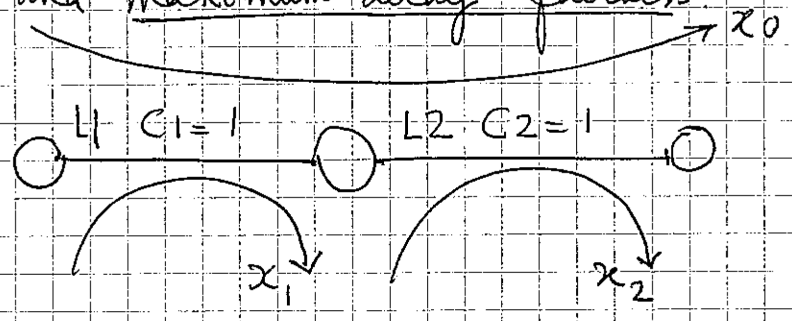
2.1. Question: For the example network in figure below, Calculate the rate allocation under proportional fairness and maximum delay fairness.



Proportional fair resource allocation problem

$$\text{maximize} \quad (\log x_0 + \log x_1 + \log x_2)$$
$$\text{s.t.} \quad x_0 + x_1 \leq 1$$
$$x_0 + x_2 \leq 1$$
$$x_0, x_1, x_2 \geq 0.$$

First note that the maximization of the utility will correspond to all the source rates x_0, x_1 and x_2 to be strictly positive i.e. $x_0, x_1, x_2 > 0$ and both the capacity constraints will be satisfied with equality. So we can ignore the non-negativity constraints, and use the Lagrange multiplier technique to solve the problem.

~~L(x)~~ The Lagrangian for the problem is

$$L(\vec{x}, \vec{\lambda}) = \log x_0 + \log x_1 + \log x_2 - \lambda_1 (x_0 + x_1 - 1) - \lambda_2 (x_0 + x_2 - 1)$$

Here $\vec{x}$ is the datarate allocation vector: $\vec{x} = (x_0, x_1, x_2)$ and $\vec{\lambda}$ is the vector of Lagrange multipliers $\vec{\lambda} = (\lambda_1, \lambda_2)$.

Setting $\frac{\partial L}{\partial x_i} = 0$, for each $i = 0, 1, 2$, yields

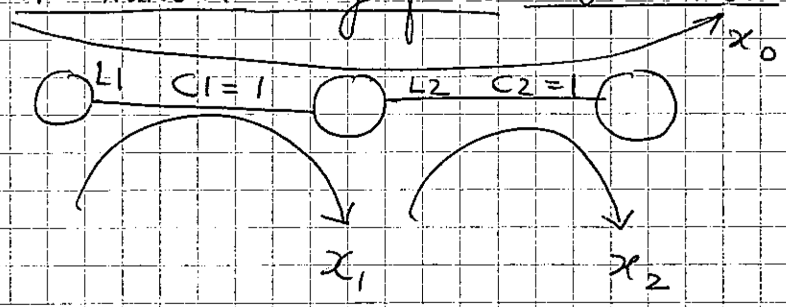
$$\frac{1}{x_0} - \lambda_1 - \lambda_2 = 0, \quad \frac{1}{x_1} - \lambda_1 = 0, \quad \frac{1}{x_2} - \lambda_2 = 0.$$

Solving, $x_0 = \frac{1}{\lambda_1 + \lambda_2}$, $x_1 = \frac{1}{\lambda_1}$, $x_2 = \frac{1}{\lambda_2}$

$$\left.\begin{aligned} x_0 + x_1 = 1 &\Rightarrow \frac{1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_1} = 1 \\ x_0 + x_2 = 1 &\Rightarrow \frac{1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_2} = 1 \end{aligned}\right\} \Rightarrow \lambda_1 = \lambda_2.$$

Set $\lambda = \lambda_1 = \lambda_2$; solving we get $x_0 = \frac{1}{3}$, $x_1 = \frac{2}{3} = x_2 = \frac{2}{3}$

2.1. Maximum delay fair allocation for the network



Utility function for max delay fairness

$$U_r(x_r) = -\frac{1}{x_r}$$

So maximum delay fair allocation problem

$$\text{maximize} \left(-\frac{1}{x_0} - \frac{1}{x_1} - \frac{1}{x_2} \right) \quad [[\text{sum of utilities}]]$$

$$\text{s.t.} \quad x_0 + x_1 \leq 1$$
$$x_0 + x_2 \leq 1$$
$$x_0, x_1, x_2 \geqslant 0.$$

As in the case for proportional fairness, we can infer that all the rates x_0, x_1 and x_2 are strictly positive ie $x_0, x_1, x_2 > 0$ and the link constraints will be satisfied with equality. The Lagrangian multiplier technique can be employed to solve the problem.

Form the Lagrangian for the problem

$$L(x, \lambda) = -\frac{1}{x_0} - \frac{1}{x_1} - \frac{1}{x_2} - \lambda_1 (x_0 + x_1 - 1) - \lambda_2 (x_0 + x_2 - 1)$$

Set $\frac{\partial L}{\partial x_i} = 0$, for $i = 0, 1, 2$ and do the calculations as in the case for proportional fairness.

Solution: $x_0 = 0.4142$, $x_1 = 0.5858$ $x_2 = 0.5858$.

2.4

Max-min fair allocation:

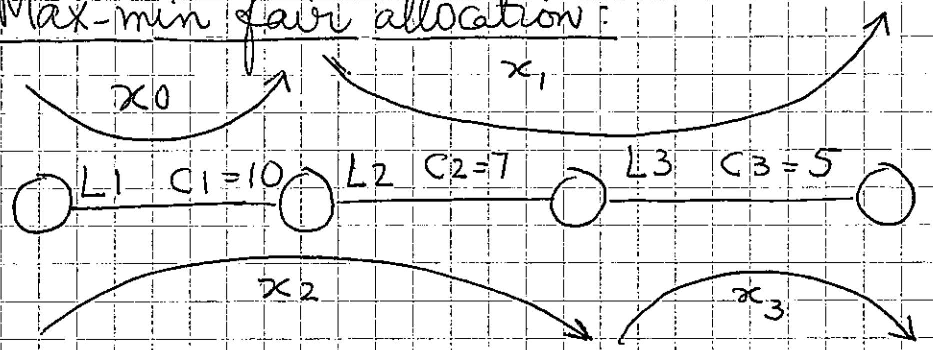


Fig: A network with four sources and three sinks.

Question: To calculate the rates for the sources using the max-min fair allocation of rates.

Solution: We first describe the max-min fair allocation algorithm.

Step 1: let n_ℓ denote the number of sources in S that use link ℓ. For each ℓ such that $c_\ell \neq 0$, define the fair share f_ℓ on link ℓ as follows:

$$f_\ell = \frac{c_\ell}{n_\ell}$$

Thus the fair share of link ℓ is obtained by dividing the link capacity among all sources equally.

Define $$Z_r = \min_{\ell \in r} f_\ell$$

ie, if in each link ℓ, the rate f_ℓ is allocated to all the sources sharing the link, then (route) r will get Z_r. Define Z_{min} to be the smallest of these rates in the network ie $Z_{min} = \min_r Z_r$.

let $\tilde{R}$ be the set of sources that have the smallest rate ie

$$\tilde{R} = \{ r \in S : Z_r = Z_{min} \}$$

For all $r \in \tilde{R}$, Z_r is allocated as the max-min rate ie

$$x_r = Z_r, \quad r \in \tilde{R}.$$

<u>Step 2</u> Set $S \leftarrow S \setminus \tilde{R}$

If S is empty, terminate; else continue.

<u>Step 3</u> For all $l \in \mathcal{L}$ ($\mathcal{L}$ - set of links)

$$c_l \leftarrow c_l - \sum_{r \in \tilde{R},\, l \in r} x_r$$

ie subtract the capacity used up by the sources that were allocated their max-min rates ~~from the~~ just earlier. (Note that the links for which $f_l = z_{min}$ will have $c_l = 0$ after this step).

Go to Step 1.

(The above algorithm is from
<u>Srikant</u> <u>Mathematics of Internet Congestion Control</u>).

For our three-link network,
$C_A = 10$, $C_B = 10$, $C_C = 7$.

The max-min allocation algorithm performs the steps

- $S = \{1, 2, 3, 4\}$, $f_A = 5$, $f_B = 3.5$, $f_C = 2.5$,
  Thus $\tilde{R} = \{2, 4\}$, and $x_2 = x_4 = 2.5$.
  The new $S = \{1, 3\}$, $C_A = 10$, $C_B = 4.5$, $C_C = 0$

- Now $f_A = 5$, $f_B = 4.5$, ~~thus~~ $\tilde{R} = \{3\}$, $x_3 = 4.5$,
  The new $S = \{1\}$, and $C_A = 5.5$, $C_B = 0$

- Finally $f_A = 5.5$, $x_1 = 5.5$. The new $S = \emptyset$,
  and the algorithm terminates $\square$

3.5. Question: To show that the function e^{ax} is convex, for $\forall a \in \mathbb{R}$

Solution: (Here we use the fact that for a differentiable convex function $f(x)$, the derivative $f'(x)$, ~~≥ 0~~ is increasing and when the convex function is twice differentiable, the second derivative $f''(x) \geq 0$.)

for the function $f(x) = e^{ax}$, we know that it is twice differentiable (indeed it is differentiable any number of times)

$f''(x) = a^2 e^{ax}$ and $a^2 e^{ax} \geq 0$, $\forall a \in \mathbb{R}$,

hence e^{ax} is convex, $\forall a \in \mathbb{R}$.

3.6 Question: The function $\log x$, $x > 0$ is concave.

Solution: equivalently, we show that $-\log x$, $x>0$ is convex.

$\because \frac{d}{dx}(-\log x) = -\frac{1}{x}$, $\frac{d^2}{dx^2}(-\log x) = \frac{1}{x^2} > 0$

~~so $-\log x$, $x>0$ is~~ so $-\log x$, $x > 0$ is convex.

3.7. Question: The set $\{x \mid Ax \leq b\}$ for suitable matrix A and vector b is convex.

Solution: Either direct verification.

(Show that x_1, x_2 lies in the given set implies $\lambda x_1 + (1-\lambda)x_2$ lies in the set for $0 \leq \lambda \leq 1$)

or argue that each of the inequalities in the set (corresponding to each row of the matrix inequality $Ax \leq b$) defines a half-space which is convex and the set in question is convex as it is the intersection of finitely many convex sets.

3.8. Similar to question 3.7.

2.2. Question: An allocation vector $\vec{x}^* = (x_s^*, s \in S)$ is <u>proportionally fair</u> if it is feasible (i.e. $x^* \geq 0$, $Ax^* \leq C$) and if for any other feasible vector $\vec{x}$, the aggregate of proportional changes is zero or negative:

$$\sum \frac{x_s - x_s^*}{x_s^*} \leq 0.$$

Prove that allocations based on the utility function

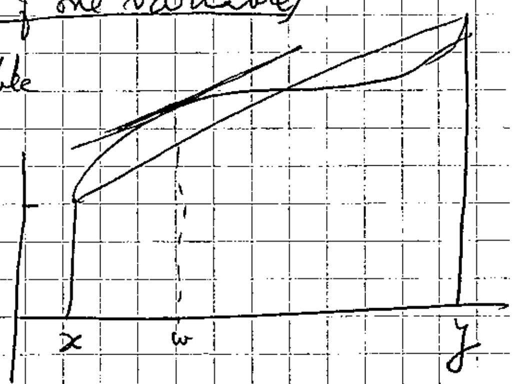
$$U_r(x_r) = \log x_r \quad \text{is proportionally fair.}$$

<u>Solution</u>:

We begin with Mean Value Theorem of Calculus for functions of several variables which is a generalization of the Mean Value Theorem of Calculus for functions of one variable.

<u>Mean Value Theorem of Calculus (for functions of one variable)</u>

let f be a continuously differentiable function of one variable. Then



$$\frac{f(y) - f(x)}{y - x} = \text{slope of } f \text{ at some intermediate point } w$$

$$= f'(w)$$

i.e. $f(y) = f(x) + f'(w)(y - x) \qquad x < w < y$

<u>Mean Value Theorem (for functions of several variables)</u>

For a continuously differentiable function $f(\vec{x})$, we have

$$f(\vec{y}) = f(\vec{x}) + \nabla f(\vec{w}) \cdot (\vec{y} - \vec{x}), \quad \text{where}$$

$$w = x + \lambda(y - x), \text{ for some } \lambda \in [0, 1].$$

Using the MVT, we can show the optimality condition of a maximization problem.

2.2. Contd. Lemma (Optimality Condition) If x^* is the solution of $\max_x f(x)$, where $f(\cdot)$ is continuously differentiable, then $(x - x^*) \cdot \nabla f(x^*) \leq 0$

Proof: Assume that x^* is optimum but the condition is not satisfied. Therefore, there exists $\tilde{x}$ such that

$$(\tilde{x} - x^*) \cdot \nabla f(x^*) > 0$$

For any $\epsilon > 0$, by the Mean Value Thm, we have

$$f(x^* + \epsilon(\tilde{x} - x^*)) = f(x^*) + \epsilon \nabla f(x^* + \lambda\epsilon(x - x^*)) \cdot (x - x^*)$$

Since ϵ is arbitrary, we can pick $\epsilon \to 0$ and hence we have $\nabla f(x^* + \lambda\epsilon(x - x^*)) \cdot (x - x^*)$

$$\longrightarrow \nabla f(x^*) \cdot (x - x^*) > 0 \text{ as } \nabla f(x) \text{ is continuous.}$$

This gives us

$$f(x^* + \epsilon(x - x^*)) > f(x^*)$$

which contradicts the assumption that $f(x^*)$ is maximum.

Claim $U_r(x_r) = \log x_r$ corresponds to proportional fair allocation.

Proof Using Lemma ~~(Opt~~ above

$$(\vec{x} - \vec{x}^*) \cdot \nabla \left(\sum_{r \in S} U_r(x_r) \right) \bigg|_{x^*} \leq 0$$

$$\Rightarrow \sum \frac{\partial U_r(x_r)}{\partial x_r} \bigg|_{x_r^*} (x_r - x_r^*) \leq 0$$

$$\Rightarrow \sum \frac{x_r - x_r^*}{x_r^*} \leq 0 \cdot$$ which shows $U_r(x_r) = \log x_r$ is proportionally fair.

Note: Our solution is based on the online notes 'Fairness' by Prof Samuel Cheng.

2.3 Question: Show that $\log x_r$ utility function can be obtained from the general α-fair utility function (given below) corresponding to the case $\alpha \to 1$.

$$U_r(x_r) = \frac{x_r^{1-\alpha}}{1-\alpha}.$$

Solution: $U_r'(x_r)$, the derivative of $U_r(x_r)$, can be defined at $\alpha = 1$ in an indirect manner, as the function $U_r(x_r)$ is not defined at $\alpha = 1$.

$$U_r'(x_r) = (1-\alpha) \cdot \frac{x_r^{1-\alpha-1}}{(1-\alpha)} = x^{-\alpha}$$

$$\lim_{\alpha \to 1} U_r'(x_r) = x^{-1} = \frac{1}{x_r}.$$

The function $U_r(x_r)$ corresponding to the derivative $U_r'(x_r) = \frac{1}{x_r}$ is $\underline{U_r(x_r) = \log x_r}$

So $U_r(x_r) = \log x_r$ is the utility function corresponding to $\alpha = 1$ in the general α-fair utility function $U_r(x_r) = \frac{x_r^{1-\alpha}}{1-\alpha}$.

$\square$

Mathematical Modelling for Computer Networks
Spring 2013

Exercise 2: Due on 5th April 2013.

Write your answers to the questions briefly and clearly. Please bring a printout (or a handwritten copy) of your answers to the class.

1. For example network shown in Figure 1, find out the rate allocations under proportional fairness and minimum potential delay fairness. You can use simulation packages like CVX in MATLAB (see the Section Experiments in lecture notes: Utility, Fairness and Optimization in Resource Allocation) or some other simulation tools. Verify the results by hand calculation.

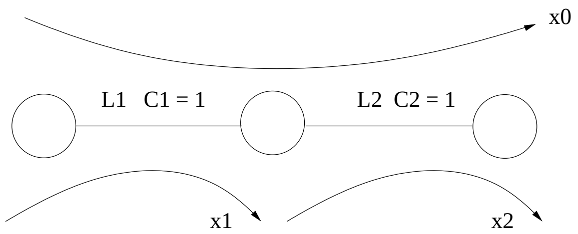


Figure 1: A network with three sources and two links

2. An allocation vector $\boldsymbol{x}^* = (x_s^*, s \in S)$ is *proportionally fair* if it is feasible (i.e., $\boldsymbol{x}^* \geq 0$ and $A\boldsymbol{x}^* \leq C$) and if for any other feasible vector $\boldsymbol{x}$, the aggregate of proportional changes is zero or negative:

$$\sum_{s \in S} \frac{x_s - x_s^*}{x_s^*} \leq 0$$

Prove that the allocation based on the utility function $\mathcal{U}_r(x_r) = log\, x_r$ is proportionally fair.

3. Prove that the $log\, x_r$ utility function for proportional fairness can be obtained from the general α-fairness utility function (given below) corresponding to the case $\alpha \to 1$.

$$U_r(x_r) = \frac{x_r^{1-\alpha}}{1-\alpha}$$

4. The algorithm for Max-min fair allocation is given in page 527-528, Chapter 6, Flow Control of the book Data Networks by Bertsekas and Gallager (available online at http://web.mit.edu/dimitrib/www/datanets.html)

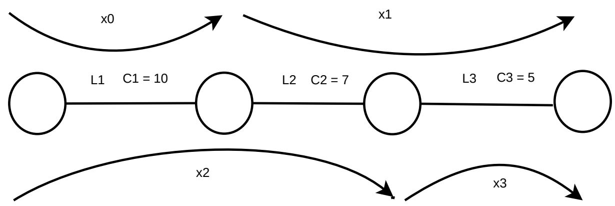


Figure 2: A network with four sources and three links

For the example network shown in Figure 2, calculate the rates for the sources using max-min fair allocation. Show the intermediate results of the allocation resulting from the execution of the algorithm.

5. Prove that convexity of the function $f(x) = e^{ax}$, for every $a \in \mathbb{R}$.

6. Prove that the function $log\, x$, $x > 0$ is concave.

7. Show that the set $\{x \mid Ax \leq b\}$ for suitable matrix A and vector b is convex.

8. Show that the set $\{x \in \mathbb{R}^n : g_i(x) \leq b_i,\ 1 \leq i \leq m\}$ is convex when $g_i : \mathcal{X} \to \mathbb{R}$ are convex functions defined over a convex set $\mathcal{X} \in \mathbb{R}^n$. Here $b_i,\ i = 1, \leq m$ are arbitrary real numbers.