6.1 State j is <u>accessible</u> from state i if $P_{ij}^{n} > 0$ for some $n \geq 0$. (Notation $i \to j$).

(Here P is the one-step state transition matrix of the Markov chain)

<u>Claim</u>: This setting (~~State~~ j is accessible from state i) implies that

State j is accessible from state i iff starting from state i it is possible that the process will ever enter state j.

<u>Solution</u>:

<u>Hint</u>: If j is not accessible from i (ie $i \not\to j$) then the probability $\mathbb{P}$(ever entering j | starting from i) $= 0$.

$\mathbb{P}$(ever entering j | starting from i)

$$= P\left\{ \bigcup_{n=0}^{\infty} (X_n = j \mid X_0 = i) \right\}$$

$$\leq \sum_{n=0}^{\infty} \mathbb{P}(X_n = j \mid X_0 = i) \qquad [[*1]]$$

$$= \sum P_{ij}^{n} \qquad [[(i \not\to j \text{ implies that } P_{ij}^{n} = 0, \forall n \geq 0)]]$$

$$= 0$$

<u>Note</u> $[[*1]]$ $\mathbb{P}\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(E_n)$

for any countable family of events E_n.

This inequality is called <u>union bound</u> and is of wide applicability in probabilistic calculations.

6.2. Two states i and j that are accessible from each other ($i \to j$ and $j \to i$) <u>communicate</u> with each other.

<u>Notation</u> $i \leftrightarrow j$. ie $i \leftrightarrow j := i \to j$ and $j \to i$

<u>Claim</u>: The communication relation is an equivalence relation ie it is reflexive, symmetric and transitive.

<u>Solution</u>:

That the communication relation is <u>reflexive</u> and <u>symmetric</u> follows almost immediately from the definition.

<u>reflexive</u>: $i \leftrightarrow i, \ \forall i \in S$ (S - State space)

<u>Symmetric</u>: $i \leftrightarrow j \Rightarrow j \leftrightarrow i$

<u>transitive</u>: $i \leftrightarrow j \wedge j \leftrightarrow k \Rightarrow i \leftrightarrow k$

let ~~there~~ exist integers m, n s.t. $P_{ij}^{n} > 0$, $P_{jk}^{m} > 0$

by (Chapman-Kolmogorov equation)

$$P_{ik}^{n+m} = \sum_{r=0}^{\infty} P_{ir}^{n} P_{rk}^{m} \geq P_{ij}^{n} P_{jk}^{m} > 0$$

Hence state k is <u>accessible</u> from state i, $i \to k$

Similarly, we can show that state i is accessible from state k ie $k \to i$.

Hence, states i and k communicate, $i \leftrightarrow k$

$\square$

6.3. Given a Markov chain with states $S = \{0, 1, 2\}$ and the transition matrix
$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

Question: (To) determine if the markov chain is irreducible.

Solution: A Markov chain is irreducible if $P_{ij}^{n} > 0$, for all states i and j, for some $n \geq 0$.

Given that $P_{02} = 0$ and $P_{20} = 0$ (from the P matrix) we can see that

$0 \xrightarrow{1/2} 1 \xrightarrow{1/4} 2$ implies $P_{02}^{2} = \frac{1}{8} > 0$

~~$2 \xrightarrow{1/3} 1 \xrightarrow{1/2} 0$ implies P~~

$2 \xrightarrow{1/3} 1 \xrightarrow{1/2} 0$ implies $P_{20}^{2} = \frac{1}{6} > 0$

So, clearly from any state one can reach any other state in one & more steps, showing that the give Markov chain is irreducible.

6.4 Question: Let f_i be the probability that starting in state i, the process will reenter state i. State i is recurrent if $f_i = 1$ and transient if $f_i < 1$.

a): If state i is recurrent, then starting from state i, the process will reenter state i again and again, and indeed, infinitely often.

b) If State i is transient, then starting in state i, the number of time periods the process will return to state i is a geometric ~~progression~~ distribution with mean $\frac{1}{1-f_i}$

Solution (a) ~~Suppose~~ Suppose the process ~~starts~~ in state i and i is recurrent. Hence with probability 1, the process will eventually reenter state i. By the definition of Markov chain, it follows that the process will be starting all over again when it reenters state i, and therefore, state i will be revisited eventually again. The repetition of the argument shows that « if state i is recurrent, then, starting from state i, the process will reenter state i again and again - in deed, infinitely often »

Solution (b) Suppose state i is transient. Hence, each time the process enters state i, there is a positive probability, $1-f_i$ that it will never again enter that state. Hence, starting in state the probability that the process will be in state i for exactly n time periods is given by $f_i^{n-1}(1-f_i)$, $n \geq 1$. This implies, that « if state i is transient, then starting in state i, the number of time periods that the process will be in state i has a geometric distribution with finite mean $1/(1-f_i)$ »

6.5 a) State i is <u>recurrent</u> if $\sum_{n=1}^{\infty} P_{ii}^{n} = \infty$

b) State i is <u>transient</u> if $\sum_{n=1}^{\infty} P_{ii}^{n} < \infty$.

<u>Solution</u> let $A_n = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{if } X_n \neq i \end{cases}$

(Thus A_n is an indicator function that 'signals' the occurrence of the process entering the state i in step n).

Thus $\sum_{n=0}^{\infty} A_n$ represents the number of visits of the process to state i.

Clearly, $\mathbb{E}\left[\sum_{n=0}^{\infty} A_n \,\middle|\, X_0 = i\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[A_n \,\middle|\, X_0 = i\right]$

$$= \sum_{n=0}^{\infty} \mathbb{P}\left\{X_n = i \,\middle|\, X_0 = i\right\}$$

$$= \sum_{n=0}^{\infty} \mathbb{P}_{ii}^{n}$$

This yields

a) State i <u>recurrent</u> if $\sum_{n=1}^{\infty} P_{ii}^{n} = \infty$

b) State i <u>transient</u> if $\sum_{n=1}^{\infty} P_{ii}^{n} < \infty$

6.6. Claim: If state i is recurrent and state i communicates with state j, then state j is recurrent

Proof $i \leftrightarrow j \implies P_{ij}^{k} > 0$ and $P_{ji}^{m} > 0$ for some integers k and m.

As $$P_{jj}^{m+n+k} \geq P_{ji}^{m} P_{ii}^{n} P_{ij}^{k},$$

$$\sum_{n=1}^{\infty} P_{jj}^{m+n+k} \geq P_{ji}^{m} P_{ij}^{k} \sum_{n=1}^{\infty} P_{ii}^{n} = \infty$$

which implies that state j is recurrent.

Note So recurrence is a (communication) class property. (It can be shown that transience is also a class property).

6.7. Given a Markov chain with state space $S = \{0, 1, 2, 3\}$ having the transition matrix $\mathbb{P}$:

$$\mathbb{P} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Question: Identify the transient and recurrent states of the given Markov chain.

Solution: Easy to check that all states communicate, and hence, (since the chain is a finite state Markov chain), all states must be recurrent (for in a finite state Markov chain, there exists at least one recurrent state).

Mathematical Modelling for Computer Networks
Spring 2013

Exercise 6: Due on 26th April 2013.

Write your answers to the questions briefly and clearly. Please bring a printout (or a handwritten copy) of your answers to the class. You may refer to the book Introduction to Probability by Grinstead and Snell (`http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/book.html`)

1. State j is *accessible* from state i if $P_{ij}^n > 0$ for some $n \geq 0$ (notation $i \to j$). Show that this implies that state j is accessible from state i iff starting from state i it is possible that the process will ever enter state j.
   Hint: Show that if j is not accessible from state i, then the probability P (ever entering $j \mid$ starting from i) $= 0$

2. Two states i and j that are accessible from each other ($i \to j$ and $j \to i$) *communicate* with each other. Notation ($i \leftrightarrow j$). Show that the communication relation is an equivalence relation i.e, it is reflexive, symmetric and transitive.

3. Determine if the Markov chain with the given matrix P is irreducible.
$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

4. Let f_i be the probability that starting in state i , the process will reenter state i. State i is *recurrent* if $f_i = 1$ and *transient* if $f_i < 1$. Show that if state i is recurrent, then starting in state i, the process will reenter the state i again and again, and indeed infinitely often.

   Show that if the state i is transient, then starting in state i, the number of time periods the process will return to state i is a geometric distribution with mean $\frac{1}{1-f_i}$

5. Show that state i is recurrent if $\sum\limits_{n=1}^{\infty} P_{ii}^n = \infty$ and transient if $\sum\limits_{n=1}^{\infty} P_{ii}^n < \infty$

6. Show that if state i is recurrent and state i communicates with state j, then state j is recurrent.

7. Let a Markov chain with states 0, 1, 2, 3 have the transisition matrix P as follows.
$$P = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

   Determine which states are transient and which are recurrent