exceptional Lie type. The generic character degrees of these groups are all covered by Lübeck's data ([42, 45]). We proceed as follows:

1. Compute the exact values of $s_n(\mathcal{E})$ up to $n = 10^7$ using Lübeck's data and the ATLAS of Finite Groups.
2. Arrange the Lusztig polynomials of $\mathcal{E}$ into classes parametrised by $\nu = (a, d, \varepsilon)$, as was done in the case of classical groups, and use this parametrisation to compute bounds for $s_n(\mathcal{E})$ when $n > 10^7$.

With classical types, we dealt with each Lie type separately, taking sums over ranks. Now we sum over all exceptional types, the set of which we denote Exc.

As in the classical case, we let $q_1 = 49$ and deal with groups of smaller field size separately. With $q \geq q_1$, the classification of the Lusztig polynomials works out the same way as with classical groups. However, instead of dealing with each type separately, we collect together all degree polynomials. For class ν, the final multiplicity polynomial h_ν will then be the sum of maximal multiplicities from each Lie type, again dropping the negative terms. The procedure is analogous to the one in the classical case if we imagine all exceptional types to have the same Dynkin letter but different rank.

Note, however, that with the so-called Suzuki and Ree types ${}^2B_2$, ${}^2F_4$ and ${}^2G_2$, the indeterminate q in Lübeck's polynomial data denotes the square root of the field size, and hence is not an integer. To mend this, we recompute the polynomials for these types, using as new indeterminate $\hat{q} = q/\sqrt{2}$ for ${}^2B_2$ and ${}^2F_4$, and $\hat{q} = q/\sqrt{3}$ for ${}^2G_2$. Now, $\hat{q}$ always assumes integral values. For example, to compute the character degrees of the group ${}^2B_2(8)$, we need to substitute $\hat{q} = \sqrt{8}/\sqrt{2} = 2$ into the new polynomials.

Using these new polynomials in the classification everything works as before, and we have the following lemma corresponding to Lemma 4.7.

Lemma 4.11. *Write $H(q)$ for a regular universal covering group of an exceptional Lie type H. With the notation analogous to Lemma 4.7, we have*

$$\sum_{H\in\mathrm{Exc}}\sum_{q\geq q_1} r_n\big(H(q)\big) \;\leq\; \sum_{\nu;\,N_\nu\leq n} h_\nu\big((n/a)^{1/d}+\varepsilon\big)$$

for all $n > 1$.

We can now proceed directly to proving the main theorem.

Proof of Theorem 4.3. Let us first deal with the exceptional covering groups. The character degrees of $F_4(2)$, $G_2(2)$, $G_2(3)$, $G_2(4)$, ${}^2B_2(8)$ and ${}^2F_4(2)$ can be found in the Atlas. On the other hand, the Schur multiplier of the simple group of type ${}^2E_6(2)$ is an abelian group of type $2^2\cdot 3$. Every character of the universal covering group has a central subgroup of order 2 in its kernel, and all three factor groups are isomorphic via an outer automorphism. (See [45, Chapter 3.3].) We denote the simple group by ${}^2\breve{E}_6(2)$, and the factor group by $6.{}^2\breve{E}_6(2)$.

From the above, we know that all character degrees of the universal covering group ${}^2E_6(2)$ appear also as degrees of $6.{}^2\breve{E}_6(2)$ (with the same multiplicities). Lübeck has computed the character table of $6.{}^2\breve{E}_6(2)$, and the data is available on his website [43]. From the data we see that the three smallest non-trivial character degrees of $6.{}^2\breve{E}_6(2)$ are 1938, 2432 and 45696, each appearing with multiplicity 1. As the conjugacy class number of $6.{}^2\breve{E}_6(2)$ is 542, we can make the following estimate for $n \geq 45696$:

$$r_n\big({}^2E_6(2)\big) \leq 2\cdot 542 - 3 = 1081.$$

For s in $\{1,\ 2/3,\ 1/2\}$, we are interested in upper bounds of

$$Q_n(s) = \frac{1}{n^s}\sum_{H\in\mathrm{Exc}}\sum_{q} r_n\big(H(q)\big). \tag{4.23}$$

When $n < n_0 = 45696$, we can use Lübeck's polynomial data together with the Atlas and the discussion above to compute the exact values of $Q_n(s)$. The maximum is found to be 1 for $s = 1$, $7/7^{2/3} \approx 1.91294$ for $s = 2/3$ and $7/7^{1/2} \approx 2.64576$ for $s = 1/2$. (There are three 7-dimensional representations of $G_2(2)$ and four of ${}^2G_2(3)$.)

Let $n_1 = 10^7$ and write $\mathrm{Exc}' = \mathrm{Exc} \setminus \{{}^2E_6\}$. For $n_0 < n \leq n_1$, we can compute the values of

$$\bar{Q}_n(s) = \frac{1}{n^s}\left(\sum_{H\in\mathrm{Exc}'}\sum_{q} r_n\big(H(q)\big) + \sum_{q>2} r_n\big({}^2E_6(q)\big)\right). \tag{4.24}$$

By the above discussion, $\bar{Q}_n(s) + 1081/n$ is an upper bound to $Q_n(s)$ in this range.

For $n > n_1$, we examine small and large values of q separately. Set $n_2 = 10^{20}$. We can compute the exact values of

$$R_n^1(s) = \frac{1}{n^s}\left(\sum_{H\in\mathrm{Exc}'}\sum_{q<q_1} r_n\big(H(q)\big) + \sum_{2<q<q_1} r_n\big({}^2E_6(q)\big)\right). \tag{4.25}$$

for all $n < n_2$. We know that $R_n^1(s)$ will eventually become zero, so we also compute

$$\bar{R}^1(s) = \max_{n>n_2} R_n^1(s). \tag{4.26}$$

For large q, define

$$R_n^2(s) = \frac{1}{n^s}\sum_{\nu;\, N_\nu\leq n} h_\nu\big((n/a)^{1/d} + \varepsilon\big), \tag{4.27}$$

where the notation is as in Lemma 4.11. By that lemma, we know that

$$\frac{1}{n^s}\sum_{H\in\mathrm{Exc}}\sum_{q\geq q_1} r_n\big(H(q)\big) \leq R_n^2(s)$$

for all n. The values of $R_n^2(s)$ can be computed for any specific n. For the ultimate behaviour, we use the following upper bound for $R_n^2(s)$:

$$\bar{R}_n^2(s) = \frac{1}{n^s}\sum_{\nu} h_\nu\big((n/a)^{1/d} + \varepsilon\big). \tag{4.28}$$

As in the classical case, $\bar{R}_n^2(s)$ is decreasing in n. Namely, the exponent n in each term $h_\nu((n/a)^{1/d} + \varepsilon)$ is at most $1/2$ which is the smallest value of s considered. (As a matter of fact, for some terms coming from groups of type ${}^2B_2$ the exponent is exactly $1/2$, so we could not do better than $s = 1/2$ here.)

Finally, we define the function F as follows:

$$F_s(n) = \begin{cases} \bar{Q}_n(s) + 1081/n & \text{when } n_0 \leq n \leq n_1 \\ R_n^1(s) + R_n^2(s) + 1081/n & \text{when } n_1 < n < n_2 \\ \bar{R}^1(s) + \bar{R}_n^2(s) + 1081/n & \text{when } n \geq n_2. \end{cases}$$

Now, $F_s(n)$ is an upper bound to $Q_n(s)$ for $n \geq n_0$. Computation shows that the maximum of $F_s(n)$ for $n_0 \leq n < n_2$ is less than 0.023657 for $s = 1$, less than 0.84575 for $s = 2/3$ and less than 5.05693 for $s = 1/2$. On the other hand, $F_1(n_2) < 10^{-10}$, $F_{2/3}(n_2) < 0.001$ and $F_{1/2}(n_2) < 1$, and when $n \geq n_2$, we see that $F_s(n)$ is descending for each s.

For $s \in \{1, 2/3\}$, let $c_{\mathcal{E},s}$ be the maximum of $Q_n(s)$ for $n < n_0$ obtained above, and let $c_{\mathcal{E},1/2}$ be the maximum of $F_{1/2}(n)$ for $n_0 \leq n < n_2$. Now, for each s, the value of $c_{\mathcal{E},s}$ is an upper bound to $Q_n(s)$ for all n. It follows that $s_n(\mathcal{E}) \leq c_{\mathcal{E},s} n^s$, which proves the claim. $\square$

5. Cross-characteristic representation growth of finite quasisimple groups of Lie type

5.1. Statement of results

In this chapter, we investigate the same groups of Lie type as in Chapter 4, but this time we are counting modular representations instead of complex ones. This means that the representation space will be over a field of positive characteristic ℓ. Moreover, we will only look at such groups of Lie type that are defined over a field of characteristic different to ℓ. This is what is meant by *cross-characteristic* representations. The results of this chapter will be used in Chapter 6 to bound the number of conjugacy classes of maximal subgroups in finite classical groups.

Let us first fix our notation. It is similar to the notation used in the previous chapter, but this time we need to include the characteristic of the representation space. To this end, assume that G is a finite quasisimple group whose simple quotient is a group of Lie type defined over a field of characteristic p. Define then $r_n(G, \ell)$ to be the number of inequivalent irreducible n-dimensional representations of G over the algebraic closure of the finite prime field of characteristic $\ell \neq p$. Also, write $r_n^f(G, \ell)$ for the number of such said representations that are in addition faithful. If $\mathcal{L}$ is a family of finite quasisimple groups of Lie type, denote

$$s_n(\mathcal{L}, \ell) = \sum_{G \in \mathcal{L}} r_n^f(G, \ell).$$

As in the previous chapter, we will present upper bounds for the growth of $s_n(\mathcal{L}, \ell)$ for different families of groups of Lie type. This time, however, we only strive to get linear bounds. The bounds will have no dependence on ℓ.

Regarding classical groups, we concern ourselves with the same families of quasisimple groups as before:

A_1 : linear groups in dimension 2
A' : linear groups in dimension at least 3
${}^2A$: unitary groups in dimensions at least 3
B : orthogonal groups in odd dimension ≥ 7 over a field of odd size
C : symplectic groups in dimension at least 4
D : orthogonal groups of plus type in even dimension ≥ 8
${}^2D$: orthogonal groups of minus type in even dimension ≥ 8.

Theorem 5.1. *For any prime ℓ, we have*

$$s_n(A_1, \ell) \leq \tfrac{8}{3}n \qquad \text{for } n > 1,$$

and

$$s_n(A_1, \ell) \leq n + 3 \qquad \text{for } n > 12.$$

Remark. This theorem is almost identical to Theorem 4.1, and the proof is likewise similar. We know the constant in the first bound to be smallest possible, but the tightness of the second one depends on a number-theoretical problem.

Theorem 5.2. *Let $\mathcal{L}$ denote one of the classical families of finite quasisimple groups mentioned above. For all $n > 1$ and for any prime ℓ, we have*

$$s_n(\mathcal{L}, \ell) \leq c_{\mathcal{L}}\, n,$$

where the constants $c_{\mathcal{L}}$ are as shown in Table 5.1.

$\mathcal{L}$:	A_1	A'	^2A	B	C	D	^2D
$c_{\mathcal{L}}$:	8/3	1.5484	2.8783	0.9859	2.8750	1.5135	1.7969

Table 5.1: Bounding constants for classical groups.

Bounds in the last theorem are probably far from optimal (except for A_1), as can be judged from the known maximal values of $s_n(\mathcal{L}, \ell)/n$ for $n \leq 250$ (see Table 5.7 on page 123). Also, it should be possible to obtain sublinear bounds for the growth in most cases, as was done with complex representations, but we have not performed the necessary computations.

The next theorem deals with the exceptional types. Let $\mathcal{E}$ denote the class of finite quasisimple groups with simple quotient an exceptional group of Lie type, excluding the groups $G_2(2)'$ and $^2G_2(3)'$. (The groups are excluded because they are isomorphic to $\mathrm{SU}_3(3)$ and $\mathrm{SL}_2(8)$, respectively, and we do not wish to count them twice in Corollary 5.4. Notice that this differs from our approach in Theorem 4.3 with complex representations.)

Theorem 5.3. *For all $n > 1$ and for any prime ℓ, we have*

$$s_n(\mathcal{E}, \ell) < 1.2795n.$$

It is straightforward to add together all the constants pertaining to different families to get a universal bound for the cross-characteristic representation growth of quasisimple groups of Lie type.

Corollary 5.4. *Let $\mathcal{H}$ denote the class of all finite quasisimple groups of Lie type. For all $n > 1$ and for any prime ℓ, we have*

$$s_n(\mathcal{H}, \ell) < 15.6n.$$

The results are based on explicit information on representation degrees below 250, provided by G. Hiss and G. Malle in [21] and [23], together with generic forms of minimal degrees, called *gap results*, for which there is a good survey article [59] by P. H. Tiep. From time to time, we will refer to the Atlas of Finite Groups ([6]) and to the Atlas of Brauer Characters (so-called "Modular Atlas", [30]). We shall also need some information on conjugacy class numbers, provided by F. Lübeck in his online data ([42]) and by J. Fulman and R. Guralnick in [11].

With the universal covering groups, we use the same notation as in the previous chapter. Namely, we shall write $H_r(q)$ for the universal covering group of a simple group of Lie type H having rank r and being defined over a field of size q. For example, $A_2(3)$ is the group $\mathrm{SL}_3(3)$, and $E_6(4)$ is the triple cover of the finite simple group of Lie type E_6 over the field of four elements. Notice that this notation differs from the one used in Section 2.2, and also from the one used in the Atlases.

As explained in the end of Section 4.4, the universal covering group of a simple group of Lie type is a fixed point group of a simply-connected simple algebraic group, apart from finitely many exceptions. As in the previous chapter, the simply-connected covering groups will be called *regular*. We list again the exceptional covers: $A_1(4)$, $A_1(9)$, $A_2(2)$, $A_2(4)$, $A_3(2)$, ${}^2A_3(2)$, ${}^2A_3(3)$, ${}^2A_5(2)$, $B_3(3)$, $C_2(2)$, $C_3(2)$, $D_4(2)$, ${}^2E_6(2)$, $F_4(2)$, $G_2(2)$, $G_2(3)$, $G_2(4)$, ${}^2B_2(8)$ and ${}^2F_4(2)$ (the Tits group).

The main results for classical families (Theorems 5.1 and 5.2) are proved in Section 5.3, and for exceptional groups (Theorem 5.3) in Section 5.4. The general methods are similar to the ones used in the previous chapter. However, compared with complex representations, we have now much less precise information available on representation degrees, mostly owing to the lack of an analogue to Deligne–Lusztig methods in modular representation theory. To get results even as good as the ones presented, we need to work a little harder in bounding the different sums that appear in the proofs.

Before the proofs, we have a look at the gap results and prove some lemmata that will become useful later on.

5.2. On gap results

For this section, write ℓ for the characteristic of the representation space, and let $H_r(q)$ be a universal covering group of a classical simple group, with the characteristic of the defining field different from ℓ. We will suppose, unless otherwise mentioned, that $H_r(q)$ is not an exceptional cover but of simply-connected type. For convenience of the general treatment, we shall also assume that $H_r(q)$ is not $A_2(3)$, $A_2(5)$, $A_3(3)$, $A_5(2)$, $A_5(3)$, ${}^2A_2(3)$, ${}^2A_2(4)$, ${}^2A_2(5)$, $C_2(3)$ or ${}^2D_4(2)$.

To get upper bounds for those ranks and field sizes that may yield representations of dimension n, we need to bound the possible representation degrees of $H_r(q)$ from below. Landazuri and Seitz found lower bounds for the representation dimensions in [34], and the bounds were subsequently improved by Seitz and Zalesskii [57]. These bounds were given for each Lie type as functions of rank and the size of the defining field. The bounds have later been slightly improved by various authors.

As with complex representations, it is generally the case that the group $H_r(q)$ has a few representations of the smallest degree n_1, maybe a few also of the degrees $n_1 + 1$ and $n_1 + 2$, after which there is a relatively large *gap* before the next degrees. Then, after a couple of degrees, there is again a gap before the next one, and so on. The size of the first gap is known for linear, unitary and symplectic groups, and the most recent results can be found in [18], [22], [17] and [19]. We will constantly refer to degrees *below* and *above the gap* when we talk about these groups, but it will be made clear in the context which degrees belong to which category. For the other classical groups, we take the first gap to exist before the smallest dimension, so that with these groups, the phrase "degrees below the gap" comes to mean the empty set.

For example, any group $A_r(q) = \mathrm{SL}_{r+1}(q)$ with r at least 2 has irreducible cross-characteristic representations of dimensions

$$\begin{aligned} &\frac{q^{r+1}-q}{q-1} - \kappa_{r,q,\ell} &= q^r + q^{r-1} + \cdots + q - \kappa_{r,q,\ell} \\ \text{and} \quad &\frac{q^{r+1}-1}{q-1} &= q^r + q^{r-1} + \cdots + q + 1, \end{aligned}$$

where $\kappa_{r,q,\ell}$ is either 0 or 1, depending on r, q and ℓ. The difference between these two dimensions is at most two, and they are both said to be "below the gap". The next dimension is "above the gap", given by a polynomial with degree at least $2r - 3$.

5.2.1. Below the gap

Now, for groups of type A_r, ${}^2A_r$ and C_r, the dimensions below the gap are given by some polynomials in q, whose degrees depend on the rank. The polynomials are listed in various sources and collected in Table 5.2, together with multiplicities.

group	degree	multiplicity	reference
$A_r(q)$, $r > 1$	$\frac{q^{r+1}-q}{q-1} - \left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}$	1	[18]
	$\frac{q^{r+1}-1}{q-1}$	$(q-1)_{l'} - 1$	
${}^2A_r(q)$	$\frac{q^{r+1}-q(-1)^r}{q+1}$	1	[22]
	$\frac{q^{r+1}+(-1)^r}{q+1}$	$(q+1)_{l'} - 1$	
$C_r(q)$, q odd	$\frac{1}{2}(q^r - 1)$	2	[17]
	$\frac{1}{2}(q^r + 1)$	2	

Table 5.2: Minimal representation degrees of some classical groups. The symbol $\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}$ means either 0 or 1, depending on r, q and ℓ.

For each classical type $\mathcal{L}$ in $\{A', {}^2A, C\}$, it is straightforward to find a lower bound for all the degree polynomials given in Table 5.2. The bounds that will be used later are given in Table 5.3 as $\varphi_{\mathcal{L},r}$. It is also easy to bound the multiplicities from above, and we list some bounds in the same table as $\psi_{\mathcal{L}}$.

Now, if n is a representation degree of one of the groups $H_r(q)$ mentioned in Table 5.2, we know that $n \geq \varphi_{\mathcal{L},r}(q)$. For each $\mathcal{L}$, this gives an upper bound to q. For example, if n is a representation degree of $A_r(q)$, then $n \geq q^r$, so $q \leq n^{1/r}$. These upper bounds are given in Table 5.3 as $q^{<}_{\mathcal{L},r}(n)$. Similarly, upper bounds can be found for the rank, and these are listed as $r^{<}_{\mathcal{L}}(n)$.

$\mathcal{L}$	$\varphi_{\mathcal{L},r}(q)$	$\psi_{\mathcal{L}}(q)$	$q^{<}_{\mathcal{L},r}(n)$	$r^{<}_{\mathcal{L}}(n)$
A'	q^r	q	$n^{1/r}$	$\log_2 n$
${}^2A$	$(q-1)^r$, $\frac{1}{2}q^r$	q	$n^{1/r}+1$, $(2n)^{1/r}$	$\log_2 n + 1$
C	$\frac{1}{3}q^r$	2		$\log_3 n + 1$

Table 5.3: Bounds for minimal representation degrees, their multiplicities and related quantities for some classical groups.

The following two lemmata will be used in bounding the number of representations with dimension below the gap. They correspond to Lemma 4.8 in the complex case.

Lemma 5.5. *Wrote $H(q)$ for a universal covering group of one of the types A_r, ${}^2A_r$, or C_r, with fixed Lie family and rank. Let f_1 and f_2 be two polynomials expressing representation degrees of $H(q)$ below the gap, as given in Table 5.2. If $f_1(q_1) = f_2(q_2)$ for some prime powers q_1 and q_2, then $q_1 = q_2$ and $f_1 = f_2$.*

Proof. Suppose first that $H(q)$ is of type A_r with $r \geq 2$. We see from Table 5.2 that for $i \in \{1, 2\}$, the following holds:

$$q^r < f_i(q) < (q+1)^r \quad \text{for all } q.$$

Hence, if $n = f_i(q)$, we must have $n^{1/r} - 1 < q < n^{1/r}$. There is, however, at most one integer q that fits, so we must have $q_1 = q_2$. Also, the difference between the two minimal degree polynomials of A_r is uniformly either 1 or 2, so that $f_1(q) = f_2(q)$ only if $f_1 = f_2$.

The cases for ${}^2A_r$ and C_r are handled similarly. □

Lemma 5.6. *Let $H(q)$ be as in the previous lemma. Writing $r^{<}_n(H(q), \ell)$ for the number of irreducible ℓ-modular representations of $H(q)$ with dimension n below the gap, we have*

$$\sum_q r^{<}_n\big(H(q), \ell\big) < \psi_{\mathcal{L}}\big(q^{<}_{\mathcal{L},r}(n)\big),$$

with $\psi_{\mathcal{L}}$ and $q^{<}_{\mathcal{L},r}(n)$ as in Table 5.3.

Proof. By the previous lemma, for each n there can be at most one value of q, such that n is a degree of an irreducible representation of $H(q)$ below the gap. Also, there can be at most one minimal degree polynomial f such that $f(q) = n$. The multiplicity corresponding to f is bounded from above by $\psi_{\mathcal{L}}(q)$. Furthermore, the polynomial $\psi_{\mathcal{L}}$ is non-decreasing, so an upper bound is obtained by considering an upper bound to q. $\square$

5.2.2. Above the gap

After the few smallest representation degrees there is the first gap, and the next degrees are significantly larger. Lower bounds for these degrees that lie above the gap are here called the *gap bounds*, and they are again polynomials in q with degree depending on r. We list the relevant information on the gap bounds in Table 5.4. (The bounds might not hold for some groups that were excluded in the beginning of this section.) Notice that for the groups not appearing in Table 5.2, we take the gap bound to be the Landazuri–Seitz–Zalesskii bound (or one of its refinements) for the smallest dimension of a non-trivial irreducible representation.

In Table 5.5, we list as $\Gamma_{\mathcal{L},r}(q)$ some lower bounds for the gap bounds. As with degrees below the gap, these bounds yield upper bounds for those values of q, for which n may be a representation degree above the gap. The bounds for q are listed as $q^{>}_{\mathcal{L},r}(n)$. Similarly, we get upper bounds for the ranks and list those as $r^{>}_{\mathcal{L}}(n)$.

The following lemmata are used in bounding the number of representations with dimension above the gap. The first one appeared already as Lemma 4.9 in the previous chapter, but we include it here for easier reference.

Lemma 5.7. *Suppose K is a strictly increasing function on the integers and Q is a positive integer. For summing the values of $K(q)$ over prime powers q, we have the following estimate:*

$$\sum_{q \le Q} K(q) \;<\; K(Q) + \frac{1}{2}\sum_{i=3}^{Q-1} K(i) + \sum_{i=1}^{[\log_2 Q]} K(2^i).$$

(Here $[x]$ denotes the integral part of x.)

Lemma 5.8. *Keep the notation of the previous lemma. Assume additionally that K is a polynomial function, and denote $K(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_s x^s$. Then we have*

$$\sum_{q \le Q} K(q) < K(Q) + \frac{1}{2}\int_0^Q K(x)\,\mathrm{d}x + \alpha_0 \log_2 Q + \sum_{j=1}^{s} \frac{2^j \alpha_j Q^j}{2^j - 1}.$$

group	gap bound	remarks	ref.
$A'_r(q)$	$(q^r-1)\left(\frac{q^{r-1}-q}{q-1}-\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}\right)$	$r \geq 4$	[18]
	$(q-1)(q^2-1)/\gcd(3,q-1)$	$r=2$	
	$(q-1)(q^3-1)/\gcd(2,q-1)$	$r=3$	
${}^2A_r(q)$	$\frac{(q^{r+1}+1)(q^r-q^2)}{(q^2-1)(q+1)}-1$	r even	[17]
$r \geq 4$	$\frac{(q^{r+1}-1)(q^r-q)}{(q^2-1)(q+1)}$	r odd	
${}^2A_r(q)$	$\frac{1}{6}(q-1)(q^2+3q+2)$	$r=2,\, 3 \mid (q+1)$	[22]
$r<4$	$\frac{1}{3}(2q^3-q^2+2q-3)$	$r=2,\, 3 \nmid (q+1)$	
	$\frac{(q^2+1)(q^2-q+1)}{\gcd(2,q-1)}-1$	$r=3$	
$B_r(q)$	$\frac{(q^r-1)(q^r-q)}{q^2-1}$	$q=3,\, r\geq 4$	[24]
	$\frac{q^{2r}-1}{q^2-1}-2$	$q>3$	
$C_r(q)$	$\frac{(q^r-1)(q^r-q)}{2(q+1)}$		[17, 19]
$D_r(q)$	$\frac{(q^r-1)(q^{r-1}-1)}{q^2-1}$	$q \leq 3$	[24]
	$\frac{(q^r-1)(q^{r-1}+q)}{q^2-1}-2$	$q>3$	
${}^2D_r(q)$	$\frac{(q^r+1)(q^{r-1}-q)}{q^2-1}-1$	$r \geq 6$	[24]
	1026	$r=4,\, q=4$	
	151	$r=5,\, q=2$	
	2376	$r=5,\, q=3$	

Table 5.4: Gap bounds for the classical groups.

Proof. The result follows directly from the lemma above. The first two terms on the right hand side are obvious. For the last two, write $M = [\log_2 Q]$, and consider

$$\sum_{i=1}^{M} K(2^i) = \sum_{i=1}^{M}\left(\alpha_0 + \sum_{j=1}^{s}\alpha_j 2^{ij}\right) = \alpha_0 M + \sum_{j=1}^{s}\alpha_j \sum_{i=1}^{M}(2^j)^i.$$

It only remains to apply the formula for a geometric sum. □

5.3. Proving the main results on classical groups

In this section, we prove Theorems 5.1 and 5.2. For technical reasons, we will handle type A_1 separately in the end of the section.

Let $\mathcal{L}$ stand for one of the classical families A', ${}^2A$, B, C, D or ${}^2D$, and write $H_r(q)$ for a universal covering group of type $\mathcal{L}$, having rank r and being defined over a field of

$\mathcal{L}$	$\Gamma_{\mathcal{L},r}(q)$	$q^{>}_{\mathcal{L},r}(n)$	$r^{>}_{\mathcal{L}}(n)$	remarks
A'	q^{2r-2}	$n^{1/(2r-2)}$	$\frac{1}{2}\log_2 n + 1$	$r \geq 4$
	$\frac{1}{4}q^3$	$(4n)^{1/3}$		$r = 2,\ q \geq 7$
	$\frac{1}{3}q^4$	$(3n)^{1/4}$		$r = 3,\ q \geq 4$
${}^2A$	$\frac{1}{2}q^{2r-2}$	$(2n)^{1/(2r-2)}$	$\frac{1}{2}\log_2 n + \frac{3}{2}$	$r \geq 4$
	$\frac{1}{6}q^3$	$(6n)^{1/3}$		$r = 2$
	$\frac{2}{5}q^4$	$(5n/2)^{1/4}$		$r = 3,\ q \geq 4$
B	q^{2r-2}	$n^{1/(2r-2)}$	$\frac{1}{2}\log_3 n + 1$	$r \geq 3$
C	$\frac{1}{4}q^{2r-1}$	$(4n)^{1/(2r-1)}$	$\frac{1}{2}\log_2 n + \frac{3}{2}$	
$D, {}^2D$	q^{2r-3}	$n^{1/(2r-3)}$	$\frac{1}{2}\log_2 n + \frac{3}{2}$	$r \geq 4$

Table 5.5: Bounds for the gap bounds and related quantities for classical groups.

size q. We take ℓ, the characteristic of the representation space, to be fixed, and suppress the notation by writing $r_n(H,\ell) = r_n(H)$ and $s_n(H,\ell) = r_n(H)$.

Proof of Theorem 5.2 (excluding type A_1). We want to find an upper bound for the ratio

$$Q_n(\mathcal{L}) = \frac{1}{n}\sum_r \sum_q r_n\big(H_r(q)\big).^1$$

For each classical type $\mathcal{L}$, some small ranks and field sizes have been disregarded because of isomorphisms between the small groups. For example, for family B we take the smallest applicable rank to be 3, since $B_1(q) \cong A_1(q)$ and $B_2(q) \cong C_2(q)$ for every q. Also, we will not consider groups of type B in even characteristic, as $B_r(2^k) \cong C_r(2^k)$ for all r. Furthermore, the groups in A_1 are handled in a separate proof after this one. For each type $\mathcal{L}$, the smallest applicable rank and field size are denoted $r_0 = r_0(\mathcal{L})$ and $q_0 = q_0(\mathcal{L})$, respectively, and they are recalled in Table 5.6.

$\mathcal{L}:$	A'	${}^2A$	B	C	D	${}^2D$
$r_0:$	2	2	3	2	4	4
$q_0:$	2	2	3	2	2	2

Table 5.6: Smallest applicable ranks and field sizes.

Let us first compute the maxima of $Q_n(\mathcal{L})$ for $n \leq 250$. Here we use the tables of Hiss and Malle from [23]. The results are shown in Table 5.7. As these values are

[1] In this section, sums over r and q are to be taken over positive integers and prime powers, respectively.

all less than the corresponding bounds given in the statement of the theorem, we may henceforth assume that $n > 250$.

A'	${}^2A$	B	C	D	${}^2D$
1/2	2/3	2/27	3/13	1/8	1/33

Table 5.7: Maximal values of $Q_n(\mathcal{L})$ for $n \leq 250$.

The following is a general discussion of the structure of the proof, which will later be carried out case-by-case for each classical family. In the general discussion, we silently assume that the universal covering groups are of regular type, and moreover, do not appear in the list of groups excluded in the beginning of Section 5.2.

We first consider those groups that may yield a representation degree n below the gap. This only applies to families A', ${}^2A$ and C. We let $r_n^<(H_r(q))$ denote the number of irreducible representations of $H_r(q)$ with dimension n below the gap, and desire a bound for the ratio

$$Q_n^<(\mathcal{L}) = \frac{1}{n} \sum_r \sum_q r_n^<(H_r(q)).$$

For a given rank r, let f_r denote the polynomial giving the smallest representation degree as exhibited in Table 5.2. Now, if $f_r(q_0) > n$, there is no group of type $\mathcal{L}$ having rank r that would have an irreducible representation of dimension n. On the other hand, if $f_r(q_0) \leq n$, the number of such representations is at most $\psi(q_r^<(n))$, according to Lemma 4.8. We obtain the following upper bound for $Q_n^<(\mathcal{L})$:

$$R_n^<(\mathcal{L}) = \frac{1}{n} \sum_{r \geq r_0}{}' \psi\big(q_r^<(n)\big). \tag{5.1}$$

The prime ($'$) stands for taking the summand that corresponds to r into account only when $f_r(q_0) \leq n$ (this makes the sum finite).

The values of the expression[1] $R_n^<$ can be computed explicitly for all values up to a desired n_0. After n_0, notice that ψ is linear and $q_r^<(n)$ has the form $(an)^{1/r}$. This means that the function $r \mapsto \psi(q_r^<(n))$ is decreasing and convex for all $n > n_0$, so we may use a trapezial estimate to bound the sum in (5.1) from above (see Figure 5.1). This estimate is explained in the following.

The sum in (5.1) is taken up to the biggest rank r for which $f_r(q_0) \leq n$. Denote this rank $\hat{r}(n)$. Since $f_r(q_0)$ is strictly increasing in r, the value of $\hat{r}(n)$ is also increasing in n. So, for $n > n_0$, we are taking the sum at least up to $r = \hat{r}(n_0)$, denoted simply R_0. Write

$$M_r(n) = \psi\big(q_r^<(n)\big),$$

[1] Henceforth, we shall mostly suppress the parameter $\mathcal{L}$.

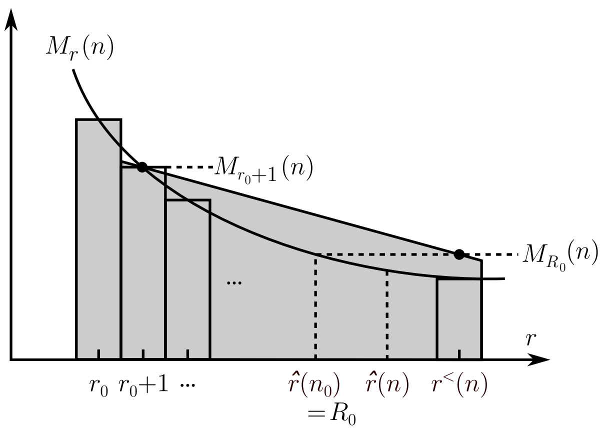

Figure 5.1: The trapezium used to obtain (5.2). Notice how the convexity of $M_r(n)$ guarantees that the shaded area is large enough to cover all the rectangles, even if $R_0 = r^<(n)$.

and let $r^<(n)$ denote the upper bound for the rank as given in Table 5.3. Setting up a trapezium as in Figure 5.1, we get the following upper bound to $R_n^<(\mathcal{L})$:

$$\bar{R}_n^<(\mathcal{L}) = \frac{1}{n}\left(M_{r_0}(n) + \big(r^<(n) - r_0\big)\frac{M_{r_0+1}(n) + M_{R_0}(n)}{2}\right). \tag{5.2}$$

Observe that $r^<(n)$ has the form $\log n + b$, whence it follows that $\bar{R}_n^<(\mathcal{L})$ will be decreasing after some n. We will take care to choose n_0 above this point for each $\mathcal{L}$.

Let us then write $r_n^>(H_r(q))$ for the number of irreducible representations of $H_r(q)$ with dimension n greater than the gap bound. We are looking for a bound to

$$Q_n^>(\mathcal{L}) = \frac{1}{n}\sum_r \sum_q r_n^>\big(H_r(q)\big).$$

This bound will consist of two terms: $R_n^1(\mathcal{L})$ and $R_n^>(\mathcal{L})$, corresponding to small and large ranks, respectively.

For small ranks, that is, with r below some suitably chosen $r_1 = r_1(\mathcal{L})$, we can use precise conjugacy class numbers obtainable from Lübeck's online data ([42]) to bound $r_n^>(H_r(q))$ from above. (The values of $r_1(\mathcal{L})$ will be chosen so as to obtain the best results. However, we will have $r_1(\mathcal{L}) \le 8$ in each case.) Write $k_r(q)$ for the class number of $H_r(q)$, and define

$$R_n^1(\mathcal{L}) = \frac{1}{n}\sum_{r=r_0}^{r_1-1}\sideset{}{'}\sum_{q\ge q_0} k_r(q). \tag{5.3}$$

The inner (dashed) sum is taken over those q, for which the gap bound of $H_r(q)$ is at most n, and there are only finitely many such q.

The exact values of R_n^1 can be computed for n up to any desired n_0. For larger values, we use a polynomial upper bound $K_r(q) = \sum_j \alpha_{r,j} q^j$ for the class number of $H_r(q)$.

Although the class numbers themselves are given by polynomials in q, there can be finitely many different polynomials, each applicable to a certain congruence class of q. To obtain an upper bound, we simply choose the maximal one among these polynomials, leaving out all negative terms. Then all the coefficients $\alpha_{r,j}$ will be non-negative integers, and it can be observed that the degree of $K_r(q)$ equals r.

Let $q_r^>(n)$ be the upper bound to q as given in Table 5.3. The expression $R_n^1(\mathcal{L})$ can now be bounded from above, in accordance with Lemma 4.10, by

$$\bar{R}_n^1(\mathcal{L}) = \frac{1}{n}\sum_{r=r_0}^{r_1-1}\Bigg(K_r\big(q_r^>(n)\big) + \frac{1}{2}\int_0^{q_r^>(n)} K_r(x)\,dx + \alpha_0\log_2 q_r^>(n) + \sum_{j\geq 1}\frac{\alpha_{r,j}\big(2q_r^>(n)\big)^j}{2^j-1}\Bigg). \quad (5.4)$$

The important thing to note here is that for all types $\mathcal{L}$ and all ranks r, the largest power of n inside the outermost brackets is at most 1. (This can be seen by comparing the expression for $q_r^>(n)$ with the degree of $K_r(q)$, which is r.) With all coefficients $\alpha_{r,j}$ non-negative, we can then be sure that $\bar{R}_n^1$ is decreasing in n when $\log_2 q_r^>(n)/n$ is. Since $q_r^>(n)$ is of the form $(an)^{1/d_r}$ for some a and d_r, this happens when $n \geq e/a$.

For large ranks, that is, for $r \geq r_1$, we need to use generic upper bounds for conjugacy class numbers. Suitable bounds were found by Fulman and Guralnick in [11]. The bounds have the form $b_r(q) = q^r + B_{\mathcal{L}}q^{r-1}$, with $B_{\mathcal{L}}$ as in Table 5.8. We write

$$R_n^2(\mathcal{L}) = \frac{1}{n}\sum_{r\geq r_1}\sum_{q\geq q_0}{}' \left(q^r + Bq^{r-1}\right).$$

The inner sum is again taken over those q, for which the gap bound of $H_r(q)$ is at most n.

$\mathcal{L}:$	A	${}^2A$	B	C	D	${}^2D$
$B_{\mathcal{L}}:$	3	15	22	30	32	32

Table 5.8: Constants in the Fulman–Guralnick bounds for conjugacy class numbers of classical groups. (Copy of Table 4.19.)

Also the values of R_n^2 can be computed exactly up to n_0. After that, we use an upper bound that results from substituting $b_r(q)$ in place of $K_r(q)$ in equation (5.4). When simplified, this becomes

$$\frac{1}{n}\sum_{r=r_1}^{[r^>(n)]}\Lambda_r(n),$$

where

$$\Lambda_r(n) = \frac{1}{2r+2}\,q_r^>(n)^{r+1} + \left(\frac{2^{r+1}-1}{2^r-1} + \frac{B_{\mathcal{L}}}{2r}\right)q_r^>(n)^r + \frac{(2^r-1)B_{\mathcal{L}}}{2^{r-1}-1}\,q_r^>(n)^{r-1}.$$

Notice that $q_r^>(n)$ has the form $(an)^{1/(br-c)}$ for some positive integers b and c. This makes every term in $\Lambda_r(n)$ decreasing in r (for any fixed n), except possibly the final $(an)^{\frac{r-1}{br-c}}$, which is increasing if and only if $b > c$. In this exceptional case (which occurs only when $\mathcal{L} = C$), we replace $(an)^{\frac{r-1}{br-c}}$ by its limit $(an)^{1/b}$. Then we can use the simple estimate

$$\bar{R}^2_{\mathcal{L}}(n) = \frac{r^>(n) - r_1 + 1}{n}\,\Lambda_{r_1}(n) \tag{5.5}$$

as an upper bound to $R^2_n(\mathcal{L})$. Here, $r^>(n)$ is the upper bound for the rank as given in Table 5.5, and it has the form $a\log n + b$. Since the highest power of n in $\Lambda_{r_1}(n)$ is less than 1, this means that $\bar{R}^2_{\mathcal{L}}(n)$ will become decreasing after some n.

Finally, the ratio Q_n is bounded above by the sum of $R_n^<$, R_n^1 and R_n^2. We shall finish the proof by computing the maximal values of these quantities for each classical family separately. We also add the values arising from groups with exceptional covers individually in each case.

Case $\mathcal{L} = A'$. Let us first assume that (r, q) is none of $(2, 2)$, $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 2)$, $(3, 3)$, $(5, 2)$ or $(5, 3)$. According to equation (5.1) and Table 5.3, we have

$$R_n^< = {\sum_{r\geq r_0}}' \frac{1}{n^{1-1/r}}\,,$$

where the summand corresponding to r is taken into account if and only if $n \geq 2^{r+1} - 3$ (except for $r = 5$ where $q \geq 4$, so n has to be at least 1363, which is above 250). The exact values of $R_n^<$ will be computed up to $n_0 = 650000$. At n_0, the biggest r in the sum is 18, so looking at equation (5.2), we have

$$R_n^< < \bar{R}_n^< = \frac{1}{n^{1/2}} + \left(\frac{1}{2}\log_2 n - 1\right)\left(\frac{1}{n^{2/3}} + \frac{1}{n^{17/18}}\right)$$

for $n > n_0$. This expression is decreasing after n_0.

Let us then bound $Q_n^>$. We set $r_1 = 5$. For ranks less than r_1, we use the conjugacy class numbers obtainable from Lübeck's data to determine the values of R_n^1. The values of R_n^1 are computed up to n_0. After this, we substitute $q^>_{A',r}(n)$ from Table 5.5 into equation (5.4) to get a decreasing expression $\bar{R}_n^1$ that can be used as an upper bound.

We move on to large ranks. Write $\Gamma_r(q)$ for the gap bound of $A_r(q)$. For rank 5, we have assumed that $q \geq 4$, and $\Gamma_5(4) = 84909$. Thus, the smallest appearing gap bound is $\Gamma_6(2) = 1827$. For values of n from 1827 up to n_0, we will compute R_n^2, and afterwards we can substitute values from Tables 5.8 and 5.5 into equation (5.5) to obtain

$$\bar{R}_n^2 = \left(\frac{1}{2}\log_2 n - 3\right)\left(\frac{1}{12n^{1/4}} + \frac{723}{310n^{3/8}} + \frac{31}{5n^{1/2}}\right).$$

This expression is decreasing for $n > n_0$.

Now, for n from 251 up to n_0, we can compute the exact values of

$$F(n) = R_n^< + R_n^1 + R_n^2,$$

and this bounds Q_n from above. We are still, however, avoiding the exceptional cases. From the ATLASes, we find that the universal covering groups $A_2(2)$, $A_2(3)$, $A_2(4)$, $A_2(5)$ and $A_3(2)$ have no irreducible representations of dimension over 250, so we may forget about them. For $A_3(3)$, we extract the necessary data from the ATLASes for directly computing $r_n(A_3(3))$ for all n. We find that the maximal value of $r_n(A_3(3))/n$ over all n is less than 0.01924.

For the remaining two groups, we use the following information obtainable from [18] and from the conjugacy class numbers of the groups, the latter of which are 60 for $A_5(2)$ and 396 for $A_5(3)$.

group	degree	multiplicity (at most)
$A_5(2)$	≥ 526	57
$A_5(3)$	362 or 363	1
	364	1
	≥ 6292	393

Having added to $F(n)$ the values of $r_n(A_3(3))/n$ (for all n), $57/n$ for $n \geq 526$, $1/n$ for $n \in \{362, 363, 364\}$, and $393/n$ for $n \geq 6292$, we find that the biggest total sum below n_0 appears at $n = 251$, and is less than 1.5484. More precisely, we have $R^<_{251} = 0.115$, $R^1_{251} = 1.435$, and $R^2_{251} = r_{251}(A_3(3)) = 0$.

For $n > n_0$, we use

$$\bar{F}(n) = \bar{R}^<_n + \bar{R}^1_n + \bar{R}^2_n$$

as a decreasing upper bound to $F(n)$. We have $\bar{F}(n_0) < 1.5267$, so even with the additions coming from the exceptional cases, we see that $F(n) < F(251)$ when $n > n_0$. Since according to Table 5.7, we have $Q_{A'}(n) < F(251)$ also for $n \leq 250$, we know that the obtained upper bound works for all n.

Case $\mathcal{L} = {}^2A$. We assume that (r, q) is none of $(2, 2)$ (soluble), $(2, 3)$, $(2, 4)$, $(2, 5)$, $(3, 2)$, $(3, 3)$ or $(5, 2)$. Proceeding in the same way as in the case $\mathcal{L} = A'$, we read from Table 5.3 that

$$R^<_n = {\sum_{r \geq r_0}}' \left(\frac{1}{n^{1-1/r}} + \frac{1}{n} \right),$$

where each summand is taken into account when n is at least $(2^{r+1} - 2)/3$ if r is even, or $(2^{r+1} - 1)/3$ if r is odd. The values of $R^<_n$ will be computed explicitly up to $n_0 = 80000$. At n_0, the biggest rank is 16, so we use

$$\bar{R}^<_n = \frac{\sqrt{2}}{n^{1/2}} + (\log_2 n - 1) \left(\frac{1}{(2n)^{2/3}} + \frac{1}{(2n)^{15/16}} \right).$$

(Here we applied the second value given for $q^<_{{}^2A,r}(n)$ in Table 5.3.) This expression is decreasing for $n > n_0$.

To bound $Q_n^>$, we set $r_1 = 7$. For ranks less than r_1, we use Lübeck's class number polynomials to compute R_n^1 up to $n = n_0$. After this, we use $\bar{R}_n^1$, defined in (5.4), as a decreasing upper bound for R_n^1.

For $r \geq r_1$, the smallest gap bound is 3570 (the corresponding group is ${}^2A_7(2)$). For values of n from 3570 up to n_0, we compute R_n^2, and afterwards we use

$$\bar{R}_n^2 = (\log_2 n - 9)\left(\frac{1}{16 \cdot (2n)^{1/3}} + \frac{5475}{1778 \cdot (2n)^{5/12}} + \frac{635}{21 \cdot (2n)^{1/2}}\right).$$

This is decreasing in n for $n > n_0$.

For the exceptional cases, ${}^2A_2(3)$, ${}^2A_2(4)$, ${}^2A_2(5)$ and ${}^2A_3(2)$ have all representation degrees below 250, so these groups can be ignored. The values of $r_n\big({}^2A_3(3)\big)/n$ can be computed using the Atlases, and the maximum is 0.03572. Finally, group ${}^2A_5(2)$ has 131 non-trivial conjugacy classes. Now, the values of

$$F(n) = R_n^< + R_n^1 + R_n^2 + \frac{r_n\big({}^2A_3(3)\big)}{n} + \frac{131}{n}$$

are computed up to n_0, and the maximal value is $F(272) < 2.8783$. (More precisely, $R_{272}^< = 0.17$, $R_{272}^1 = 2.24$ and $R_{272}^2 = r_{272}({}^2A_3(3)) = 0$.) After n_0, the upper bound

$$\bar{F}(n) = \bar{R}_n^< + \bar{R}_n^1 + \bar{R}_n^2 + 0.03572 + 131/n$$

is decreasing, and $\bar{F}(n_0) < 2.873$.

Case $\mathcal{L} = B$. For the orthogonal groups, we have no gap results. Assume that (r, q) is not $(3, 3)$. Also, we are assuming that $r \geq 3$ and $q \geq 3$, since otherwise we would be in coincidence with the symplectic groups. We set $r_1 = 5$. For ranks less than r_1, we compute $R_B^1(n)$ up to $n_0 = 58000$. When $n > n_0$, we use $\bar{R}_B^1(n)$ as a decreasing upper bound to $R_B^1(n)$.

For $r \geq r_1$, the smallest gap bound is 7260 (the corresponding group is $B_5(3)$). For values of n from 7260 up to n_0, we can compute R_n^2, and afterwards we use the upper bound

$$\bar{R}_n^2 = \left(\frac{1}{2}\log_3 n - 3\right)\left(\frac{1}{12n^{1/4}} + \frac{656}{155n^{3/8}} + \frac{682}{15n^{1/2}}\right).$$

This is decreasing for $n > n_0$.

The exceptional case $B_3(3)$ has 87 non-trivial conjugacy classes. The values of

$$F(n) = R_n^1 + R_n^2 + \frac{87}{n}$$

are computed up to n_0, and the maximal value is $F(780) < 0.9859$. (We have $R_{780}^1 = 0.88$ and $R_{780}^2 = 0$.) After n_0, we apply

$$\bar{F}(n) = \bar{R}_n^1 + \bar{R}_n^2 + \frac{87}{n},$$

which is decreasing and has $\bar{F}(n_0) < F(780)$.

Case $\mathcal{L} = C$. Assume (r, q) is none of $(2, 2)$, $(2, 3)$ or $(3, 2)$. Below the gap bound given in Table 5.4, representation degrees exist only for groups with odd q. Referring to Table 5.3, we have

$$R_n^< = a_n \cdot \frac{2}{n},$$

where a_n is the number of ranks $r \geq 2$ such that $n \geq (3^r - 1)/2$. The values of $R_n^<$ will be computed up to $n_0 = 140000$. By using the bounds in Table 5.3 to estimate $a_n \leq \log_3(n)$, we obtain the upper bound

$$\bar{R}_n^< = \frac{2 \log_3 n}{n}.$$

This is decreasing for $n > n_0$.

For $Q_n^>$, we set $r_1 = 7$. For ranks less than r_1, we compute R_n^1 up to n_0, and when $n > n_0$, we use $\bar{R}_n^1$ as a decreasing upper bound to R_n^1.

For $r \geq r_1$, the smallest gap bound is 2667 (the corresponding group is $C_7(2)$). For values of n from 2667 up to n_0, we can compute R_n^2, and afterwards we use the upper bound

$$\bar{R}_n^2 = (\log_2 n - 9) \left(\frac{1}{8 \cdot (4n)^{5/13}} + \frac{7380}{889 \cdot (4n)^{6/13}} + \frac{5080}{21 n^{1/2}} \right).$$

(Note that the last term is chosen according to the discussion on page 126.) This bound is decreasing for $n > n_0$.

For the exceptional cases, $C_2(2)$ and $C_2(3)$ have all degrees below 250. The values of $r_n(C_3(2))/n$ can be computed from the Atlases, and the maximum is 0.01072. The values of

$$F(n) = R_n^< + R_n^1 + R_n^2 + \frac{r_n(C_3(3))}{n}$$

are computed up to n_0, and the maximal value is $F(288) = 2.8750$. (More precisely, $R_{288}^< = 1/36$, $R_{288}^1 = 205/72$ and $R_{288}^2 = r_{288}(C_3(3)) = 0$.) After n_0, the upper bound $\bar{F}(n) = \bar{R}_n^< + \bar{R}_n^1 + \bar{R}_n^2$ is decreasing, and $\bar{F}(n_0) < F(288)$.

Case $\mathcal{L} = D$. Assume (r, q) is not $(4, 2)$. There are no gap results. We set $r_1 = 8$. For ranks less than r_1, we compute R_n^1 up to $n_0 = 222000$. When $n > n_0$, we use $\bar{R}_n^1$ as a decreasing upper bound to R_n^1.

For $r \geq r_1$, the smallest gap bound is 11048 (the corresponding group is $D_8(2)$). For values of n from 11048 up to n_0, we can compute R_n^2, and afterwards we use the upper bound

$$\bar{R}_n^2 = (\log_2 n - 11) \left(\frac{1}{36 n^{4/13}} + \frac{1021}{510 n^{5/13}} + \frac{4080}{127 n^{6/13}} \right).$$

This is decreasing for $n > n_0$.

The exceptional group $D_4(2)$ is covered in the Modular Atlas, so we can compute the values of $r_n(D_4(2))/n$, the maximum of which is 0.005792. The values of

$$F(n) = R_n^1 + R_n^2 + \frac{r_n(D_4(2))}{n}$$

are computed up to n_0, and the maximal value is $F(298) < 1.5135$. (More precisely, we have $R^1_{298} = F(298)$ and $R^2_{298} = r_{298}(D_4(2)) = 0$.) After n_0, the upper bound $\bar{F}(n) = \bar{R}^1_n + \bar{R}^2_n + 0.005792$ is decreasing, and $\bar{F}(n_0) < F(298)$.

Case $\mathcal{L} = {}^2D$. Assume (r, q) is not $(4, 2)$. There are no gap results. We set $r_1 = 7$. For ranks less than r_1, we compute R^1_n up to $n_0 = 220000$. When $n > n_0$, we use $\bar{R}^1_n$ as a decreasing upper bound to R^1_n.

For $r \geq r_1$, the smallest gap bound is 2663 (the corresponding group is ${}^2D_7(2)$). For values of n from 2663 up to n_0, we can compute R^2_n, and afterwards we use the upper bound

$$\bar{R}^2_n = (\log_2 n - 9)\left(\frac{1}{32n^{3/11}} + \frac{3817}{1778n^{4/11}} + \frac{2032}{63n^{5/11}}\right).$$

This is decreasing for $n > n_0$.

The representation degrees of ${}^2D_4(2)$ are given in the Atlases. The maximum of $r_n({}^2D_4(2))/n$ is 0.004202. The values of

$$F(n) = R^1_n + R^2_n + \frac{r_n({}^2D_4(2))}{n}$$

are computed up to n_0, and the maximal value is $F(251) < 1.7969$. (More precisely, we have $R^1_{251} = F(251)$ and $R^2_{251} = r_{251}({}^2D_4(2)) = 0$.) After n_0, the upper bound $\bar{F}(n) = \bar{R}^1_n + \bar{R}^2_n + 0.004202$ is decreasing, and $\bar{F}(n_0) < F(251)$.

We have thus checked that all the bounds given in the statement of the theorem hold, except possibly for $\mathcal{L} = A_1$. $\square$

We still need to check the case of rank one linear groups.

Proof of Theorem 5.1. For $n \leq 250$, the bounds given in the statement of the theorem hold according to the tables of Hiss and Malle ([21]). Moreover, it can be verified that the largest value of $s_n(A_1)/n$ for $n \leq 250$ is 8/3, and this can be obtained only at $n = 3$. Similarly, the second largest value 7/3 can be obtained only at $n = 6$, and the third largest value 3/2 only at $n = 4$ and $n = 12$.

We may now assume that $q > 250$, since the maximal dimension of an irreducible representation of $A_1(q)$ is $q + 1$. With this assumption, the universal cover of $A_1(q)$ is $\mathrm{SL}_2(q)$. For this group, the complex character degrees and their multiplicities are listed in Table 5.9.

The cross-characteristic decomposition numbers for the simple groups of type $\mathrm{PSL}_2(q)$ are given in [2]. If q is even, $\mathrm{SL}_2(q)$ is isomorphic to $\mathrm{PSL}_2(q)$, and if ℓ is even, the scalar -1 in $\mathrm{SL}_2(q)$ acts trivially in the representation, so $\mathrm{SL}_2(q)$ has no faithful representations. Assume then that q and ℓ be odd. When q is not a power of ℓ, the Sylow ℓ-subgroups of $\mathrm{SL}_2(q)$ are cyclic. Now Dade's theorems can be used (see e.g. [9, §68]), and in this case they tell us that the decomposition numbers are all either 0 or 1, there are at most two irreducible Brauer characters in each block, and the set of irreducible ℓ-Brauer characters is a subset of the irreducible complex characters (restricted to p-regular elements). Therefore, it is enough to consider the case with $\ell = 0$.

q even		q odd	
degree	multiplicity	degree	multiplicity
$q-1$	$q/2$	$q-1$	$(q-1)/2$
q	1	q	3
$q+1$	$(q-2)/2$	$q+1$	$(q-3)/2$
		$(q-1)/2$	2
		$(q+1)/2$	2

Table 5.9: Non-trivial complex character degrees of $\mathrm{SL}_2(q)$. (Copy of Table 4.22.)

We can read off from the character table that for any degree n, there are at most $n+3$ characters of this degree, of any groups of type SL_2. (Equality is obtained only if n, $n+1$, $n-1$, $2n+1$ and $2n-1$ are all prime powers.) Furthermore, when $n \geq 251$, we have $(n+3)/n < 8/3$, so the upper bound for $n \leq 250$ holds globally. This proves Theorem 5.1, and also completes the proof of Theorem 5.2 above. □

5.4. Proving the main result on exceptional groups

In this section, we shall prove Theorem 5.3. For universal covering groups $H(q)$ of exceptional Lie type, the rank is always bounded and there are no gap results. We will use Lübeck's conjugacy class numbers and the (sharpened) Landazuri–Seitz–Zalesskii bounds for the minimal representation degrees. The latter are listed in Table 5.10, where Φ_k denotes the k'th cyclotomic polynomial in q. As with classical groups (in Section 5.2), these lower bounds lead to upper bounds for the values of q for which $H(q)$ can have a representation of degree n. Such upper bounds that will be used later are listed in Table 5.11 as $q_H^{<}(n)$.

For technical reasons, we shall deal with the Suzuki and Ree types ${}^2B_2$ and ${}^2G_2$ separately. Write therefore Exc' for the set of remaining exceptional Lie types (i.e., the set of symbols ${}^3D_4$, E_6, E_7, E_8, ${}^2E_6$, F_4, ${}^2F_4$ and G_2).

Let $H(q)$ be a regular universal covering group of a simple group of an exceptional Lie type $H \in \mathrm{Exc}'$. Mimicking the previous section, we denote

$$Q_n(\mathcal{E}) = \frac{1}{n} \sum_{H \in \mathrm{Exc}'} \sum_{q} r_n\big(H(q)\big).$$

Write $f_H(q)$ for the lower bound of the smallest representation dimension of $H(q)$ as given in Table 5.10. Now, $Q_n(\mathcal{E})$ is bounded above by

$$R_n(\mathcal{E}) = \frac{1}{n} \sum_{H \in \mathrm{Exc}'} \sum_{q}{}' k_H(q),$$

where $k_H(q)$ is the conjugacy class number of $H(q)$, and the dashed sum is taken over those q for which $H(q)$ is defined and $f_H(q) \leq n$.

group	bound for rep. degree	remark	ref.
${}^2B_2(q)$	$(q-1)\sqrt{q/2}$		[34]
${}^3D_4(q)$	q^5-q^3+q-1		[47]
$E_6(q)$	$q(q^4+1)(q^6+q^3+1)-1$		[25]
${}^2E_6(q)$	$(q^5+q)(q^6-q^3+1)-2$		[47]
$E_7(q)$	$q\Phi_7\Phi_{12}\Phi_{14}-2$		[25]
$E_8(q)$	$q\Phi_4^2\Phi_8\Phi_{12}\Phi_{20}\Phi_{24}-3$		[25]
$F_4(q)$	$\frac{1}{2}q^7(q^3-1)(q-1)$	q even	[34]
	$q^6(q^2-1)$	q odd	[57]
${}^2F_4(q)$	$q^4(q-1)\sqrt{q/2}$		[34]
$G_2(q)$	$q^2(q^2+1)$	$q\equiv 0 \pmod 3$	[59]
	q^3	$q\equiv 1 \pmod 3$	
	q^3-1	$q\equiv 2 \pmod 3$	
${}^2G_2(q)$	$q(q-1)$		[34]

Table 5.10: Lower bounds for representation degrees of the universal covering groups of exceptional groups of Lie type.

For large n, we shall need the estimate $R_n(\mathcal{E}) < \bar{R}_n(\mathcal{E})$, where $\bar{R}_n(\mathcal{E})$ is defined analogously to $\bar{R}^1_n(\mathcal{L})$ on page 125, using $q^{>}_H(n)$ from Table 5.11 instead of $q^{>}_{\mathcal{L},r}(n)$. The expression $\bar{R}_n(\mathcal{E})$ becomes decreasing eventually. The main reason for this is that the ratio $\deg(k_H)/(\deg(f_H)+1)$ is at most 1 for $H \in \mathrm{Exc}'$. (Note that for $H = {}^2F_4$, we see that f_H is not actually a polynomial in q; in this case we say that $\deg(f_H) = 11/2$.)

We will now establish the claimed bound for the representation growth of exceptional groups of Lie type. Let us first deal with the excluded types.

Lemma 5.9. *For the Suzuki and Ree groups, if $n > 250$, we have*

$$\sum_q r_n\big({}^2B_2(q)\big) < \sqrt{n},$$

and

$$\sum_q r_n\big({}^2G_2(q)\big) < \sqrt[3]{n}.$$

Proof. Firstly, we can confirm from the Atlases that ${}^2B_2(8)$ has all irreducible representation degrees below 250. Recall that we exclude ${}^2G_2(2)$ and ${}^2G_2(3)$ because they are isomorphic to classical groups. (Notice also that ${}^2B_2(2)$ is soluble.) Thus we may assume that $q \geq 32$ for type ${}^2B_2$ and $q \geq 27$ for ${}^2G_2$. With these assumptions, the universal covering groups are isomorphic to the simple groups.

H	$q_H^{>}(n)$	remark
${}^3D_4$	$(4/3 \cdot n)^{1/5}$	
E_6	$n^{1/11}$	
${}^2E_6$	$(3/2 \cdot n)^{1/11}$	$q \geq 3$
E_7	$n^{1/17}$	
E_8	$n^{1/29}$	
F_4	$(9/8 \cdot n)^{1/8}$	
${}^2F_4$	$(2n)^{2/11}$	$q \geq 8$
G_2	$(125/124 \cdot n)^{1/3}$	$q \geq 5$

Table 5.11: Upper bounds for such q for which $H(q)$ of exceptional Lie type may have a representation of degree n. (See the text for details.)

Let us begin with the groups ${}^2B_2(q)$. The generic complex character table of ${}^2B_2(q)$ was introduced in [58], and the decomposition matrices for ℓ-modular representations of these groups are given in [3]. From this information, we see that the set of irreducible Brauer characters is a subset of the irreducible complex characters (restricted to p-regular elements), except when ℓ divides $q - \sqrt{2q} + 1$. In the latter case, there appears an additional irreducible Brauer character with degree $q^2 - 1$. The possible ℓ-modular degrees are all listed in Table 5.12. The multiplicities in the table are for complex characters (except for the degree $q^2 - 1$), as this gives an upper bound for them.

degree	multiplicity	class
$(q-1)\sqrt{q/2}$	2	A
$(q - \sqrt{2q} + 1)(q-1)$	$(q + \sqrt{2q})/4$	
$q^2 - 1$	1	
q^2	1	B
$q^2 + 1$	$q/2 - 1$	
$(q + \sqrt{2q} + 1)(q-1)$	$(q - \sqrt{2q})/4$	

Table 5.12: Non-trivial Brauer character degrees of ${}^2B_2(q)$.

The character degrees of ${}^2B_2(q)$ are given by functions of q that are also polynomials in $\sqrt{q}$. These functions can be divided into two classes according to their polynomial degree, as shown in the last column of Table 5.12. The values of the functions in class B are increasing and strictly between $\frac{3}{4}q^2$ and $2q^2$. Suppose now that f_1 and f_2 are two functions from class B, and that $f_1(q_1) = f_2(q_2)$ for some $q_1 < q_2$. As ${}^2B_2(q)$ is only defined for odd powers of 2, we know that $q_2 \geq 4q_1$. It follows that

$$f_1(q_1) < 2q_1^2 \leq \frac{1}{8}q_2^2 < \frac{3}{4}q_2^2 < f_2(q_2),$$

which is against the assumption. This means that if $f_1(q_1) = f_2(q_2)$ for two functions from class B, we must have $q_1 = q_2$. However, it is easily checked that for any q, different functions in class B give different values. Hence it follows that if n is a character degree of ${}^2B_2(q)$ and $n \neq (q-1)\sqrt{q/2}$, then q is fixed and $n = f(q)$ for a unique function in class B.

In the case just described (where n is a value of a function from class B), we know that $n > \frac{3}{4}q^2$, so that $q < 2\sqrt{n}/\sqrt{3}$. Substituting this to the largest multiplicity in class B and adding the 2 possible characters coming from class A, we find that

$$\sum_{q \geq 32} r_n\big({}^2B_2(q)\big) < \frac{1}{2\sqrt{3}}\sqrt{n} + \frac{1}{2\sqrt[4]{3}}\sqrt[4]{n} + 2.$$

This gives the first result.

In case of ${}^2G_2$, a similar analysis can be made. The complex character degrees are given in [62]. For $\ell = 2$, the decomposition matrices can be found in [35], and for odd ℓ, they can be inferred from the Brauer trees presented in [20]. It turns out that in addition to the complex characters, there appears a Brauer character of degree $q^2 - q$ if $\ell = 2$, one of degree $q^3 - 1$ if ℓ divides $q^2 - \sqrt{3}q + 1$, and one of degree $(q-1)(q+2\sqrt{q/3}+1)(q-\sqrt{3q}+1)$ if $\ell = 2$ or ℓ is odd and divides $q+1$. All these degrees and their multiplicities (or upper bounds to them) are given in Table 5.13.

degree	multiplicity	class
$q^2 - q$	1	A
$q^2 - q + 1$	1	
$\sqrt{q/3}(q-1)(q-\sqrt{3q}+1)/2$	2	
$\sqrt{q/3}(q-1)(q+\sqrt{3q}+1)/2$	2	B
$\sqrt{q/3}(q^2-1)$	2	
$(q^2-1)(q-\sqrt{3q}+1)$	$(q+\sqrt{3q})/6$	
$(q-1)(q+2\sqrt{q/3}+1)(q-\sqrt{3q}+1)$	1	
$(q-1)(q^2-q+1)$	$(q-3)/6$	
$q(q^2-2+1)$	1	C
$q^3 - 1$	1	
q^3	1	
$q^3 + 1$	$(q-3)/2$	
$(q^2-1)(q+\sqrt{3q}+1)$	$(q-\sqrt{3q})/6$	

Table 5.13: Non-trivial Brauer character degrees of ${}^2G_2(q)$.

The degree functions are divided into three classes. We notice that in class C, the values of the functions are strictly between $\frac{2}{3}q^3$ and $2q^3$. Now, suppose that f_1 and f_2 are two functions from class C, such that $f_1(q_1) = f_2(q_2)$ for some $q_1 < q_2$. As $q_2 \geq 9q_1$, we get

$$f_1(q_1) < 2q_1^3 \leq \frac{2}{729}q_2^3 < \frac{2}{3}q_2^3 < f_2(q_2),$$

a contradiction. As before, we can conclude that if the value n is a character degree of ${}^2G_2(q)$ given by a function in class C, then that function is unique, and so is q. Also, in this case we would have $n > \frac{2}{3}q^3$, so that $q < \sqrt[3]{3n}/\sqrt[3]{2}$. Substituting this upper bound into the largest multiplicity and adding the 3 degrees coming from classes A and B (also in these classes only one function can have the value n), we get the following estimate:

$$\sum_{q \geq 27} r_n\big({}^2G_2(q)\big) < \frac{\sqrt[3]{3}}{2\sqrt[3]{2}}\sqrt[3]{n} + \frac{3}{2}.$$

This gives the second result. □

Proof of Theorem 5.3. Recall that we are not considering groups $G_2(2)$ or ${}^2G_2(3)$ here, as they are isomorphic to classical groups. From the tables of Hiss and Malle ([21]), we find that the stated bound holds for all $n \leq 250$.

Assume that $n > 250$. We first take care of the following exceptional cases: ${}^2E_6(2)$, $F_4(2)$, $G_2(3)$, $G_2(4)$ and ${}^2F_4(2)$. The universal covering groups ${}^2F_4(2)$, $G_2(3)$ and $G_2(4)$ can be found in the ATLASes. Letting $\mathcal{X}$ denote the set of these four groups, we can compute

$$\max_n \sum_{H \in \mathcal{X}} \frac{r_n(H)}{n} = 0.03389.$$

Of those groups that are not covered in the Modular ATLAS, $F_4(2)$ has 95 conjugacy classes. On the other hand, as explained in Section 4.5 of the previous chapter, the group ${}^2E_6(2)$ is a 12-fold cover of the simple group, and F. Lübeck has computed the character table of the 6-fold subcover ([43]). From his data, we find that the subcover has 542 conjugacy classes, so ${}^2E_6(2)$ has at most 1084. The smallest non-trivial representation degree of ${}^2E_6(2)$ is at least 1536, according to [34].

We use Lemma 5.9 to estimate the contribution of types ${}^2B_2$ and ${}^2G_2$. Computing the values of

$$F(n) = R_n(\mathcal{E}) + \frac{1}{n}\left(\sum_{H \in \mathcal{X}} r_n(H) + [1083] + 94 + \sqrt{n} + \sqrt[3]{n}\right)$$

up to $n_0 = 25000$, adding the [1083] only for $n \geq 1536$, we find that the maximum is $F(251) < 1.27949$. (We have $R_{251}(\mathcal{E}) = 0.82$ and $r_{251}(H) = 0$ for all $H \in \mathcal{X}$.) Afterwards, the function

$$\bar{F}(n) = \bar{R}_n(\mathcal{E}) + 0.03389 + \frac{1083 + 94}{n} + \frac{1}{\sqrt{n}} + \frac{1}{n^{2/3}}$$

is decreasing, $F(n) < \bar{F}(n)$, and $\bar{F}(n_0) < F(251)$.

As $F(n)$ is an upper bound to $Q_n(\mathcal{E})$ for $n > 250$, we see that the bound stated in the theorem holds also for $n > 250$. This proves the claim. □

6. Application: Bounding the number of conjugacy classes of maximal subgroups in classical groups

6.1. Two results

Consider first finite groups. Let G_0 be a finite simple group of Lie type, and let G be an almost simple group with G_0 as the socle. In other words, let G be such that $G_0 \leq G \leq \mathrm{Aut}(G_0)$. In their paper [16], R. Guralnick, M. Larsen and P. H. Tiep use information on modular representation growth of finite quasisimple groups to find an asymptotic bound for the number $m(G)$ of conjugacy classes of maximal subgroups of G. This bound is given as

$$m(G) < ar^6 + br \log \log q,$$

where a and b are unknown constants, and r and q are the rank of G_0 and the size of its defining field, respectively ([16, Theorem 1.2]).

Using results derived in Chapter 5, we can sharpen the result of Guralnick et al. in the case where the socle G_0 is a classical group.

Theorem 6.1. *Assume G is a finite almost simple group with socle a classical group of dimension n over the field $\mathbb{F}_q$. Let $m(G)$ denote the number of conjugacy classes of maximal subgroups of G not containing the socle. Then*

$$m(G) < 2n^{5.2} + n \log_2 \log_2 q.$$

Consider then the simple algebraic groups in characteristic zero. Using our Corollary 4.4, Guralnick, Larsen and C. Manack managed in [14] to prove the following bound.

Theorem 6.2 (Guralnick, Larsen, Manack)**.** *Let G be a simple algebraic group of rank r over an algebraically closed field of characteristic zero. Then the number of conjugacy classes of maximal closed subgroups of G is $O(r)$.*

We shall prove Theorem 6.1 in Section 6.3, after first presenting some general theory and auxiliary results in Section 6.2. In Section 6.4, we give a brief account on how our result was used to prove Theorem 6.2.

6.2. Maximal subgroups in finite classical groups

Theorem 6.1 will be proved along the same lines as Theorem 6.3 in [16]. Let G be an almost simple group having as its socle a classical simple group G_0 preserving a classical form $\mathbf{f}$ on the n-dimensional $\mathbb{F}_q$-space V (see Section 2.3 for more information on the classical groups). With this notation, we have $G_0 \trianglelefteq G \leq \mathrm{Aut}(G_0)$, and apart from certain known exceptions, the automorphism group $\mathrm{Aut}(G_0)$ is $\mathrm{P\Gamma} = \mathrm{P\Gamma}_{\mathbf{f}}(V)$, the group of projective semilinear transformations preserving $\mathbf{f}$ up to scalar multiplication. (For the exact definition of $\mathrm{P\Gamma}$, see Section 2.3.) The exceptions occur when G_0 is one of $\mathrm{PSL}_n(q)$, $\mathrm{Sp}_4(2^k)$ or $\mathrm{P\Omega}_8^+(q)$, with additional (graph) automorphisms of order 2, 2 and 3, respectively. (See [31, Theorem 2.1.4].)

Let M be a maximal subgroup of G not containing G_0. It is known that the intersection $M \cap G_0$ is non-trivial (see e.g. [37, page 395]), so we must have $M = N_G(M \cap G_0)$. It then follows that the intersection $M \cap G_0$ completely determines M, and moreover, two maximal subgroups M_1 and M_2 are conjugate in G precisely when the intersections $M_1 \cap G_0$ and $M_2 \cap G_0$ are. Thus, to find an upper bound for the number of G-conjugacy classes of maximal subgroups M not containing G_0, it suffices to bound the number of G_0-classes of the intersections $M \cap G_0$.

Let Δ stand for the group of similarities of V preserving $\mathbf{f}$, and let $\mathrm{P}\Delta$ be the corresponding projective group. (For the exact definitions of Δ and $\mathrm{P}\Delta$, see Section 2.3.) It is often the number of $\mathrm{P}\Delta$-classes of $G_0 \cap M$ which is the easiest to bound. When this is the case, we need to insert a factor of $\sigma = [\mathrm{P}\Delta : G_0]$ to account for the splitting of $\mathrm{P}\Delta$-classes under G_0. In the linear and unitary groups, σ is at most the dimension of V, otherwise it is at most 8 (see Table 2.1.D in [31]).

Assume now that $\mathrm{Aut}(G_0) = \mathrm{P\Gamma}$ (the general case). To estimate the number of G_0-classes of maximal subgroups, we make use of Aschbacher's Theorem on maximal subgroups of classical groups ([1], see also [31, Section 1.2]). By that theorem, M either belongs to one of the so-called *geometrical families* $\mathcal{C}_1$–$\mathcal{C}_8$, or in a further family consisting of almost simple groups. This additional family will be called $\mathcal{S}$.

Let us look at $\mathcal{S}$ more closely. By Aschbacher's Theorem, any maximal subgroup in this family is itself an almost simple group whose non-abelian simple socle acts absolutely irreducibly on V. Let M be one of these maximal subgroups and let M_0 denote its socle. Then we have $M_0 \leq M \cap G_0 \lhd M$ and $N_G(M_0) = M$. As before, it follows that it is enough to consider the G_0-conjugacy classes of the socles. A diagram of the situation is presented in Figure 6.1.

The socle M_0, via its action on V, corresponds to an absolutely irreducible modular projective representation of a simple group isomorphic to it. The projective representation of the simple group in turn lifts to a linear representation of the universal covering group $\tilde{M}_0$, and equivalent linear representations correspond to subgroups conjugate under $\mathrm{GL}(V)$. Furthermore, the conjugating element will always reside in Δ (see e.g. [31, Corollary 2.10.4]).

The number of cross-characteristic representations of groups of Lie type was bounded in Chapter 5. Assume now that the characteristic ℓ of the representation divides q.

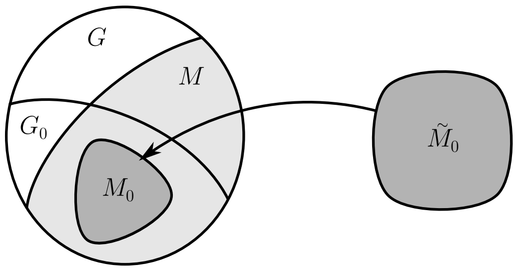

Figure 6.1: The situation when M belongs to $\mathcal{S}$. The maximal subgroup M is determined by the socle M_0, which in turn corresponds to a linear representation of a quasisimple covering group $\tilde{M}_0$ of M_0.

In this case, we will apply Steinberg's Tensor Product Theorem, which states that an irreducible representation of a simply-connected quasisimple group of Lie type is a tensor product of Frobenius twists of so-called restricted representations ([31, Theorem 5.4.5]). For the exceptional covering groups, which are not of the simply-connected type, we note that the ℓ-part of the Schur multiplier of M_0 belongs to the kernel of the representation. From the list of Schur multipliers of groups of Lie type (see Tables 2.3–2.5), we find that dividing out the said ℓ-part, the remaining part is just the regular Schur multiplier. (We may exclude the covering group of $C_2(2)' \cong A_6$.) It follows that any representation of an exceptional covering group is a lift of a representation of the simply-connected group.

Imitating [16], we now divide the non-geometrical subgroups into five subfamilies according to the nature of the socle M_0 and the representation:

- $\mathcal{S}_1$ The socle is an alternating group.
- $\mathcal{S}_2$ The socle is a sporadic group.
- $\mathcal{S}_3$ The socle is a group of Lie type with defining characteristic not dividing q.
- $\mathcal{S}_4$ The socle is a group of Lie type with defining characteristic dividing q, and the representation is not restricted.
- $\mathcal{S}_5$ The socle is a group of Lie type with defining characteristic dividing q, and the representation is restricted.

For each subfamily $\mathcal{S}_i$, we want to bound the number of inequivalent irreducible projective representations of simple groups in that subfamily from above, as this gives a bound to the number of PΔ-conjugacy classes of maximal subgroups appearing in $\mathcal{S}_i$. As already mentioned, these projective representations of the simple groups can also be viewed as linear representations of their covering groups.

Next, we present some lemmata that will be useful for the coming calculations. The first is a result of F. Lübeck.

Lemma 6.3 ([44], Theorems 4.4 and 5.1)**.** *Let H be the universal covering group of a simple classical group, and let ρ be a restricted, absolutely irreducible representation of H in the defining characteristic. If H is of type A_r or ${}^2A_r$ for some r, then either*

$\dim(\rho) > r^3/8$, *or otherwise* ρ *is the natural representation or its dual, or one of at most 3 dual pairs of representations. If* H *is of another classical type of rank* r*, then one of the following holds:*

(a) *If* $r > 11$*, then either* $\dim(\rho) > r^3$*, or otherwise* ρ *is the natural representation or one of at most 2 representations.*
(b) *If* $r \leq 11$*, then then either* $\dim(\rho) > r^3$*, or otherwise* ρ *is the natural representation or* $\dim(\rho)$ *is one of at most 5 representations.*

The second lemma gives some information on modular representations of alternating groups and their covers.

Lemma 6.4. *Let* $\tilde{A}_d$ *denote the universal covering group of a simple alternating group* A_d*. The number of* d *such that there exists an irreducible* ℓ*-modular representation of* $\tilde{A}_d$ *of dimension* n *is less than* $2\sqrt{n} + 4.33\ln(4n) + 14$*.*

Proof. For dimensions $n \leq 250$, Hiss and Malle have listed all the representations of finite quasisimple groups in [21], so the claim can be readily checked in this case.

Assume that $n > 250$. When an irreducible representation of the symmetric group S_d is restricted to A_d, it may stay irreducible or split into two irreducible representations of half the dimension. Define $f(d) = \frac{1}{2}(d-1)(d-2)$. By [28, Theorem 7], there are only three ℓ-regular partitions that correspond to non-trivial irreducible ℓ-modular representations of S_d of dimension less than $f(d)$, provided that $d \geq 15$. Depending on ℓ, there are two possibilities for the dimension of each of these representations (see [28, Appendix]). Taking also into account the possibility that some of these exceptional representations may split in A_d, we can say that there are at most 12 possible values for the dimension of an irreducible representation of A_d below $f(d)/2$.

Now, assuming that n is a dimension of an irreducible representation of A_d, at least one of the following must hold:

(1) $n \geq f(d)/2$
(2) $d \geq 15$ and the value of n is one of 12 possibilities
(3) $d < 15$.

On the other hand, having $d < 15$ without $n \geq f(d)/2$ contradicts our assumption that $n > 250$. Hence either (1) or (2) holds, so the number of possible values of d, for which there exists an n-dimensional irreducible modular representation of A_d, is at most 12 plus the number of d such that $f(d)/2 \leq n$. As f is strictly increasing, the sum is at most $f^{-1}(2n) + 12 < 2\sqrt{n} + 14$.

Let us then look at faithful representations of $\tilde{A}_d$. We may assume that $d > 9$, since otherwise all irreducible representations of $\tilde{A}_d$ have dimension less than 250. From [61], we find that the smallest dimension of a faithful irreducible representation of $\tilde{A}_d$ is at least $g(d) = 2^{\lfloor (d-s-1)/2 \rfloor}$, where s is the number of 1's in the binary representation of d. Estimating $g(d) > 0.25 \cdot 1.26^d$ (cf. proof of Lemma 3.22 on page 79), we find that the number of d, such that there exists a faithful n-dimensional irreducible representation of $\tilde{A}_d$, is less than $4.33\ln(4n)$. The claim follows. ∎

The final lemma deals with some of the exceptional automorphisms.

Lemma 6.5. *Let G and G_0 be as above.*

(a) Suppose $G_0 = \mathrm{Sp}_4(2^k)$ and $G \not\leq \mathrm{P}\Gamma$. The number of G-conjugacy classes of maximal subgroups M of G, such that $G_0 \not\leq M$ and M does not appear in Aschbacher's families $\mathcal{C}_5$ or $\mathcal{S}$, is at most 5.

(b) Suppose $G_0 = \mathrm{P}\Omega_8^+(q)$. The number of G-conjugacy classes of maximal subgroups M of G, such that $G_0 \not\leq M$ and M does not appear in class $\mathcal{C}_5$, is at most 44.

Proof. (a) In [1, Section 14], a new system of families $\mathcal{C}_1'$–$\mathcal{C}_5'$ is given, which contains all maximal subgroups not included in $\mathcal{S}$. Moreover, $\mathcal{C}_4'$ is the same family as the old family $\mathcal{C}_5$. It is then proved that there is only one $\mathrm{Aut}(G_0)$-conjugacy class in each of $\mathcal{C}_1'$, $\mathcal{C}_3'$ and $\mathcal{C}_5'$. In $\mathcal{C}_2'$, there are at most two $\mathrm{Aut}(G_0)$-classes. In fact, the proof also shows that the classes do not split under G. This gives the result.

(b) In this case, the number of conjugacy classes can be counted from [32, Section 1.5], where P. Kleidman lists all conjugacy classes of maximal subgroups of G. □

6.3. Proving the main result on finite classical groups

Now we are in a position to prove Theorem 6.1. We retain the notation from the previous section.

Proof of Theorem 6.1. We assume that G is not a group mentioned in parts (a) or (b) of Lemma 6.5. These cases are dealt with at the end of the proof. The only other groups G containing exceptional automorphisms are the automorphism groups of $G_0 = \mathrm{PSL}_n(q)$. For these groups, Aschbacher presents another family, $\mathcal{C}_1'$, that contains the additional maximal subgroups. We will follow the example of [31] and include the new family $\mathcal{C}_1'$ in $\mathcal{C}_1$ in this case.

In accordance with the discussion in the previous section, we want to bound the number of G_0-conjugacy classes of groups $M \cap G_0$, where M is a maximal subgroup of G. We will examine each Aschbacher family separately, bounding the number of conjugacy classes with M belonging to that family. Afterwards, all the bounds will be added together. Let us first look at the geometrical families.

Subfield family. Family $\mathcal{C}_5$ contains groups defined over a subfield of $\mathbb{F}_q$ of prime index. By [15, Lemma 2.1], the number of $\mathrm{P}\Delta$-conjugacy classes of maximal subgroups (intersected with G_0) is at most $\log_2 \log_2 q + 1$. We need to add a factor of σ to account for the splitting of $\mathrm{P}\Delta$-classes under G_0, as explained above, so the total number of the G_0-conjugacy classes of maximal subgroups of the subfield family becomes at most $\sigma(\log_2 \log_2 q + 1)$.

Other geometrical families. We refer again to [15, Lemma 2.1]. There, an upper bound to the number of $\mathrm{P}\Delta$-classes of maximal subgroups of non-subfield type is given as

$$\frac{3n}{2} + 4d(n) + \pi(n) + 3\log_2 n + 8,$$

where $d(n)$ is the number of divisors of n, and $\pi(n)$ the number of prime divisors. (This also includes the family $\mathcal{C}_1'$.) Using the crude estimate $\pi(n) \leq d(n) \leq \log_2 n$ and multiplying by σ to account for splitting under G_0, we find that the number of G_0-classes of maximal subgroups is at most $\sigma(\frac{3}{2}n + 8\log_2 n + 8)$ in this case.

Let us then turn to the exceptional family $\mathcal{S}$. We will deal with each subfamily $\mathcal{S}_i$ separately.

Case $\mathcal{S}_1$. By Lemma 6.4 above, the number of non-isomorphic A_d that can yield an n-dimensional irreducible projective representation is at most $2\sqrt{n} + 4.33\ln(4n) + 14$. On the other hand, it was shown in Theorem 1.1.(ii) of [16] that the number $R_n(A_d)$ of n-dimensional irreducible projective representations of A_d is less than $n^{2.5}$. Hence,

$$\sum_d R_n(A_d) < 2n^3 + 4.33n^{2.5}\ln(4n) + 14n^{2.5}.$$

As explained above, the number of inequivalent irreducible projective representations can be used as an upper bound to PΔ-conjugacy classes of maximal subgroups in the family $\mathcal{S}$. We still need to take into account the splitting of PΔ-classes under G_0 by inserting a factor of σ. Hence, the number of conjugacy classes of maximal subgroups of this type is bounded by $\sigma(2n^3 + 4.33n^{2.5}\ln(4n) + 14n^{2.5})$.

Case $\mathcal{S}_2$. We divide the sporadic groups into two sets. The first one contains those for which complete information on representation degrees is available in the Atlases. We call these the *small* sporadic groups. For small sporadic groups, we list the maximal multiplicity of any representation degree of the universal covering group in Table 6.1. Adding these up will give an upper bound for the total number of n-dimensional representations of small sporadic groups, for any fixed n.

group	M_{11}	M_{12}	M_{22}	M_{23}	M_{24}	J_1	J_2	J_3	HS	McL
multiplicity	3	3	8	3	3	4	2	6	3	6

Table 6.1: Maximal multiplicities of representation degrees (over all characteristics) for the universal covering groups of small sporadic groups.

The other set contains the *big* sporadic groups. For these, we shall make a crude approximation based solely on conjugacy class numbers of the universal covering groups. These are listed in Table 6.2. Adding them up will give a bound for the total number of representations.

The smallest representation degree of a big sporadic group is at least 12 (the group is the Suzuki group, see [31, Proposition. 5.3.8]). In conclusion, we can bound the number of PΔ-classes in subfamily $\mathcal{S}_2$ by 41, when $n < 12$, and by 1988 afterwards. This needs to be multiplied by σ to account for splitting.

Case $\mathcal{S}_3$. This subfamily corresponds to cross-characteristic representations of finite quasisimple groups of Lie type. This is where our previous results are used. Namely, by Corollary 5.4, the total number of equivalence classes of such representations is less

group	J_4	He	Ru	Suz	$O'N$	Co_1	Co_2	Co_3
class number	62	33	61	210	80	167	60	42

group	Fi_{22}	Fi_{23}	Fi'_{24}	HN	Ly	Th	BM	M
class number	282	98	256	54	53	48	247	194

Table 6.2: Class numbers of the universal covering groups of big sporadic groups.

than $16n$. As before, we need to add a factor of σ to account for the splitting of the corresponding PΔ-conjugacy classes.

Case $\mathcal{S}_4$. By Steinberg's Tensor Product Theorem, the representation is a non-trivial tensor product of twisted restricted representations. Denote $G_0 = \mathrm{P}Cl_{y^s}(q)$, a simple classical group of dimension y^s over the field $\mathbb{F}_q$. Now, from Corollary 6 of [56] it follows that M normalises a classical subgroup of G_0 of the form $\mathrm{P}Cl_y(q^s)$. These subgroups are completely described in [53], and we can read from Table 1B of that work that the number of subgroups of this type is at most $a+2$, where a is the number of ways to write the dimension n as a power y^s. Thus, the number of subgroups of the form $\mathrm{P}Cl_y(q^s)$ is bounded above by $\log_2 \log_2 n + 2$. This number also bounds the number of conjugacy classes of M.

Case $\mathcal{S}_5$. In this case, the socle of M is the image in G_0 of a projective representation of a finite simple group $H = H_{r'}(q')$ of Lie type. The homomorphism is injective, being a homomorphism of a simple group, so we can identify H with the socle. The representation is characterised by a highest weight λ, as explained e.g. in [31, Section 5.4]. (For more details on this theory, see [48].) We write $V(\lambda)$ for the module of the representation.

Assume first that $G_0 = \mathrm{PSL}_n(q)$. We know that H does not preserve a non-degenerate alternating, Hermitian or quadratic form, for otherwise M would belong to Aschbacher's family $\mathcal{C}_8$ which consists of classical subgroups of G.

If $H = \mathrm{PSL}_{r'+1}(q')$, we must have $q = q'$. Namely, if it were that $q' > q$, then Proposition 5.4.6 of [31] would tell us that the representation is not restricted. On the other hand, we cannot have $q' < q$, because M is not in the subfield family $\mathcal{C}_5$.

By Lemma 6.3, either $(r')^3/8 < n$, or the representation belongs to one of at most 3 dual pairs of representations. In the former case, r' can be one of at most $2n^{1/3}$ possibilities, and by [16, Theorem 1.1.(i)], $\mathrm{PSL}_{r'+1}(q)$ has at most $n^{3.8}$ restricted representations of degree n. This gives altogether at most

$$2n^{1/3+3.8} + 3$$

conjugacy classes of subgroups under PΔ.

If $H = \mathrm{PSU}_{r'+1}(q')$, there is no corresponding subgroup of G. Namely, it follows from Propositions 5.4.2 and 5.4.3 presented in [31], that the dual module $V(\lambda)^*$ is isomorphic to either $V(\lambda)$ or $V(\lambda)^{\psi}$, where ψ is the involutory automorphism of $\mathbb{F}_q = \mathbb{F}_{(q')^2}$. In the first case the group would fix a non-degenerate bilinear form, and in the latter case it would fix a Hermitian form (see [31, Lemma 2.10.15]). Similarly, H cannot be of type B or C as groups of these types have only self-dual representations.

If H is of one of the two remaining orthogonal types, there are at most 2 possibilities for q' (either $q' = q$ or possibly $q' = q^{1/2}$). Here, Lemma 6.3 tells us that either $(r')^3 < n$ or n is one of at most 5 different possibilities. In the former case, there are at most $n^{1/3}$ possibilities for r', and by [16, Theorem 1.1.(i)], the number of restricted n-dimensional representations of $H_{r'}(q')$ is at most $n^{2.5}$. Thus the number of PΔ-conjugacy classes of maximal subgroups of types D and ${}^2D$ is at most $2(2n^{1/3+2.5} + 5)$.

For the exceptional Lie types, only E_6 and ${}^2E_6$ have other than self-dual representations ([31, Proposition 5.4.3]). There are at most two possibilities for q' with the twisted type, and the number of restricted n-dimensional representations of each group is at most $n^{2.5}$ by [16, Theorem 1.1.(i)]. Hence, the number of PΔ-conjugacy classes is at most $3n^{2.5}$.

As a conclusion, the number of PΔ-conjugacy classes has been bounded in the case $G_0 = \mathrm{PSL}_n(q)$. To bound the number of G_0-classes, we multiply the original bound by σ, which in this case equals n. Hence, we get the following bound for the number of G_0-classes:

$$b_1(n) = 2n^{5.14} + 4n^{3.84} + 3n^{3.5} + 13n.$$

Similar analysis can be performed when G_0 is of any other classical type. In the case $G_0 = \mathrm{PSU}_n(q)$, the main difference is that we cannot have $H = \mathrm{PSL}_{r'+1}(q')$. This is because we would need the dual module $V(\lambda)^*$ to be isomorphic to $V(\lambda)^\psi$, where ψ is the involutory automorphism of $\mathbb{F}_{q^2}$, but this would lead to an impossible equation on the weights as λ is restricted. Instead, we may well have $H = \mathrm{PSU}_{r'+1}(q')$, and the estimates lead to the same bound $b_1(n)$ as in the previous case.

On the other hand, when G_0 is neither linear nor unitary, we have $\sigma \leq 8$, and we can make the estimate simpler by assuming that H can be of any Lie type and saying that there are always at most two possibilities for the value of q'. The full details are omitted, but the intermediate results are gathered in the following table:

H	no. of Lie types	bound
$\mathrm{PSL}_{r'}(q')$	1	$4n^{1/3+3.8} + 3$
$\mathrm{PSU}_{r'}(q')$	1	$4n^{1/3+3.8} + 3$
other classical	4	$2n^{1/3+2.5} + 5$
exceptional	10	$2n^{2.5}$

Multiplied by 8, the bound becomes

$$b_2(n) = 64n^{4.14} + 64n^{2.84} + 160n^{2.5} + 208.$$

For $n \geq 32$, we have $b_1(n) > b_2(n)$. When $n < 32$, we can use Lemma 6.3 together with additional data given in [44] to bound the number of representations in the same way as above. Firstly, if $H_{r'}(q')$ is of type A or ${}^2A$, we must have $r' \leq 6$, and we can check from the tables in [44, Appendix A] that there are at most 5 representations of any particular dimension for both types (counting dual representations only once). For ^2A, there are two possibilities for q, so that we get at most 15 representations of the same degree.

Similarly, for the other classical types, we have r' at most 3, so the types D can ^2D can be left out. We get at most 4 representations of the same degree. For the exceptional types, we get at most 8 representations of the same degree, of which 4 come from types E_6 and $^2E_6$ and 4 from G_2 and $^2G_2$. Saying, for simplicity, that q' can have at most 2 values for each exceptional type, we get 16 representations. Thus, there are altogether at most 35 restricted representations of any degree n less than 32, so there are at most 35σ conjugacy classes under G_0, when $n < 32$.

Conclusion. Continue assuming that the automorphism group of G is of generic type (that is, $\mathrm{Aut}(G) = \mathrm{P}\Gamma$). Suppose first that $n \geq 32$, so that $\sigma \leq n$ for all types of G. Using this estimate for σ, we collect into Table 6.3 the partial results obtained for each Aschbacher family $\mathcal{C}_i$ and subfamily $\mathcal{S}_i$. Adding up the partial results, it is easy to check computationally that the bound given in the statement of the theorem holds for $n \geq 32$.

family	bound
subfield family $\mathcal{C}_5$	$n(\log_2 \log_2 q + 1)$
other geometrical families	$\frac{3}{2}n^2 + 8n \log_2 n + 8n$
$\mathcal{S}_1$	$2n^4 + 4.33n^{3.5} \ln(4n) + 14n^{3.5}$
$\mathcal{S}_2$	$1988n$
$\mathcal{S}_3$	$16n^2$
$\mathcal{S}_4$	$\log_2 \log_2 n + 2$
$\mathcal{S}_5$	$2n^{5.14} + 4n^{3.84} + 3n^{3.5} + 13n$

Table 6.3: Upper bounds for the number of conjugacy classes of different types of maximal subgroups of classical groups.

For smaller values of n, we apply the other bound obtained for $\mathcal{S}_5$, namely 35σ. For $n \geq 8$, we still have $\sigma \leq n$, so we may substitute $35n$ in the place of the expression in the last row of Table 6.3. Adding up, we see that the bound given in the statement of the theorem holds for $8 \leq n < 32$. Finally, P. Kleidman has in his PhD thesis ([33]) determined all maximal subgroups of classical groups in dimension at most 11, and from this we can read that the bound given in the statement holds even for $n < 8$.

It only remains to check the groups with exceptional automorphisms. For this, we use Lemma 6.5. The number of conjugacy classes of maximal subgroups of subfield type (family $\mathcal{C}_5$) is bounded by $n(\log_2 \log_2 q + 1)$, just like in the general case. Apart from these classes, the lemma gives 5 additional classes for $G_0 = \mathrm{Sp}_4(2^k)$, and 44 classes for $G_0 = \mathrm{P}\Omega_8^+(q)$. Evidently, the statement of the theorem holds in this case as well. This concludes the proof. $\square$

6.4. Description of the result on algebraic groups

For classical algebraic groups, Martin Liebeck and Gary Seitz proved in [38] a theorem that corresponds to Aschbacher's Theorem on finite classical groups. Let K be an algebraically closed field of characteristic $p \geq 0$, and let G be a classical simple algebraic group of rank r (i.e. G is either $\mathrm{SL}_{r+1}(K)$ for $r > 0$, $\mathrm{Sp}_{2r}(K)$ for $r > 1$, $\mathrm{SO}_{2r}(K)$ for $r > 3$, or $\mathrm{SO}_{2r+1}(K)$ for $r > 2$ and $p \neq 2$). Furthermore, let H be a proper closed subgroup of G. Now, the Liebeck–Seitz Theorem states that either H belongs to one of six *geometrical families* of subgroups, denoted $\mathcal{C}_1$–$\mathcal{C}_6$, or to one of two additional families, denoted $\mathcal{S}_1$ and $\mathcal{S}_2$ (see [14]). In [38], the authors also prove Aschbacher's Theorem as a corollary to the Liebeck–Seitz Theorem.

In their paper ([14]), Guralnick, Larsen and Manack state that the number of conjugacy classes of subgroups in the geometrical families $\mathcal{C}_1$–$\mathcal{C}_6$ is at most $n + 3\log n + 3$, where n is the dimension of the defining space of G. On the other hand, the family $\mathcal{S}_2$ consists of normalisers of irreducible simple algebraic subgroups, and for this family the authors prove the following: for any $\varepsilon > 0$, the number of conjugacy classes of maximal subgroups in $\mathcal{S}_2$ is $O(n^{\varepsilon})$.

The family $\mathcal{S}_1$, on the other hand, consists of normalisers of irreducible finite quasisimple groups. At the moment of publication of the paper by Guralnick et al., we had shown that the number of irreducible characteristic zero representations of finite quasisimple groups is $O(n)$, and this was used by the abovementioned authors to bound the number of conjugacy classes in $\mathcal{S}_1$.

Theorem 6.2 is now proved as follows. If G is a simple algebraic group of exceptional Lie type, the rank of G is bounded, so there is nothing to prove. Otherwise G is of classical type defined in some space of dimension n, and the above remarks show that the total number of conjugacy classes of maximal subgroups is $O(n)$. For each classical type, we have $n \sim r$, so the claim holds.

As a final remark, we note that our sharpened result in Corollary 4.4 can now be used to obtain a numerical bound for the number of conjugacy classes in family $\mathcal{S}_1$.

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