Mathematical Modelling for Computer Networks- Part I	Spring 2013 (Period 4)

Congestion Control 1: The Chiu-Jain Model

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Abstract

This lesson presents the Chiu-Jain model of congestion control. The Chiu-Jain model [CJ89] abstracts the congestion control problem by presenting how a simple network comprising of a *single* link of a fixed capacity can be shared by two sources. It elucidates how the Additive Increase Multiplicative Decrease (AIMD) law used by the two sources to adapt their sending rates to the feedback from the network on whether the link is congested or not, leads to a stable equilibrium point of network operation which is both fair and efficient. Moreover this model clarifies several basic features of a typical congestion control algorithm used in the Internet.

1.1 Chiu-Jain model

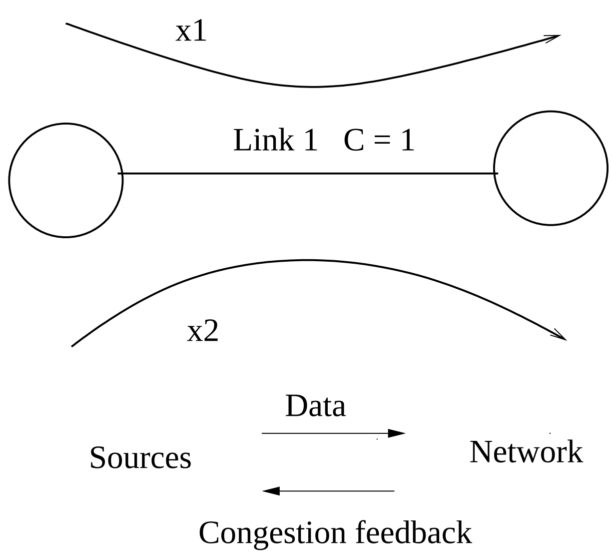

Figure 1.1: Resource allocation: Chiu-Jain model

Figure 1.1 shows two sources share a common link that has a capacity c packets/sec. Let x_i be the rate at which source i sends packet into the network, for $i = 1, 2$. The link provides feedback to the sources to indicate whether the link access rate $x_1 + x_2$ exceeds the link capacity or not. The term *congestion* refers to the situation where the link access rate exceeds the link capacity. The feedback signal from the link to the sources is $I(x_1 + x_2 > c)$, the indicator function of the event $(x_1 + x_2 > c)$. It takes the value 1 when the event $(x_1 + x_2 > c)$ is true and the value 0 when the event is false. The *congestion control problem* here is to adapt the sending rate of the sources to the feedback signal so that the link can shared fairly and fully utilized corresponding to convergence of the sending rates of the sources to a stable operating point, which realizes the unique equilibrium of the network.

In response to the congestion signal, the sources adjust their sending rates according to the differential equation

$$\dot{x}_i = \alpha I(x_1 + x_2 \leq c) - \beta x_i I(x_1 + x_2 > c) \;\; for\, i \in \{1, 2\} \tag{1.1}$$

Here $\dot{x}_i$ refers to the time derivative of x_i i.e., $\frac{dx_i}{dt}$ and α and β are positive constants.

The equation (1.1) says that if the total arrival rate at the link does not exceed the capacity c,then a source increases its sending rate at a constant rate α (additive increase) and if the link arrival rate exceeds the link capacity, then the sending rate is decreased multiplicatively (as $\dot{x}$ is proportional to $-x_i$) with β as the constant of proportionality. Note that the two events $x_1 + x_2 \leq c$ and $x_1 + x_2 > c$ are complementary, in the sense that at any instant exactly one of them is true. An assumption implicit in the model is that the network delays are negligible so that the feedback is modelled as instantaneous.

To study the behaviour of the system, we set the variable $y = x_1 - x_2$ which leads to a simplified differential equation involving y obtained from the equation (1.1) by simple algebra.

$$\dot{y} = -\beta y I(x_1 + x_2 > c) \tag{1.2}$$

So when $x_1 + x_2 \leq c$, $\dot{y} = 0$, indicating that y does not change with time, and so $x_1 - x_2$ remains a constant. However (1.1) implies that x_1 and x_2 increase steadily under this condition. So when $x_1 + x_2 \leq c$, both x_1 and x_2 increase steadily at the same rate while maintaining their difference constant. In the case when $x_1 + x_2 > c$, equation (1.2) indicates that y evolves to reduce the difference between x_1 and x_2 and as $t \to \infty$, $x_1 + x_2 \to c$ and $y = (x_1 - x_2) \to 0$. Thus in the steady state, the network attains the equilibrium where the link is fully utilized as $x_1 + x_2 \to c$ and is equally shared by the two senders as $x_1 - x_2 \to 0$.

1.2 Observations on the Chiu-Jain model

Several features of the simple Chiu-Jain model are noteworthy as they reflect the characteristics that we desire in any congestion control algorithm designed to operate in a complex network like the Internet.

1.2.1 Congestion Control as a Resource Sharing principle/mechanism

The sources adapt their sending rate to the extent of congestion in the network by decreasing the sending rates if the link arrival rate is in excess of the link capacity and by increasing the sending rate if the link arrival rate below the link capacity. Note that the dynamic allocation of resources (such as link capacity in this this) is fundamental in deriving the benefits of packet switching.

1.2.2 Congestion Feedback

The congestion control algorithm responds to feedback from the network about the presence or absence of congestion in the form of the congestion signal $I(x_1+x_2 > c)$ obtained from the congestion event $(x_1 + x_2 > c)$. The amount of feedback is minimal, it is a single bit of information indicating whether the link arrival rate exceeds the link capacity or not. If the link merely drops packets, the receiver can detect the loss of packets and the inform the source about the presence of congestion in the network.

1.2.3 Operation of the Congestion Control Algorithm

The congestion control algorithm *steers* the network towards an operating point which corresponds to a unique stable equilibrium for the operation of the network which is both efficient and fair. A good congestion control algorithm should provide a rate region that is as large as possible while supporting (some form of) fairness in allocating the rates to the different users.

1.2.4 Decentralised Operation

Each source (congestion controller) utilizes one-bit feedback from the network and the different sources need not communicate with one another. A link can signal congestion based on the total arrival rate at the link.

1.3 Experiments

In this section we describe MATLAB experiments to study the equilibrium convergence behaviour of the differential equation (1.1). A first step here is an appropriate discretization of the differential equation to obtain a difference equation that can be implemented as a computer program.

The difference equation is obtained from the original differential equation as follows.

The original differential equation (1.1) is

$$\dot{x}_i = \alpha I(x_1 + x_2 \leq c) - \beta x_i I(x_1 + x_2 > c) \;\; for\, i \in \{1, 2\}$$

The main step here is to replace the continuous derivative by its discrete counterpart : $\frac{dx}{dt} \approx \frac{x(t+\Delta t)-x(t)}{\Delta t}$

In the difference quotient, we replace $x(t+\Delta t)$ by $x(k+1)$, $x(t)$ by $x(k)$ and $\Delta t \, \text{by} \, \delta$ and then substitute and simplify to get

$$x(k+1) = x(k) + \alpha\delta I(x_1 + x_2 \leq c) - \beta\delta x_i I(x_1 + x_2 > c) \qquad (1.3)$$

In our experiments, we choose $c = 1, \alpha = 1, \beta = 0.5 \; and \; \delta = 0.05$. The initial values are $x_1 = 0.3 \; and \; x_2 = 0.1$

We have implemented the difference equation in MATLAB and plotted the resulting equilibrium convergence behaviour. The experiment confirms that at the equilibrium, the link is fully utilized $(c = 1)$ and both sources share the link equally $(x_1 = x_2 = 0.5)$.

Our description of the Chiu-Jain algorithm and the experiments given here closely follow the approach given in the book [S04].

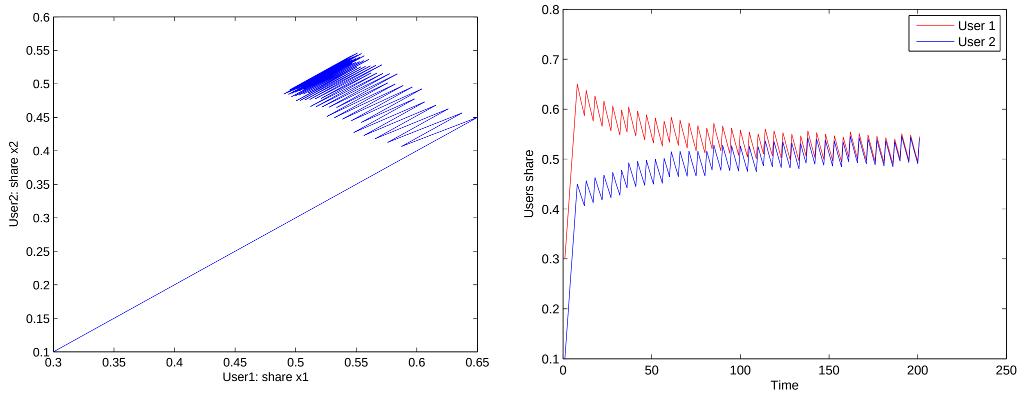

Figure 1.2: Rate evolution of the the Chiu-Jain Algorithm for two sources sharing a single link of capacity one. Starting from the point (0.3,0.1), the system moves towards the point (0.5,0.5).

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% Chiu-Jain Algorithm - MATLAB code

size=100;
x1=zeros(size,1);
x1(1)=0.3;
x2=zeros(size,1);
x2(1)=0.1;


c=1;
delta=0.05;
beta=0.5;

for i=1:1:size
  if (x1(i)+ x2(i))<=c
    x1(i+1)=x1(i)+delta;
    x2(i+1)=x2(i)+delta;
  else
    x1(i+1)=x1(i)-delta* beta * x1(i);
    x2(i+1)=x2(i)-delta* beta * x2(i);
  end
end

figure;
set(gcf,'Color','white');
plot(x1,x2);

xlabel('User x1');
ylabel('User x2');
```

We encourage the reader to perform these experiments for different choices of the initial values and different values of the parameters β δ. Instead of MATLAB, any programming environment/language can be used.

Acknowledgements

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References

[CJ89] Dah-Ming Chiu and Raj Jain, “Analysis of the increase and decrease algorithms for congestion avoidance in computer networks,” *Journal of Computer Networks and ISDN Systems*, Volume 17 Issue 1, June 1989 Pages 1 - 14.

[S04] R. Srikant, ”The Mathematics of Internet Congestion Control”. *Birkhauser*, 2004.