3.2 Question Consider the primal problem

$$\min f(x)$$
$$\text{s.t. } g_i(x) \le 0, \quad 1 \le i \le m, \quad x \ge 0$$

where $f: \mathbb{R}^n \to \mathbb{R}$ and $g_i: \mathbb{R}^n \to \mathbb{R}, \forall i$ are convex and differentiable functions over $\mathbb{R}^n$.

This is a convex optimization problem.

Define for $\lambda \in \mathbb{R}^m_+$

$$\theta(\lambda) = \inf \left\{ x \ge 0 : f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \right\}$$

$\theta(\lambda)$ is the Lagrangian dual function and is obtained by relaxing the constraints $g_i(x) \le 0, 1 \le i \le m$

a) Show that $\theta(\lambda)$ is a concave function on $\lambda \in \mathbb{R}^m_+$

b) Show that if $\lambda \ge 0$ and if x satisfies the constraints of the primal problem, then

$$\theta(\lambda) \le f(x).$$

Solution (a): $\theta(\lambda)$ is the pointwise infimum of a family of affine functions in the variables $\lambda_1, \ldots \lambda_m$.

So $\theta(\lambda)$ is a concave function.

Note that the conclusion $\theta(\lambda)$ is a concave function (in variables $\lambda_1, \ldots \lambda_m$) is valid regardless of the functions $f(\cdot)$ and $g_i(\cdot)$ being convex or not.

3.2(b) (b) <u>Claim</u>: if $\lambda \geqslant 0$ and $\tilde{x}$ satisfies the constraints of the primal problem then

$$\theta(\lambda) \leq f(\tilde{x}).$$

<u>Proof</u>: given that $\tilde{x}$ is a feasible point for the primal problem (ie $\tilde{x}$ satisfies the constraints of the primal)

so $\quad g_i(\tilde{x}) \leq 0$.

$\lambda \succeq 0 \quad$ (ie $\lambda = (\lambda_1, \dots \lambda_m)$, $\lambda_i \geqslant 0, \forall i$).

yields $\quad \sum \lambda_i g_i(\tilde{x}) \leq 0$

So $L(\tilde{x}, \lambda) = f(\tilde{x}) + \sum \lambda_i g_i(x) \leq f(\tilde{x})$

So $\theta(\vec{\lambda}) = \inf_{x \in D} L(x, \lambda) \leq L(\tilde{x}, \lambda) \leq f(\tilde{x})$.

Since <u>$\theta(\lambda) \leq f(\tilde{x})$ holds</u> for every feasible point,
(we can even conclude that
$\theta(\lambda) \leq p^*$ where p^* is the optimal value of the primal problem)

$\square$.

3.3. Question: To formulate the Lagrangian dual function for the primal problem given below:

$$\min \ (x_1-5)^2+(x_2-5)^2$$

$$\text{s.t.} \begin{cases} x_1^2+x_2^2-5 \le 0 \\ \frac{1}{2}x_1+x_2-2 \le 0 \\ -x_1 \le 0 \\ -x_2 \le 0 \\ x \in \mathbb{R}^2 \end{cases}$$

Standard form

$\min f(x)$

s.t $g_i(x) \le 0 \quad 1 \le i \le m$

$x \ge 0$

General dual problem:

let $\lambda \in \mathbb{R}^m_+$

$\theta(\lambda) = \inf\{x \ge 0 : f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)\}$

$\theta(\lambda)$: Lagrangian dual function

Dual Problem: $\max_{\lambda \ge 0} \theta(\lambda)$

$$\theta(\lambda) = \inf\left\{x \ge 0 : ((x_1-5)^2+(x_2-5)^2) + \lambda_1(x_1^2+x_2^2-5) + \lambda_2\left(\tfrac{1}{2}x_1+x_2-2\right)\right\}$$

Differentiating w.r.t. x_1 and x_2, we get

$$\begin{cases} 2(x_1-5)+2\lambda_1 x_1 + \frac{\lambda_2}{2} = 0 \\ 2(x_2-5)+2\lambda_1 x_2 + \lambda_2 = 0 \end{cases}$$

Solving,

$$x_1 = \frac{20-\lambda_2}{4(1+\lambda_1)}, \quad x_2 = \frac{10-\lambda_2}{2(1+\lambda_1)}$$

Substituting for x_1 and x_2 in $\theta(\lambda)$, we get

$$\theta(\lambda) = \frac{720\lambda_1 + 88\lambda_2 - 32\lambda_1\lambda_2 - 80\lambda_1^2 - 5\lambda_2^2}{16(1+\lambda_1)}.$$

We know that $\theta(\lambda)$ is a Concave function. (See question 3.2) Differentiating (w.r.t λ_1 and λ_2)

$$720 - 120\lambda_2 - 80\lambda_1^2 - 160\lambda_1 + 5\lambda_2^2 = 0$$

$$88 - 32\lambda_1 - 10\lambda_2 = 0$$

Solving these equations, we get the optimal solution

$$\lambda_1^* = \frac{2}{3}, \quad \lambda_2^* = \frac{20}{3}.$$

Substituting these, we get the optimal solution

$$x_1^* = 2, \quad x_2^* = 1$$

The optimal value $f(x^*) = 25$ $\square$.

Mathematical Modelling for Computer Networks
Spring 2013

Exercise 3: Due on 5th April 2013.

Write your answers to the questions briefly and clearly. Please bring a printout (or a handwritten copy) of your answers to the class.

1. TCP Vegas is a delay based algorithm that uses queueing delay to infer congestion in the network. Read the sections 4.4, 4.5 and 4.6 of P14 (S. Shakkottai and R. Srikant. Network Optimization and Control. Now publishers, 2007) and derive the utility function of TCP Vegas.

2. Consider the primal problem

$$\begin{aligned} \min \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \;\; 1 \leq i \leq m \\ & x \geq 0 \end{aligned}$$

   Where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ are convex and differentiable functions over $\mathbb{R}^n$. Note that this is a convex optimization problem.

   For this general primal problem, we can define for $\lambda \in \mathbb{R}^m$, $\lambda \geq 0$,

$$\Theta(\lambda) \,=\, inf\{x \geq 0 :\, f(x) \,+\, \sum_i^m \lambda_i g_i(x)\}$$

   $\Theta(\lambda)$ is called the *Lagrangian dual function* and is obtained by relaxing the constraints $g_i(x) \leq 0,\;\; 1 \leq i \leq m$.
   (a) Show that $\Theta(\lambda)$ is a concave function on $\lambda \in \mathbb{R}^m$, $\lambda \geq 0$
   (b) Show that if $\lambda \geq 0$ and if x satisfies the constraints of the primal problem, then $\Theta(\lambda) \,\leq\, f(x)$.

   The Dual problem is given by

$$\max \Theta(\lambda),\;\; \lambda \geq 0$$

   We now apply the formalism given above to the problem in question 3.

3. Formulate the Lagrangian dual function for the example problem given in Section 3.3, the Karush-Kuhn-Tucker (KKT) Conditions, of the notes on Convex Optimization. Solve for the Lagrangian multipliers and find the optimal solution and the optimal value of the function $f(x)$.