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Counting Linear Extensions of Sparse Posets

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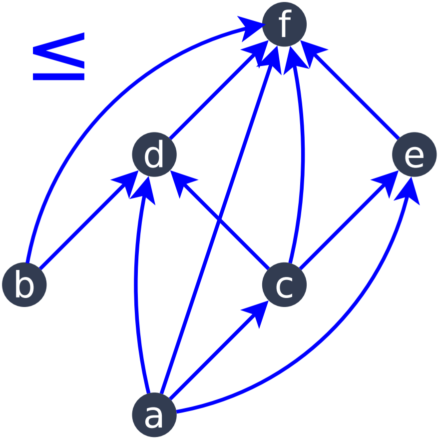
Partially ordered set (poset)

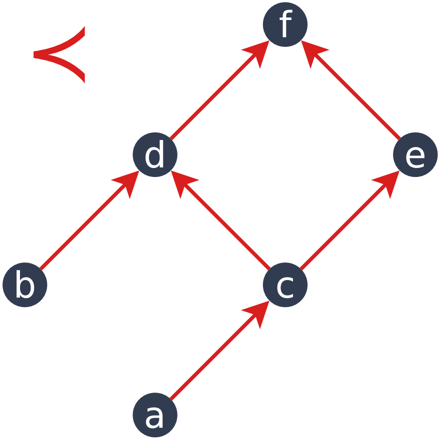
A set *P* with an antisymmetric, transitive relation



Partially ordered set (poset)

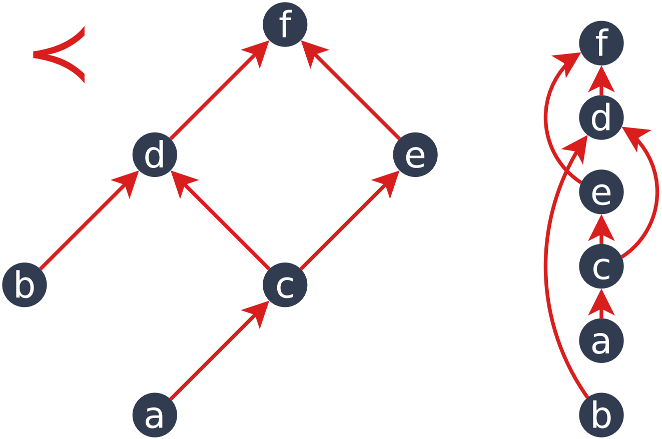
Cover graph

$\prec$



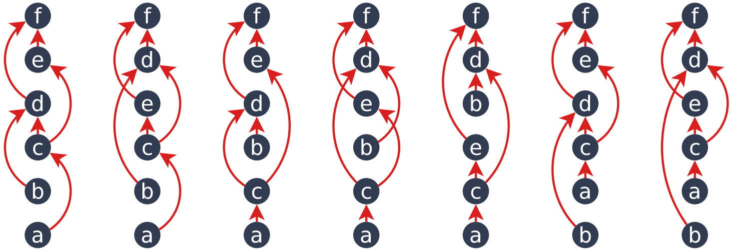
Linear extensions

A *linear extension* = order preserving permutation



Counting linear extensions

#P-complete (Brightwell & Winkler, '91)



Applications: sequence analysis, preference reasoning, sorting, learning probabilistic models, ...

Counting linear extensions

Currently we can do $O(2^n n)$ time for n elements.

We give two algorithms, based on

1. recursion (exploiting low connectivity)
2. variable elimination (exploiting low treewidth)

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Recursive counting

$\ell(P)$ = the number of linear extensions of poset P

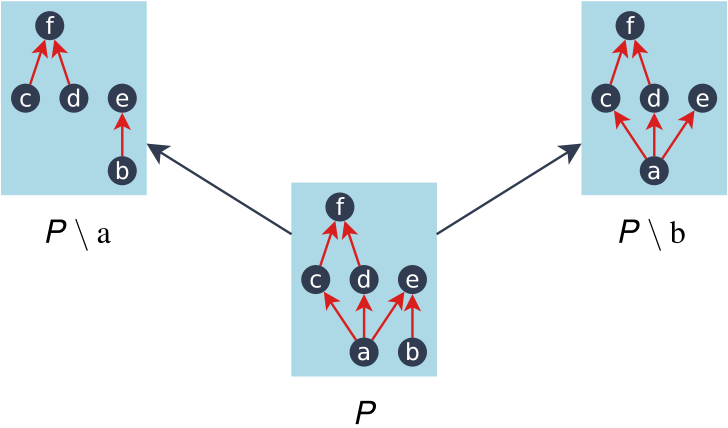
Rule 1 $$\ell(P) = \sum_{x \in \min(P)} \ell(P \setminus x)$$

Rule 2 $$\ell(P) = \sum_{(D,U)} \ell(D) \cdot \ell(U)$$

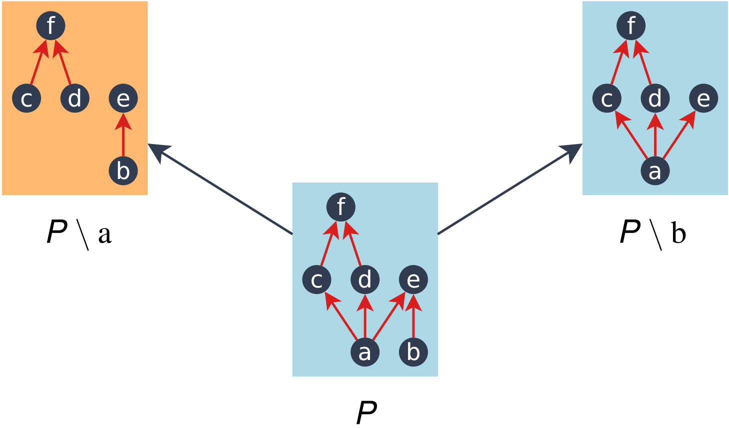
Rule 3 $$\ell(P) = \prod_{i=1}^{k} \ell(S_i)$$

Rule 4 $$\ell(P) = \ell(A) \cdot \ell(B) \cdot \binom{|P|}{|A|}$$

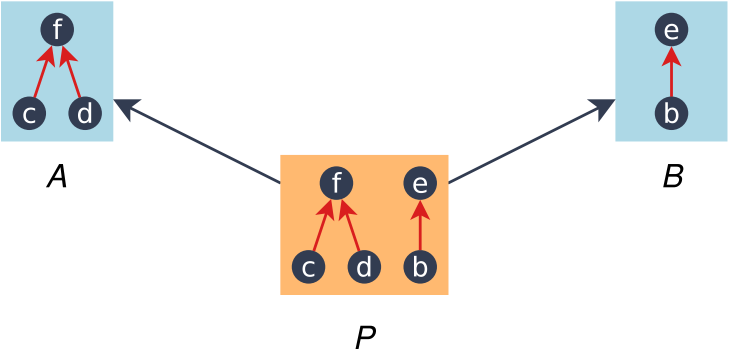
Recursive counting

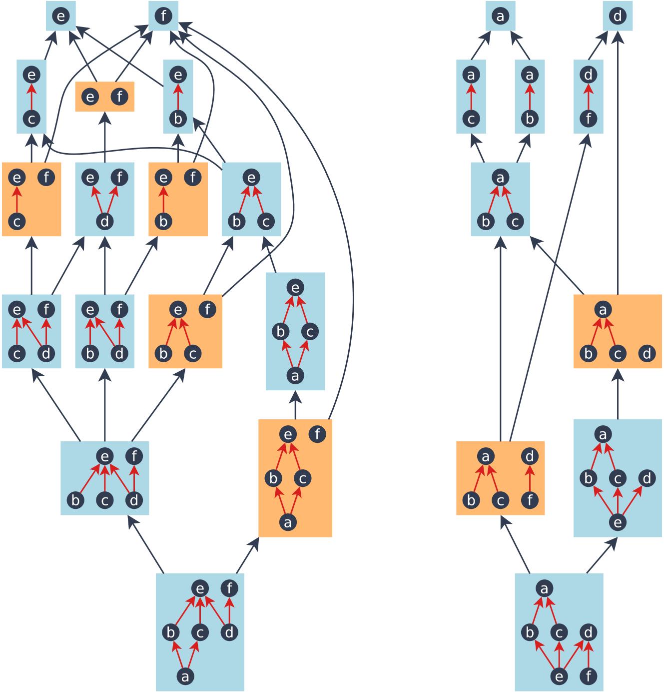


Recursive counting



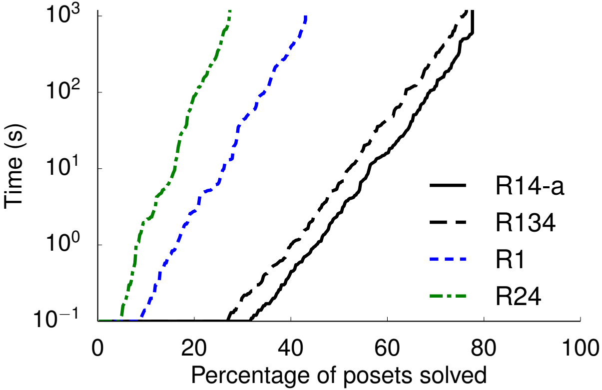
Recursive counting





Recursive counting

Experiments on sparse posets for $n = 30, \ldots, 100$



Counting linear extensions

Currently we can do $O(2^n n)$ time for n elements.

We give two algorithms, based on

1. recursion (exploiting low connectivity)
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Variable elimination

$$\sum_{a,b,c,d,e,f} \phi_1(a,b,d)\,\phi_2(a,c)\,\phi_3(b,c,e)\,\phi_4(d,f)$$

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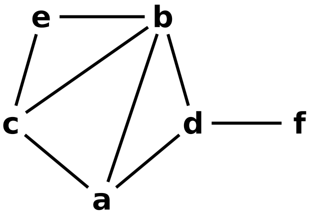
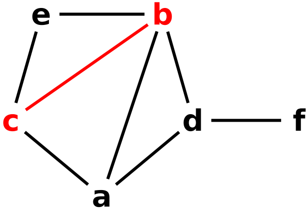


Variable elimination

$$\sum_{a,b,c,d,e,f} \phi_1(a,b,d)\ \phi_2(a,c)\ \phi_3(b,c,e)\ \phi_4(d,f)$$



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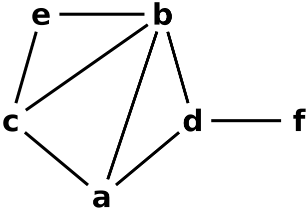

Polynomial time for bounded **treewidth**

Variable elimination

For every permutation $\sigma : P \to [n]$ define

$$\Phi(\sigma) = \prod_{x \prec y} [\sigma_x < \sigma_y]$$

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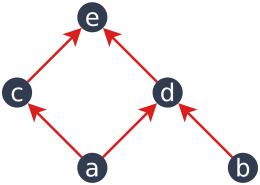

$$\Phi(\sigma) = [\sigma_a < \sigma_c]\,[\sigma_a < \sigma_d]\,[\sigma_b < \sigma_d]\,[\sigma_c < \sigma_e]\,[\sigma_d < \sigma_e]$$

Variable elimination

For every permutation $\sigma : P \to [n]$ define

$$\Phi(\sigma) = \prod_{x \prec y} [\sigma_x < \sigma_y]$$

Then,

$$\Phi(\sigma) = \begin{cases} 1, \text{ if } \sigma \text{ is a linear extension,} \\ 0, \text{ otherwise.} \end{cases}$$

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As a consequence

$$\ell(P) = \sum_{\substack{\sigma : P \to [n] \\ \text{bijection}}} \Phi(\sigma)$$

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Can't apply variable elimination because of the bijectivity constraint.

Variable elimination

$$\ell(P) = \sum_{\substack{\sigma : P \to [n] \\ \text{bijection}}} \Phi(\sigma)$$

$$= \sum_{X \subseteq [n]} (-1)^{n-|X|} \sum_{\sigma : P \to X} \Phi(\sigma)$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sum_{\sigma : P \to [k]} \Phi(\sigma)$$

Inclusion–exclusion principle

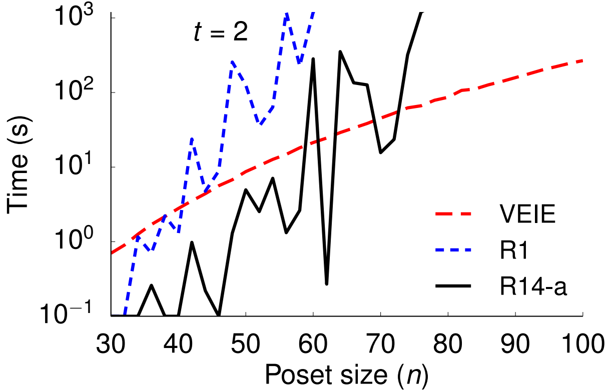
Variable elimination

$$\begin{aligned}
\ell(P) &= \sum_{\substack{\sigma : P \to [n] \\ \text{bijection}}} \Phi(\sigma) \\
&= \sum_{X \subseteq [n]} (-1)^{n-|X|} \sum_{\sigma : P \to X} \Phi(\sigma) \\
&= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \sum_{\sigma : P \to [k]} \Phi(\sigma)
\end{aligned}$$

$O(n^{t+4})$ time for treewidth t

Variable elimination

VEIE: Variable elimination via inclusion–exclusion



Summary

- **Recursion:** often fast in practice
- **Variable elimination:** polynomial time for bounded treewidth

Thank you!