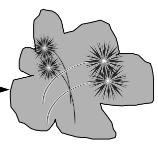
Lecture 7
Analytical Solution Method for Complex Models

Multiple Class Markov Chain Models

Convolution

Generic Solution Method

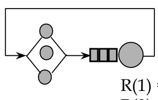
Model Parameters →  → Performance Metrics

analytical methods
simulation
  queueing network
  Petri net

exact values?
estimates?
confidence intervals?
bounds?

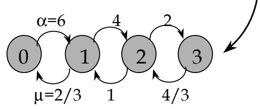
Markov Chain Solution

- Birth-Death process
- Stochastic process
- Large state space?
- Large normalizing constant?



M=3, Z = 0.5
U, R, X, N_{sys}?

R(1) = 1.5
R(2) = 1.0
R(3) = 0.75

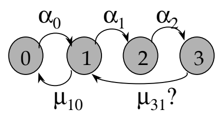


Birth-Death Process

Prob = P = 0.01   0.09   0.36   0.54
U = 1-P0 = 0.99, $X = \Sigma\mu_i P_i$ = 1.14 tps
$N_{sys} = \Sigma i P_i$ = 2.43, R = N/X = 2.13 sec

Markov Chain Solution

- 1. State description
  - finite, infinite? state space?
  - multiple classes? multiple phases?
- 2. State enumeration
- 3. Transition rates
- Balance equations
  - flow in to state = flow out of state, $\Sigma P_i = 1$
- Solve balance equations
- Use P_i's to get performance metrics $U = 1 - P_0$



Markov Example

- Central server model, Fig. 5.1 [Men 94]
- State = (n_c, n_f, n_s), $\Sigma n_i = 2$ ← number of jobs in CPU
- State enumeration: $n_i \in \{0, 1, 2\}$
  S = { (2,0,0), (1,1,0), (1,0,1), (0,2,0), (0,1,1), (0,0,2) }
  - state space S can be large (<u>very</u> large)
  - K devices, max mpl=N

$$|S| = \binom{N-K-1}{K-1} = \frac{(N-K-1)!}{N!\ (K-1)!}$$

Markov Example (contd)

- State space diagram, Fig. 5.2
- Transition rates

Rate((1,1,0) -> (0,2,0)) = μ_f = 3 ← cpu completes, request fast disk
Rate((0,2,0) -> (1,1,0)) = μ_c p = 6 * 0.5 = 3 ← fast disk completes
Rate((2,0,0) -> (1,0,1)) = μ_c (1-p) = 6 * 0.5 = 3
Rate((1,0,1) -> (2,0,0)) = μ_s = 2
….

Markov Example (contd)

- Notation P_{011} = Prob { in state (0,1,1) }
- Local balance Fig 5.3 [Men 94]
- Global balance equations

$\mu_c(1\text{-}p)P_{110} + \mu_c p P_{101} = (\mu_s + \mu_f)\, P_{011}$
…
"similarly for some other 4 states"
…
$\sum$ Pi = 1

flow in = flow out

one global balance equation is redundant!

Markov Example (contd)

- Solve balance equations
  - brute force approach
  - 6 equations, 6 unknowns, OK
  - 92378 equations, 92378 unknowns, ????
  - not always practical,
  - can be very time consuming
- Better method: transform set of equations into simpler form: local balance equations

Local Balance Equations

- Equations for local balanced transitions between neighboring states
- Each equation in terms of
  - relative device utilizations ← from transition probabilities and service rates
  - normalizing constant
    - needs to be computed
  - can be tricky
    - can be time consuming
    - "the miracle occurs here"

Relative Utilization (2)

$$\mu_f P_{110} = \boxed{\mu_c p}\, P_{200}$$
$$\mu_s P_{101} = \mu_c (1-p)\, P_{200}$$
$$\mu_f P_{020} = \mu_c p\, P_{110}$$
$$\mu_s P_{011} = \mu_c (1-p)\, P_{110}$$
$$\mu_f P_{011} = \mu_c p\, P_{101}$$
$$\mu_s P_{002} = \mu_c (1-p)\, P_{101}$$

$\sum$ P_i = 1

?

Notation:
"Relative Utilization"

$$U_f = \frac{\mu_c p}{\mu_f}$$
$$U_s = \frac{\mu_c (1-p)}{\mu_s}$$
$$U_c = 1$$

(relative to CPU, serv time & util are inversely relative to μ)

Modified Local Balance Equs (1)

$$\mu_f P_{110} = \mu_c p\, P_{200}$$
$$\mu_s P_{101} = \mu_c (1-p)\, P_{200}$$
$$\mu_f P_{020} = \mu_c p\, P_{110}$$
$$\mu_s P_{011} = \mu_c (1-p)\, P_{110}$$
$$\mu_f P_{011} = \mu_c p\, P_{101}$$
$$\mu_s P_{002} = \mu_c (1-p)\, P_{101}$$

$$U_f = \frac{\mu_c p}{\mu_f} \qquad U_s = \frac{\mu_c (1-p)}{\mu_s} \qquad U_c = 1$$

$$P_{110} = U_f P_{200}$$
$$P_{101} = U_s P_{200}$$
$$P_{020} = U_f P_{110} = U_f^2 P_{200}$$
$$P_{011} = U_s P_{110} = U_s U_f P_{200}$$
$$P_{011} = U_f P_{101} = U_f U_s P_{200}$$
$$P_{002} = U_s P_{101} = U_s^2 P_{200}$$

Normalizing Constant (2)

$$P_{110} = U_f P_{200}$$
$$P_{101} = U_s P_{200}$$
$$P_{020} = U_f P_{110} = U_f^2 P_{200}$$
$$P_{011} = U_s P_{110} = U_s U_f P_{200}$$
$$P_{011} = U_f P_{101} = U_f U_s P_{200}$$
$$P_{002} = U_s P_{101} = U_s^2 P_{200}$$

= 1

$$P_{110} = \boxed{U_c^1}\, U_f^1 U_s^0 P_{200}$$
$$P_{101} = U_c^1 U_f^0 U_s^1 P_{200}$$
$$P_{020} = U_c^0 U_f^2 U_s^0 P_{200}$$
$$P_{011} = U_c^0 U_f^1 U_s^1 P_{200}$$
$$P_{002} = U_c^0 U_f^0 U_s^2 P_{200}$$
$$\sum P_i = 1$$

$$P_{200} = \frac{1}{\sum_{\substack{t \in S \\ t=(i,j,k) \\ i+j+k=2}} U_c^i U_f^j U_s^k} = \frac{1}{G(2)}$$

← normalizing constant

2 jobs

Local Balance Equations (contd)

$$P_{ijk}(2) = \frac{U_c^i U_f^j U_s^k}{G(2)}$$ when mpl = i+j+k=2

$$P_{ijk}(N) = \frac{U_c^i U_f^j U_s^k}{G(N)}$$ when mpl = i+j+k=N

$$G(N) = \sum_{\substack{t \in S \\ t=(i,j,k) \\ i+j+k=N}} U_c^i U_f^j U_s^k$$

product form

sum over all states (complex)

Performance Measures (1)

CPU utilization for mpl=2 $\neq$ CPU relative utilization

$$U_c(2) = \sum_{i \geq 1} P_{ijk} = P_{110} + P_{101} + P_{200}$$

$$= \frac{\sum_{\substack{(i,j,k) \\ \boxed{i \geq 1} \\ i+j+k=2}} U_c^i U_f^j U_s^k}{G(2)} = \frac{U_c \sum_{\substack{(i,j,k) \\ i+j+k=\boxed{1}}} U_c^i U_f^j U_s^k}{G(2)} = \frac{U_c G(1)}{G(2)}$$

$$U_c(N) = U_c \frac{G(N-1)}{G(N)}$$

Performance Measures (contd) (3)

$$U_c(N) = U_c \frac{G(N-1)}{G(N)}$$

$1/S_c$

$$X_c(N) \overset{Little}{=} U_c(N)\mu_c = \mu_c U_c \frac{G(N-1)}{G(N)}$$

$$\bar{n}_c(2) = \sum_{\substack{i,j,k \\ i>0}} iP_{ijk} = \sum_{\substack{i,j,k \\ i=1}} iP_{ijk} + \sum_{\substack{i,j,k \\ i=2}} iP_{ijk} \overset{homework}{=} U_c^1 \frac{G(1)}{G(2)} + U_c^2 \frac{G(0)}{G(2)} \quad (7\text{-}1)$$

In general, $$\bar{n}_c(N) = \sum_{m=1}^{N} U_c^m \frac{G(N-m)}{G(N)}$$

Performance Measures (contd) (3)

mean device response time: $$R_c(N) \overset{Little}{=} \frac{\bar{n}_c(N)}{X_c(N)} = \frac{\sum_{m=1}^{N} U_c^{m-1} G(N-m)}{\mu_c G(N-1)}$$

mean device residence time: $$R_c'(N) = V_c R_c(N)$$

mean system response time: $$R(N) = \sum_c R_c'(N) = \sum_c V_c R_c(N)$$

mean system throughput: $$X_0(N) \overset{FFL}{=} \frac{X_c(N)}{V_c(N)}$$

How to Compute Relative U_c's and G (2)

in steady state: vector X_i · matrix P_{ij} = vector X_i

transition matrix

relative flow through each device

Solve for X, use Little to get U_c's: $$U_c \overset{Little}{=} X_c S_c = \frac{X_c}{\mu_c}$$

(many solutions for relative X's)

And, finally get $$G(N) = \sum_{\substack{t \in S \\ t=(i,j,k) \\ i+j+k=N}} U_c^i U_f^j U_s^k$$

Example (4)

(relative X_i, relative U_i)

$$(X_c, X_f, X_s) \begin{bmatrix} 0 & p & 1-p \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = (X_c, X_f, X_s)$$

$$X_f + X_s = X_c \qquad X_c = \mu_c$$
$$pX_c = X_f \qquad X_f = \mu_c p$$
$$(1-p)X_c = X_s \qquad X_s = \mu_c(1-p)$$

$$U_c = X_c / \mu_c = 1$$
$$U_f = \mu_c p / \mu_f = 0.5\mu_c / \mu_f = 1$$
$$U_s = \mu_c(1-p) / \mu_s = 0.5\mu_c / \mu_s = 1.5$$

$$G(2) = \sum_{\substack{i,j,k \\ N=2}} 1^i 1^j 1.5^k = \underset{(1,0,1)}{1.5} + \underset{(0,1,1)}{1.5} + \underset{(0,0,2)}{2.25} = 5.25$$

State Space Explosion

K	N	$\lvert S\rvert$
3	2	$\binom{4}{2} = \frac{4!}{2!2!} = 6$
10	3	$\binom{12}{9} = 220$
3	10	$\binom{12}{2} = 66$
10	10	$\binom{19}{9} = 92378$

Convolution Algorithm

- How to compute G(N)?
  - problem: sum over all states? many states...
  - Buzen: Convolution algorithm

G(N)=g(N,K)

for population N
for K devices

$$g(2,3) = \sum_{\substack{t=(i,j,k)\\ |t|=2}} U_c^i U_f^j U_s^k = \sum_{\substack{t=(i,j,k)\\ |t|=2\\ i=0}} U_c^i U_f^j U_s^k + \sum_{\substack{t=(i,j,k)\\ |t|=2\\ i>0}} U_c^i U_f^j U_s^k$$

$$= \sum_{\substack{t=(0,j,k)\\ |t|=2}} U_c^i U_f^j U_s^k + U_c \sum_{\substack{t=(i,j,k)\\ |t|=1}} U_c^i U_f^j U_s^k = g(2,2) + U_c g(1,3)$$

  - Fig 5.4 [Men 94]

Convolution Example

Use D_i's as (relative) U_i's:

	Vi	Si	Ui=ViSi
CPU	1	10	10
Fast	0.5	20	10
Slow	0.5	30	15

Convolution Example (1/3)

g	CPU 10	Fast 10	Slow 15
0	1	1	1
1	10	20	35
2			
3			
…			

G(0)

G(1)=
20+1*15

Convolution Example (2/3)

g	CPU 10	Fast 10	Slow 15
0	1	1	1
1	10	20	35
2	100	300	825
3			
…			

G(2)=
300+
15*35

Convolution Example (3/3)

g	CPU 10	Fast 10	Slow 15
0	1	1	1
1	10	20	35
2	100	300	825
3	1000	4000	16375
…			

G(3)= 4000+ 15*825

Response Time

mean device response time

$$R_c(2) = \frac{U_c^0 G(1) + U_c^1 G(0)}{\mu_c G(1)} = \frac{35+10}{\frac{1}{10}35} = \frac{45*10}{35} = 12.9\,\text{sec}$$

$$R_f(2) = \frac{U_f^0 G(1) + U_f^1 G(0)}{\mu_f G(1)} = \frac{35+10}{\frac{1}{20}35} = \frac{45*20}{35} = 25.7\,\text{sec}$$

$$R_s(2) = \frac{35+15}{\frac{1}{30}35} = \frac{50*30}{35} = 42.9\,\text{sec}$$

$$R(2) = \sum R_i^{'}(2) = \sum V_i R_i\,(2) = 47.2\,\text{sec}$$

mean system response time

Utilization & Throughput (2)

$$U_c(2) = U_c \frac{G(1)}{G(2)} = 10\frac{35}{825} = 0.424$$

$$U_f(2) = 10\frac{35}{825} = 0.424$$

$$U_s(2) = 15\frac{35}{825} = 0.636$$

true utilization (not relative utilization)

$$X_c(2) = \mu_c U_c \frac{G(1)}{G(2)} \overset{U_c = V_c S_c}{=} V_c \frac{G(1)}{G(2)} = 1*\frac{35}{825} = 0.042$$

Little: $X_0(2)\,R\,(2) = \frac{X_c(2)}{V_c} R\,(2) = \frac{0.042}{1} 47.2 = 1.98 \approx 2$ OK

Average Number of Jobs at Each Server (in Queue and in Service)

Job distribution in network

$$\bar{n}_c(2) = U_c \frac{G(1)}{G(2)} + U_c^2 \frac{G(0)}{G(2)}$$

$$= 10*\frac{35}{825} + 100*\frac{1}{825} = \frac{450}{825} = 0.545$$

$$\bar{n}_f(2) = 0.545$$

$$\bar{n}_s(2) = 15*\frac{35}{825} + 225*\frac{1}{825} = \frac{750}{825} = 0.91$$

total = 2 OK

Device Queue Lengths

Queue = aver. population – utilization:

$$\bar{n}_c(2) - U_c(2) = 0.545 - 0.424 = 0.125$$

$$\bar{n}_f(2) - U_f(2) = 0.545 - 0.424 = 0.125$$

$$\bar{n}_s(2) - U_s(2) = 0.91 - 0.636 = 0.27$$

Convolution Summary

- Jeff Buzen
  - founded BGS Corporation with two friends
  - BGS merged to BMC Software in 1998
- Novel practical way to compute normalizing constant for closed queueing networks
- Get all standard measures
  - Qi, Ni, Ui, Ri, Xi
  - Qir, Nir, Uir, Rir, Xir
  - Xsystem, Rsystem
- Practical computational problem
  - G can overflow or underflow!

(Multiple class version exists!)

26.3.2002 31