Heuristic search

Weighted A*

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October 17, 2013
Weighted A*

Weighted A* search – unifying view and application
Rüdiger Ebendt, Rolf Drechsler, 2009

Weighted A*
- Weight the heuristic to quickly direct the search.
- Save time, get bounded suboptimality in exchange.

Outline
2. Unifying view
3. Monotone heuristic
4. Approximate BDD minimization
5. Experiments
Standard A*

\[ f(q) = g(q) + h(q) \]

Finds an optimal path if \( h \) is admissible, i.e. \( h(q) \leq h^*(q) \).
Constant inflation

**WA*: constant inflation

\[ f^\uparrow(q) = g(q) + (1 + \varepsilon)h(q) \]

where \( \varepsilon \geq 0 \).

If \( h \) is admissible, then \( WA^* \) is \( \varepsilon \)-admissible, i.e.

\[ g(q) \leq (1 + \varepsilon)C^* \]

for all expanded \( q \) where \( C^* \) is the length of an optimal path.
ε-admissibility

If $h$ is admissible, then WA* is ε-admissible.

Proof.

Let $s \ldots q \ldots t$ be an optimal path where $q$ is the first node of the path in the open list. Assume a goal state $t$ is expanded. This can happen only if

$$g(t) = f(t) \leq f(q)$$
$$= g(q) + (1 + \varepsilon)h(q)$$
$$\leq g^*(q) + (1 + \varepsilon)h^*(q)$$
$$\leq (1 + \varepsilon)(g^*(q) + h^*(q))$$
$$= (1 + \varepsilon)C^*$$
Dynamic weighting

DWA*: Dynamic weighting

\[ f^{DW}(q) = g(q) + \left( 1 + \varepsilon \cdot \left[ 1 - \frac{d(q)}{N} \right] \right) h(q) \]

where

- \( d(q) \): depth of \( q \)
- \( N \): depth of optimal solution

Idea: as the search goes deeper, emphasize the heuristic less.

How do we get \( N \)?

- Sometimes known beforehand: e.g. BDD minimization.
- Generally not known: use an upper bound.
Keep the original cost function

\[ f(q) = g(q) + h(q) \]

Instead of expanding \( q \) with the smallest \( f(q) \), define

\[ \text{FOCAL} = \left\{ q \in \text{OPEN} \mid f(q) \leq (1 + \varepsilon) \cdot \min_{r \in \text{OPEN}} f(r) \right\} \]

Use another heuristic \( h_F \) to choose a minimum from FOCAL, i.e.

\[ \hat{q} = \arg \min_{q \in \text{FOCAL}} h_F(q) \]
\[ f(q) = g(q) + h(q) \]

\[ \hat{q} = \arg \min_{q \in \text{FOCAL}} h_F(q) \]

Original idea:
- \( h \) estimates solution cost
- \( h_F \) estimates remaining search effort

Suggestions for \( h_F \):
- \( h_F = h \)
- \( h_F(q) = N - d(q) \)
Unifying view

\[ f(q) = g(q) + h(q) \]

\[ \text{FOCAL} = \left\{ q \in \text{OPEN} \mid f(q) \leq (1 + \varepsilon) \cdot \min_{r \in \text{OPEN}} f(r) \right\} \]

WA* and DWA* are actually special cases of A*$_\varepsilon$

\[ h_F(q) = f^{\uparrow}(q) = g(q) + (1 + \varepsilon)h(q) \quad \text{WA*} \]

\[ h_F(q) = f^{DW}(q) = g(q) + \left(1 + \varepsilon \cdot \left[1 - \frac{d(q)}{N}\right]\right) h(q) \quad \text{DWA*} \]
Unifying view

A_\varepsilon^* is a unifying framework.

- Any result for A_\varepsilon^* follows for WA^* and DWA^*
  - e.g. \varepsilon-admissibility
- Makes the approaches comparable (same f)
Unifying view

Concern: what if weighted A* expands many $q$ with

$$C^* \leq f(q) \leq (1 + \varepsilon)C^*$$

- Could overcome the advantages of directing the search.
- General $A^*_\varepsilon$ makes no guarantees.
- For WA* and DWA* this happens relatively rarely.
Monotone heuristic

If for every state $q$ and its descendant $q'$

$$h(q) \leq c(q, q') + h(q')$$

the heuristic is *monotone* or *consistent*.

A* with a monotone heuristic

- When $q$ is expanded, $g(q) = g^*(q)$
- Expanded states are never reopened

Does this hold for weighted A*?
Monotone heuristic

\[ \varepsilon = \frac{1}{2} \]

\[ h_F = h \]

\[ g = \]

\[ h = 0 \]

\[ f = \]

\[ FOCAL = \{ s \} \]

\[ g = \]

\[ h = 0 \]

\[ f = \]

\[ g = \]

\[ h = 0 \]

\[ f = \]

\[ s \]

2

1

1

1

b

a

c

t

1

1

1

2

\[ g = 0 \]

\[ h = 2 \]

\[ f = 0 \]

\[ g = \]

\[ h = 1 \]

\[ f = \]
Monotone heuristic

\[ \varepsilon = \frac{1}{2} \]
\[ h_F = h \]

\[
\begin{align*}
g & = 2 \\
h & = 0 \\
f & = 2
\end{align*}
\]

FOCAL = \{ a, b \}

\[
\begin{align*}
g & = 0 \\
h & = 2 \\
f & = 0
\end{align*}
\]

\[
\begin{align*}
g & = 1 \\
h & = 1 \\
f & = 2
\end{align*}
\]

\[
\begin{align*}
g & = 2 \\
h & = 0 \\
f & = 2
\end{align*}
\]

\[
\begin{align*}
g & = \frac{1}{2} \\
h & = \frac{1}{2} \\
f & = \frac{1}{2}
\end{align*}
\]
Monotone heuristic

\[ \varepsilon = \frac{1}{2} \]
\[ h_F = h \]

FOCAL = \{ b, c \}

\begin{align*}
g &= 2 \\
h &= 0 \\
f &= 2
\end{align*}

\begin{align*}
g &= 3 \\
h &= 0 \\
f &= 3
\end{align*}

\begin{align*}
g &= h = 0 \\
f &=
\end{align*}
Monotone heuristic

\[ \varepsilon = \frac{1}{2} \]
\[ h_F = h \]

\[ g = 2 \]
\[ h = 0 \]
\[ f = 2 \]

FOCAL = \{ b \}

\[ g = 4 \]
\[ h = 0 \]
\[ f = 4 \]
Monotone heuristic

\[ \varepsilon = 1/2 \]
\[ h_F = h \]

\[ g = 2 \]
\[ h = 0 \]
\[ f = 2 \]

FOCAL = \{ \text{c} \}

\[ g = 2 \]
\[ h = 0 \]
\[ f = 2 \]

\[ g = 4 \]
\[ h = 0 \]
\[ f = 4 \]
Monotone heuristic

\[ \varepsilon = \frac{1}{2} \]

\[ h_F = h \]

\[ g = 2 \]
\[ h = 0 \]
\[ f = 2 \]

FOCAL = \{ t \}

\[ g = 3 \]
\[ h = 0 \]
\[ f = 3 \]

\[ g = 0 \]
\[ h = 2 \]
\[ f = 0 \]

\[ g = 1 \]
\[ h = 1 \]
\[ f = 2 \]
Monotone heuristic

\[ \varepsilon = \frac{1}{2} \]

\[ h_F = h \]

\[ g = 2 \]
\[ h = 0 \]
\[ f = 2 \]

FOCAL = \( \emptyset \)

\[ g = 2 \]
\[ h = 0 \]
\[ f = 2 \]

\[ g = 3 \]
\[ h = 0 \]
\[ f = 3 \]
Monotone heuristic

Turns out no. However, we do get the bound

\[ g(q) \leq (1 + \varepsilon)g^*(q) + \varepsilon \cdot h(q) \]

for all expanded \( q \).

- Weighted A* benefits less from a monotone heuristic.
- Reopening may increase running times significantly.
Without reopening

What if we don’t reopen states? Simply ignore any new better path.

Turns out the $C \leq (1 + \varepsilon) C^*$ bound no longer holds. Instead, we can show $C \leq (1 + \varepsilon) \left\lfloor \frac{N}{2} \right\rfloor C^*$ where $N$ is the depth of the optimal solution.

Intuition: shortcutting requires always at least two states. Each shortcut accumulates the error by a factor of $(1 + \varepsilon)$.

WA* and DWA* are still $\varepsilon$-admissible without reopening.
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\[
C \leq (1 + \varepsilon)^{\lfloor N/2 \rfloor} C^*
\]

where $N$ is the depth of the optimal solution.
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[WA\ast and DWA\ast are still $\varepsilon$-admissible without reopening.]

K. Kangas ()
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- Each shortcut accumulates the error by a factor of $(1 + \varepsilon)$.
- WA* and DWA* are still $\varepsilon$-admissible without reopening.
Experiments

All variants were evaluated on a number of problems:

- BDD minimization
- Blocksworld
- Sliding-tile puzzle
- Depots
- Logistics
- PSR
- Satellite
- Freecell
- Driverlog
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Boolean functions

A Boolean function is a function

$$f : \{0, 1\}^n \rightarrow \{0, 1\}$$

Can be represented as a table, e.g. $n = 3$:

<table>
<thead>
<tr>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$Y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\rightarrow$ 1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\rightarrow$ 0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\rightarrow$ 0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\rightarrow$ 0</td>
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<tr>
<td>1</td>
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<td>$\rightarrow$ 1</td>
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<tr>
<td>1</td>
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<td>1</td>
<td>$\rightarrow$ 0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\rightarrow$ 1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$\rightarrow$ 1</td>
</tr>
</tbody>
</table>
Boolean decision diagrams
Boolean decision diagrams

\[ X_1 = 1 \]
\[ X_2 = 0 \]
\[ X_3 = 1 \]
\[ Y = 0 \]
Boolean decision diagrams

\[ X_1 = 0 \]
\[ X_2 = 0 \]
\[ X_3 = 0 \]
\[ Y = 1 \]
Boolean decision diagrams

Several applications
- Model checking
- Sparse-memory applications
- Planning
- Symbolic heuristic search
- Enhancing heuristic search (e.g. $A^*$)

In general we want BDDs to be as small as possible.
- Easier to read
- Take less memory
- Faster to evaluate
Boolean decision diagrams

BDDs are not unique and can often be simplified.
Boolean decision diagrams

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Boolean decision diagrams

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Boolean decision diagrams

- For a fixed permutation of variables, applying merge and deletion iteratively yields a minimal BDD.

- However, the permutation determines how small BDDs we can achieve.
Boolean decision diagrams

Optimal BDD for permutation $X_1, X_2, X_3$. 
Boolean decision diagrams

Optimal BDD for permutation $X_2, X_1, X_3$ (and $X_2, X_3, X_1$)
Boolean decision diagrams

BDD minimization problem: find an ordering of variables that yields a minimal BDD (least nodes)

- NP-hard (decision version is NP-complete)
- Can be solved exactly in $O(3^n n)$.
- Can often be solved fast with heuristic search.
Boolean decision diagrams

In particular, we can formulate it as a path search problem.

- A state is a set of variables whose position in the order has been fixed.
- Each transition fixes the position of a variable.
- A path of length $k$ defines the first $k$ variables in the ordering.
Variable orderings

A permutation corresponds to a path in the subset lattice.
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A permutation corresponds to a path in the subset lattice.
Variable orderings

A permutation corresponds to a path in the subset lattice: 4,1,3,2
Variable orderings

Finding an optimal path is equivalent to finding an optimal BDD.

\( g: \) For a path of length \( k \), the size of the first \( k \) levels of the BDD.

\( h: \) The number of cofactors: a lower bound on the size of the BDD.

Weighted A* used for approximate BDD mimimization.
Experiments

Experimental results.
Questions?