Lecture 9
Multiple Class Models

Multiclass MVA

Approximate MVA

Arrival Theorem for Multiple Classes

- With $\bar{N}$ jobs in the system, a job in class r arriving to any server i sees the server as in equilibrium with one job (himself!) removed from the system

$A_{ir}(\bar{\mathbf{N}}) = \bar{n}_i(\bar{\mathbf{N}} - \bar{\mathbf{1}}_r) = \bar{n}^A_{ir}(\bar{N})$

$(N_1, N_2, ..., N_R)$ R job classes

$(0, 0, ... ,0 , 1, 0, ..., 0)$ r[th] job class

Multiple Class MVA (4)

starting point: $A_{ir}(\bar{\mathbf{N}}) = \bar{n}_i(\bar{\mathbf{N}} - \bar{\mathbf{1}}_r) = \bar{n}^A_{ir}(\bar{N})$ $\bar{n}_{i,r}(\mathbf{0}) = 0$

residence time

$$R'_{i,r}(\bar{N}) = \begin{cases} D_{i,r}[1+\bar{n}^A_{i,r}(\bar{N})] & \textit{queueing device} \\ D_{i,r} & \textit{delay device} \end{cases} \quad \forall i,r$$

$\bar{N}+1_r$

$$X_{0,r}(\bar{N}) \underset{system}{\overset{Little}{=}} \frac{N_r}{Z_r + \sum_i R'_{i,r}(\bar{N})} \quad \forall r$$

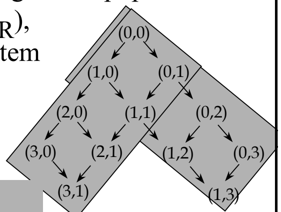
(response time law)

$$\bar{n}_{i,r}(\bar{N}) \underset{device}{\overset{Little}{=}} X_{i,r}(\bar{N}) R_{i,r}(\bar{N}) = X_{0,r}(\bar{N}) R'_{i,r}(\bar{N}) \quad \forall i,r$$

$$\bar{n}_i(\bar{N}) = \sum_r \bar{n}_{i,r}(\bar{N}) \quad \forall i$$

Multiple Class Mean Value Analysis (MVA) (2)

- Compute solutions through class population space $N=(N_1, N_2, ..., N_R)$, starting from empty system
- population (state) space can still be large!



two job classes
target population (1,3)?
target population (3,1)?
target population (6, 15, 300)?

Alg 7.2 [LZGS 84]

Example (4)

2-class closed model, Fig. 7.2 [LZGS 84]
Job classes A, B

Tbl 7.3 [LZGS 84]

(0,0) $Q_{CPU,A} = Q_{DISK,A} = Q_{CPU,B} = Q_{DISK,B} = 0$

(1,0) $R'_{CPU,A}$, $R'_{DISK,A}$, $X_{0,A}$, $Q_{CPU,A}$, $Q_{DISK,A}$

(0,1) $R'_{CPU,B}$, $R'_{DISK,B}$, $X_{0,B}$, $Q_{CPU,B}$, $Q_{DISK,B}$

(1,1) $R'_{CPU,A}$, $R'_{DISK,A}$, $R'_{CPU,B}$, $R'_{DISK,B}$, $X_{0,A}$, $X_{0,B}$, $Q_{CPU,A}$, $Q_{DISK,A}$, $Q_{CPU,B}$, $Q_{DISK,B}$

Another Simple Example

Fig. 6.1 [Men 94]

$D_{i,r}$	query r=1	update r=2
cpu i=1	0.105	0.375
d1 i=2	0.180	0.480
d3 i=3	0	0.240

Watch out for index ordering:

$D_{1,2} = D_{2,\,query}$

Population in book figures in order (Update, Query)

Simple Example (5)

update
query
(0,0) $\overline{n}_i(\overline{0}) = 0 = \overline{n}^A_{i,r}(1_r) \quad \forall i,r$

(1,0)

$R'_{1,1}(1,0) = D_{1,1}[1+0] = 0.105 \quad R'_{2,1}(1,0) = 0.180 \quad R'_{3,1}(1,0) = 0$

$X_{0,1}(1,0) = 1/(0.105+0.180) = 3.509$

$\overline{n}_{1,1}(1,0) = 3.509 * 0.105 = 0.368$

$\overline{n}_{2,1}(1,0) = 3.509 * 0.180 = 0.632 \quad \overline{n}_{3,1}(1,0) = 0$

$R'_{1,2}(1,0) = ?$ $\quad \overline{n}_{1,2}(1,0) = \overline{n}_{2,2}(1,0) = \overline{n}_{3,2}(1,0) = 0$

$\overline{n}_1(1,0) = \overline{n}_{1,1}(1,0) + \overline{n}_{1,1}(1,0) = 0.368$

$\overline{n}_2(1,0) = 0.632 \quad \overline{n}_3(1,0) = 0$

Tbl 6.3

Simple Example (contd) (6)

(0,1): $\overline{n}_1(0,1) = 0.343 \quad \overline{n}_2(0,1) = 0.438 \quad \overline{n}_3(0,1) = 0.219$

(1,1): $R'_{1,1}(1,1) = D_{1,1}\left[1+\overline{n}^A_{1,1}(1,1)\right] = D_{1,1}\left[1+\overline{n}_1(0,1)\right] = 0.105[1+0.343] = 0.141$

$R'_{2,1}(1,1) = D_{2,1}\left[1+\overline{n}_2(0,1)\right] = 0.180[1+0.438] = 0.259 \quad R'_{3,1}(1,1) = 0$

device index

$X_{0,1}(1,1) = 1/(0.141+0.258) = 2.500$

class index

total popul.

$\overline{n}_{1,1}(1,1) = 2.5 * 0.141 = 0.352$ ... $\overline{n}_{1,2}(1,1) = 0.334$ $\quad \overline{n}_1(1,1) = 0.686$

$\overline{n}_{2,1}(1,1) = 2.5 * 0.259 = 0.648$ $\quad \overline{n}_{2,2}(1,1) = 0.519$ $\quad \overline{n}_2(1,1) = 1.158$

$\overline{n}_{3,1}(1,1) = 0$ $\quad \overline{n}_{3,2}(1,1) = 0.156$ $\quad \overline{n}_3(1,1) = 0.156$

similarly for states (2,0), (3,0), (2,1), (3,1) Tbl 6.3

Simple Example (contd)

- Baseline solution — Table 6.3 [Men 94]
  - external: class *update* response time: 2.444 s
  - internal: Disk1 utilization too high — Table 6.4
- Modification: move queries to Disk2
  - move query demand from Disk 1 to Disk2
  - $D_{21} = D_{d1,query} = 0$, $D_{31} = D_{d2,query} = 0.180$

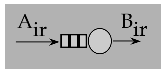
    (silly order, silly notation, sorry!)
  - solve again — Table 6.4 [Men 94]
  - class *update* response time: 1.934 s

Product Form Solution Exists: NW is Analytically Solvable

- BCMP-networks
  - Baskett, Chandy, Muntz & Palacios (1975)
- Service discipline
  - FCFS, PS, IS, LCFS-PR
- Job classes, class switching
- Service time distributions, interarrival times
  - Exponential interarrivals times for FCFS servers or for open job class
  - more generic for others (rational Laplace transformation exists)
- Load-dependent servers (LD-servers)
  - $S_{ir} = f(n_i)$ for FCFS (I.e., same for each class)
  - $S_{ir} = f(n_{ir})$ for others

Product Form Solution Exists?

- Job Flow balance
  - flow in = flow out
  - per device, per system



- One step behavior
- Device homogeneity
  - single resource possession
  - no blocking
  - independent job behavior
  - local information
  - fair service
  - routing homogeneity

$\mu_{ir} = f(n_{ir})$
$\mu_i = f(n_i)$

p_{ikr} constant
(prob for class change is load independent)

How Useful are Exact Solution Methods?

- MVA & Convolution based algorithms
- Very good for single class cases
- Might be too time consuming for multiple class cases if nr of classes or class populations are (very) large

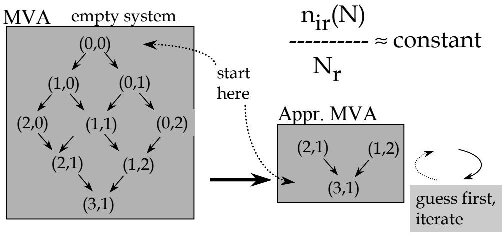
$$O(\text{MVA}) = O(K\,R \prod_r (1 + N_r))$$

K	R	N_r	#ops
3	2	2	54
5	20	1	100M
20	5	1	3200
20	5	9	10M

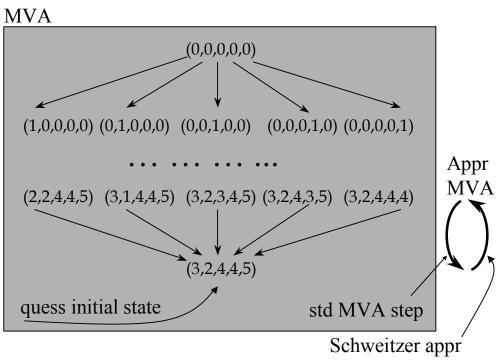
Approximate MVA (1)

- Helps to solve MVA state space problem
- Based on Schweitzer-approximation:

$$\frac{n_{ir}(N)}{N_r} \approx \text{constant}$$



MVA vs Approximate MVA



Single Class Schweitzer Approximation

$$\frac{\bar{n}_i(N)}{N} \approx \text{constant}$$

$$\therefore \frac{\bar{n}_i(N)}{N} = \frac{\bar{n}_i(N-1)}{N-1} \quad \Rightarrow \bar{n}_i(N-1) = \frac{N-1}{N}\bar{n}_i(N)$$

Single Class Approximate MVA

guess initial: $\bar{n}_i(N)$

compute: $\bar{n}_i(N-1) = \frac{N-1}{N}\bar{n}_i(N)$

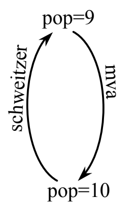
std MVA step:

$$R'_i(N) = D_i\left[1 + \bar{n}_i(N-1)\right]$$

$$X_0(N) = N \Big/ \left(Z + \sum R'_i(N)\right)$$

$$\bar{n}_i(N) = X_0(N) * R'_i(N)$$

repeat until conver-gence



Example (4)

$D_i = (10, 10, 15)$, K=3, N=2 [Fig. 5.1]

step 0 $\bar{n}_i(2) = 2/3 = 0.667 \quad \forall i$ initial guess: even distribution

step 1

$$R'_1 = D_1\left[1 + \frac{2-1}{2}\bar{n}_1\right] = 10 * 1.333 = 13.333 = R'_2 \quad R'_3 = 15 * 1.333 = 20$$

$$X_0 = \frac{2}{46.67} = 0.04286$$

$$\bar{n}_1 = 0.5714 = \bar{n}_2 \quad \bar{n}_3 = 0.8572 \quad \left(\sum \bar{n}_i = 2\right)$$

step 2

$$R'_1 = 10\left[1 + \frac{0.5714}{2}\right] = 12.857 = R'_2 \quad R'_3 = 15 * \left[1 + \frac{0.8572}{2}\right] = 21.429$$

$$X_0 = \frac{2}{47.143} = 0.042424$$

$$\bar{n}_1 = 0.5454 = \bar{n}_2 \quad \bar{n}_3 = 0.9092 \quad \left(\sum \bar{n}_i = 2\right)$$

0.5455 exact

0.9090 exact

why stop here?

Questions (3)

- Q. How to quess initial job distribution
  A: $\overline{n}_i(N) = \frac{N}{K}$ (even distribution on all devices)
- Q. Does it always converge?
  A. No. Almost always. No quarantee.
- Q. If it converges, does it converge to the right value?
  A. We hope so. It seems to do it ….

Questions (contd)

- Q. What is good measure of convergence?

  A. E.g., max relative change in $\overline{n}_i(N)$ must be less than 1%

General Stationary Iterative Method (1)

(from Numerical Linear Algebra)

Fixed point equation $\overline{n} = B\overline{n} + \overline{c}$

converges from arbitrary initial $\overline{n}^{(0)}$

if $\rho(B) = \max|\lambda_i(B)| < 1$

$\rho(B)$: spectral radius of B; $\lambda_i(B)$: i[th] eigenvalue of B

- Q. Do we check this before using approximate MVA?
  A. No.

General Theory of Iteration (1)

$\overline{n} = f(\overline{n})$

Thm:

If $\overline{n} = f(\overline{n})$ has root $\overline{\alpha}$

and $f(\overline{n})$ exists close to α, i.e., in $J = \{\overline{n} : |\overline{n} - \overline{\alpha}| < \rho\}$

and $|f'(\overline{n})| < 1 \quad \forall \overline{n} \in J$

Then (a) $\overline{n} \in J$ in each iteration

(b) $\overline{n}$ converges to $\overline{\alpha}$

(c) $\overline{\alpha}$ is the only root in J

- Q. Do we check this for approximate MVA?
  A. No.

How Good is Approximate MVA

- Pretty good for throughput and response time
- Not so good for queue lengths at heavy loads

Figs 34.1-34.3 [Jain 91]

Multiple Class Schweitzer Approximation (1)

single class:

$$\frac{\overline{n}_i(N)}{N} \approx \text{constant}$$

$$\therefore \frac{\overline{n}_i(N)}{N} = \frac{\overline{n}_i(N-1)}{N-1} \Rightarrow \overline{n}_i(N-1) = \frac{N-1}{N}\,\overline{n}_i(N)$$

multiple class:

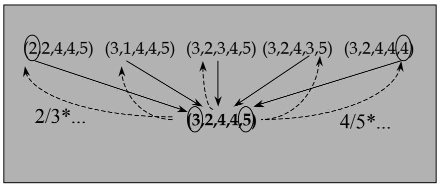
$$\frac{\overline{n}_{ir}(\overline{N})}{N_r} \approx \text{constant}$$

$$\therefore \frac{\overline{n}_{ir}(\overline{N})}{N_r} = \frac{\overline{n}_{ir}(\overline{N}-1_r)}{N_r - 1}$$

$$\Rightarrow \overline{n}_{ir}(\overline{N}-1_r) = \frac{N_r - 1}{N_r}\,\overline{n}_{ir}(\overline{N})$$

Multiple Class Approximate MVA

$$\bar{n}_{ir}(\overline{N}-1_r) = \frac{N_r - 1}{N_r}\,\bar{n}_{ir}(\overline{N})$$



Multiple Class Approximate MVA

- (1) guess initial $n_{ir}(\mathbf{N}) = \mathbf{N}_r / \mathbf{K}_r$ at target state (distribute evenly to every node visited)
- (2) compute back $n_{ir}(\mathbf{N}-\mathbf{1}_r) = (\mathbf{N}_r-\mathbf{1})/\mathbf{N}_r\, n_{ir}(\mathbf{N})$
- (3) use standard MVA step to compute new estimate of $n_i(\mathbf{N})$
- iterate steps (2) and (3) until convergence
  - usually 4-6 iterations enough
  - std output from last iteration

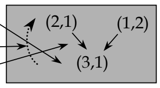


Fig. 6.5 [Men 94]

Multiple Class Approximate MVA

- Does it converge?
  - Almost always. No quarantee.
- If it converges, does it converge to the right value?
  - It seems to do good work....
- What is good measure of convergence?
  - e.g., max relative change in $n_{ir}(\overline{\mathbf{N}}) < 1\%$

Example (2)

Model: Fig. 6.1 [Men 94]

step 0

$$\bar{n}_{11}(3,1) = 1.5 \qquad \bar{n}_{12}(3,1) = 1/3 = 0.333$$
$$\bar{n}_{21}(3,1) = 1.5 \qquad \bar{n}_{22}(3,1) = 0.333$$
$$\bar{n}_{31}(3,1) = 0 \qquad \bar{n}_{32}(3,1) = 0.333$$

step 1

$$R'_{11}(3,1) = D_{11}\left[1 + \bar{n}_{11}(2,1) + \bar{n}_{12}(3,0)\right]$$
$$\overset{Schweitzer}{=} 0.105\left[1 + \frac{2}{3}\bar{n}_{11}(3,1) + \frac{0}{1}\bar{n}_{12}(3,1)\right] = 0.105\left[1 + \frac{2}{3}1.5 + 0\right] = 0.210$$
$$R'_{21}(3,1) = 0.180\left[1 + \frac{2}{3}\bar{n}_{21}(3,1) + \frac{0}{1}\bar{n}_{22}(3,1)\right] = 0.180 * 2 = 0.360$$
$$R'_{31}(3,1) = 0$$
$$R'_{12}(3,1) = \ldots \qquad R'_{22}(3,1) = \ldots \qquad R'_{32}(3,1) = \ldots$$

→ Result: Tbl 6.6, Qsolver/1 output, PMVA output

= ?

Problems with Analytical Solutions

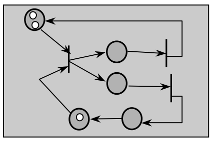
- Load dependent servers
- Memory (Figs 8.1, 8.3 [Men 94])
- Some part of system not product form
  - however, solution method is robust
- Computational problems
  - accuracy, overflow, underflow

Solution Packages (Software)

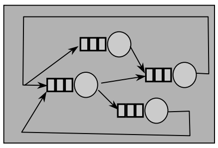
- Commercial
  - BGS, Boston, MA — Best/1
  - AT&T, Holmdale, NJ — Q+
  - SES, Austin, TX — PAWS
- University Projects
  - Jeff Brumfield, U of Austin, TX — PMVA
  - ...

Petri Nets vs. Queueing Networks

- Petri Nets
  - good for <u>concurrency</u>
  - Stochastic Petri Nets (SPN)
  - Timed Petri Nets
  - simulators

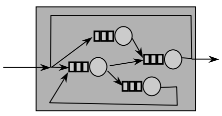


- Queuing Networks
  - good for <u>queuing</u>
  - simple bounds
    - Bottleneck Bounds (ABA)
    - Balanced Job Bounds (BJB)
  - exact solution methods
  - approximate solution methods
  - simulators



Exact Solution Methods for Queuing Networks

- Open Networks



- Closed Networks
  - Convolution
  - MVA
  - Appr. MVA