Markov Chain Model for ALOHA protocol

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Outline of the talk

- A Markov chain (MC) model for Slotted ALOHA
- Basic properties of Discrete-time Markov Chain
- Stability analysis of ALOHA

Slides on A Markov chain (MC) model for Slotted ALOHA and basic properties of Discrete-time Markov Chain are based on Chapter 4, Multiaccess Communication of the book Data Networks by Bertsekas and Gallager (available online at http://web.mit.edu/dimitrib/www/datanets.html) (B22)
Slides on stability analysis of S-ALOHA is based on Chapter 3, Example 3.3 from the book the Markov Chains: Gibbs Fields, Monte Carlo Simulation, and Queues by Bremaud, Pierre; Springer 2008 (B23)

Markov Chain model for S-ALOHA (1/5)

- Assumption: Each backlogged node retransmits with a fixed probability q_r in each successive slot until successful transmission happens.
- Assumption: No buffering at the nodes - a node has a packet to transmit which repeatedly sends until its successful transmission and any fresh packet arriving at the node while a packet is already waiting for transmission at the node is dropped at the node with impunity.
- We can model this retransmission of backlogged packets by a node as a geometric random variable X with probability of success q_r,($X \sim \mathcal{G}(q_r)$)
- In the Markov modelling of S-ALOHA, let n denote the number of backlogged nodes at the beginning of a given slot
- Each of the nodes will transmit a packet in a given slot independent of the other nodes, with probability q_r

Markov Chain model for S-ALOHA (2/5)

- In the Markov modelling of S-ALOHA, let n denote the number of backlogged nodes at the beginning of a given slot
- Each of the nodes will transmit a packet in a given slot independent of the other nodes, with probability q_r
- As there are a total of m nodes in the network, each of the $m - n$ non-backlogged nodes will transmit a packet in the given slot if one or more packets arrived at these nodes in the previous slot (Recall that newly arrived packets are transmitted immediately after their arrival in the next slot).
- These new arrivals are Poisson distributed with mean $\frac{\lambda}{m}$, so probability of no arrivals is $e^{\frac{-\lambda}{m}}$;
- This implies that the probability of an unbacklogged node transmits packets in a given slot is $q_a = 1 - e^{\frac{-\lambda}{m}}$.

Markov Chain model for S-ALOHA (3/5)

- let $Q_a(i, n)$ be the probability that i unbacklogged nodes transmit in a given slot
- let $Q_r(i, n)$ be the probability that i backlogged nodes transmit in a given slot
- $Q_a(n)$ and $Q_r(n)$ are binomially distributed as follows:
- $Q_a(n) \sim Bin(m - n, q_a)$
- $Q_r(n) \sim Bin(n, q_r)$
- we can specify the state transition probabilities in terms of $Q_a(i, n)$ and $Q_r(i, n)$ as follows.
- More explicitly

$$Q_a(i, n) = \binom{m - n}{i} (1 - q_a)^{m-n-i} q_a^i$$

$$Q_r(i, n) = \binom{n}{i} (1 - q_r)^{n-i} q_r^i$$

Markov Chain model for S-ALOHA (4/5)

- The number of backlogged packets corresponds to the *state* of the Markov chain we construct here.
- From one slot to the next, the state increases by the number of newly arrived packets in the slot, less one packet if there is a successful transmission of a packet in the state.
- A packet is transmitted successfully in a given slot, only under either of the following conditions.
  - Just one new arrival and no backlogged packets is transmitted.
  - There are no new arrivals and one backlogged packet is transmitted.

Markov Chain model for S-ALOHA (5/5)

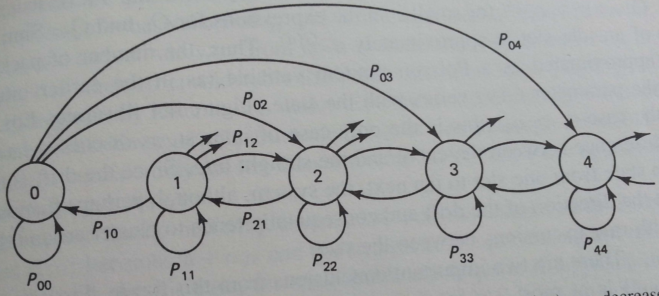

Figure 4.3 Markov chain for slotted Aloha. The state (*i.e.*, backlog) can decrease by at most one per transition, but can increase by an arbitrary amount.

Markov Chain model for S-ALOHA (1/5)

The state transition probability of going from one state to another is given by

$$P_{n,n+i} = \begin{cases} Q_a(i,n), & 2 \le i \le (m-n) \\ Q_a(i,n)[1 - Q_r(0,n)], & i = 1 \\ Q_a(1,n)Q_r(0,n) + Q_a(0,n)[1 - Q_r(1,n)], & i = 0 \\ Q_a(0,n)Q_r(1,n), & i = -1 \end{cases}$$

Markov Chain model for S-ALOHA -Equations (2/5)

- $P(n, n+i) = Q_a(i, n), \; 2 \leq i \leq (m-n)$
- This equation says that the Markov chain makes a transition from state n to state $n+i$ with probability $P(n, n+i)$ given by $Q_a(i, n)$ when i packets (between 2 and $m-n$) are transmitted in given slot.
- The justification here is that when two or more packets are transmitted in a given slot, all the packets are lost due to collision in the channel and so the number of backlogged packets goes up by i.

Markov Chain model for S-ALOHA -Equations (3/5)

- $P(n, n+i) = Q_a(i, n)[1 - Q_r(0, n)], \; i = 1$
- This equation says that the Markov chain makes a transition from state n to state $n+1$ with probability $P(n, n+1)$ corresponding to the arrival of one new packet in the given slot and one or more backlogged packets retransmitted in the slot
- Both the aforementioned events are independent and so the probability of them occurring together is given by the product of the respective probabilities.
- The justification here is that the sending of one new packet along with one or more backlogged packets results in the loss of all the packets due to collision and the newly arrived packet lost due to collision increases the number of backlogged packets awaiting retransmission by one.

Markov Chain model for S-ALOHA -Equations (4/5)

- $P(n, n+i) = Q_a(1,n)Q_r(0,n) + Q_a(0,n)[1 - Q_r(1,n)], \; i = 0$
- This equation corresponds to the case when $i = 0$, i.e when the number backlogged packets remains the same after the transition. So the equation describes the probability of remaining in state n itself.
- This equation describes the two mutually exclusive situations when the number of backlogged packets remains the same.
- In one case, one newly arrived packet is transmitted while no backlogged packets are transmitted in a given slot.
- In the other case, the converse of the above case -i.e, no newly arrived packet is transmitted while either zero or two or more backlogged packets are transmitted in the given slot.
- The justification here is that the two cases exhaust all the possible ways in which one remains in the same state of the Markov chain and in each case the two events that make up the case are independent (and so the product rule for the probabilities).

Markov Chain model for S-ALOHA -Equations (5/5)

- $P(n, n+i) = Q_a(0, n)Q_r(1, n), \; i = -1$
- This equation describes the situation when the number of backlogged packets reduces by one.
- The equation says that the probability of the backlogged packets reducing by one is precisely the product of the probability that there are no new packets transmitted and just one backlogged packet transmitted in a given slot.
- The justification for the product of the probabilities in the rule is due to the independence of the two events that occur in it.
- Note that the rules allow transition 'forward' from a given state to any number of states that lie further ahead, but allow 'backward' transition only one step back at a time.

Discrete-Time Markov Chain (DTMC)

- $\{X_n \mid n = 0, 1, 2, \ldots, \}$ be a discrete-time stochastic process.
- The process is a *Markov Chain* if whenever it is in state i there is a fixed probability P_{ij} goes to state j regardless of the process history prior to arriving at i

$$\begin{aligned} P_{ij} &= P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = 1\} \\ &= P\{X_{n+1} = j \mid X_n = i\} \end{aligned}$$

- The transition matrix P is defined such that the entry in the ith row and jth column of the matrix is P_{ij}. We denote this as $P = (P_{ij})$
- The transition matrix P is the one-step transition matrix of the Markov chain

DTMC: Transition Probability

- The transition matrix (P_{ij}) must satisfy

$$P_{ij} \geq 0, \ \sum_{j=0}^{\infty} P_{ij} = 1, \quad i = 0, 1, \ldots$$

- The above equation says that the sum of the probabilities for transition from state i to all the states of the system is 1
- The n-step transition probability P_{ij}^{n}

$$P_{ij}^{n} = P\{X_{n+m} = j \mid X_m = i\}, \quad n \geq 0, i, j \geq 0$$

- *Chapman-Kolmogorov* equations to calculate P_{ij}^{n}

$$P_{ij}^{n+m} = \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m}, \ n, m \geq 0, i, j \geq 0$$

Irreducible, aperiodic DTMC

- Two states i and j *communicate* if for some n and n', $P_{ij}^{n} > 0$ and $P_{ij}^{n'} > 0$
- A Markov chain is *irreducible*, if all its states communicate
- A Markov chain is *periodic* if there exists some integer $m \geq 1$ such that $P_{ii}^{m} > 0$ and some integer $d > 1$ such that $P_{ii}^{n} > 0$ only if n is a multiple of d
- A Markov chain is *aperiodic* if none of the states is periodic
- A probability distribution $\boldsymbol{\pi} = \{\pi_j \mid j \geq 0\}$ is a *a stationary distribution* π for the Markov chain if $\boldsymbol{\pi} = \boldsymbol{\pi} P$

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \quad j = 0, 1, \ldots$$

- Our focus is on irreducible, aperiodic Markov chains.

DTMC: Stationary distribution

- For a given irreducible, aperiodic Markov chain, whether the stationary distribution $\boldsymbol{\pi} = \boldsymbol{\pi} P$ exists is a basic question.

$$\text{Let } p_j = \lim_{n\to\infty} P\{X_n = j \mid X_0 = i\},\ i = 0, 1, \ldots$$

- p_j exists and is independent of the starting state $X_0 = i$

$$p_j = \lim_{k\to\infty} \frac{\text{Number of visits to state } j \text{ up to time } k}{k}$$

- p_j is the proportion of the time or the frequency with which the process visits j which is independent of the starting state

DTMC: Stationary distribution

Theorem

In an irreducible, aperiodic Markov chain, there are two possibilities for $p_j = \lim_{n\to\infty} P\{X_n = j \mid X_0 = i\}$*:*

1. $p_j = 0$ *for all* $j \geq 0$ *in which case the chain has no stationary distribution*
2. $p_j \geq 0$ *for all* $j \geq 0$*, in which case* $\{p_j \mid j \geq 0\}$ *is a unique stationary distribution* $\boldsymbol{\pi}$ *of the chain.*

- An example of case 1 is a queueing system where the arrival rate $\lambda \geq \mu$,the service rate, and the number of customers in the system increases to ∞. The steady-state probability p_j of having a finite number of customers j is zero
- For an irreducible, aperiodic Markov chain with *finite* states an equilibrium distribution always exists.

DTMC: Global balance equations

- Multiplying the equation $\sum_{i=0}^{\infty} P_{ji} = 1$ by π_j and using the equation $\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}$ we get

$$\pi_j \sum_{i=0}^{\infty} P_{ji} = \sum_{i=0}^{\infty} \pi_i P_{ij}, \ j = 0, 1, \ldots$$

- These equations are the *Global balance equations* (GBE)
- Here $\pi_i P_{ij}$ can be viewed as the long-term frequency of transitions from i to j
- The GBE state that at equilibrium the frequency of transitions out of j (LHS of the equations) equals the frequency of transitions into j (RHS)
- A typical approach to find the stationary distribution of an irreducible, aperiodic Markov chain to solve the GBE

DTMC: Drift and Stability

- Question: To determine whether a given irreducible, aperiodic DTMC has a stationary distribution $\boldsymbol{\pi}$ or not.
- If a stationary distribution exists, then $\pi_j > 0$ for all j and the DTMC is stable in the sense that all the states are visited infinitely often with probability 1.
- The notion of *drift* defined as follows is useful in this regard.

$$D_i = E\{X_{n+1} - X_n \mid X_n = i\} = \sum_{k=-i}^{\infty} kP_{i(i+k)} \text{ for } i = 0, 1...$$

- Roughly, the sign of D_i indicates whether, starting at some state i the state tends to increase ($D_i > 0$) or decrease ($D_i < 0$)
- Intuitively this means the chain will be stable if the drift is negative for all large enough states

DTMC: Stability

Lemma

Suppose that $D_i < \infty$ for all i, and for some scalar $\delta > 0$ and integer $i \geq 0$ we have

$$D_i = -\delta \text{ for all } i > \bar{i}$$

Then the Markov chain has a stationary distribution

Instability of S-ALOHA 1/5

- X_n be the number of backlogged packets at the beginning of slot n
- Backlogged packets behave independently
- Each packet has a probability ν of attempting retransmission in slot i
- If $X_i = k$, let $b_i(k)$ be the probability that i backlogged packets attempt to retransmit in slot n

$$b_i(k) = \binom{k}{n} \nu^i (1 - \nu)^{k-i}$$

- Let A_n be number of new packets arriving in slot n. The sequence $\{A_n\} \mid n \geq 0$ is iid such that

$$P(A_n = j) = a_j$$

$$\text{mean arrival rate}\, \lambda = E[A_n] = \sum_{i=1}^{\infty} i a_i$$

Instability of S-ALOHA 2/5

Assume that $a_0 + a_1 \in (0,1)$ and $\{X_n\}_{n\geq 0}$ is an irreducible HMC. Its transition probabilities are

$$P_{i,j} = \begin{cases} b_1(i)a_0 & \text{if } j = i - 1, \\ [1 - b_1(i)]a_0 + b_0(i)a_1 & \text{if } j = i, \\ [1 - b_0(i)]a_1 & \text{if } j = i + 1, \\ a_{j-1} & \text{if } j \geq i + 2, \end{cases} \tag{1}$$

- 1st line: one of the i backlogged packets succeeded to retransmit with probability $b_1(i)$, no new packets
- 2nd line: one of the two events-no new packets(probability a_0) and zero or strictly more than two retransmission requests from the backlog and zero retransmission request from the backlog and and one new packet
- The objective is to show that the system using Bernoulli retransmission policy is not *stable* in the sense that chain $\{X_n\}_{n\geq 0}$ is not *positive recurrent*
- To prove instability, contradict the existence of a stationary distribution π

Instability of S-ALOHA 3/5

If a stationary distribution has existed, it should satisfy the balance equations

$$\pi(i) = \pi(i)\{[1 - b_1(i)]a_0 + b_0(i)a_1\} + \pi(i-1)[1 - b_0(i-1)]a_1$$
$$+\pi(i+1)b_1(i+1)a_0 + \sum_{l=2}^{\infty}\pi(i-l)a_l$$

where $\pi(j) = 0$ if $j < 0$. We get

$$P_N = \sum_{i=0}^{N}\pi(i)$$

Summing up the balance equations from $i = 0$ to N we get

$$P_N = \pi(N)b_0(N)a_1 + \pi(N+1)b_1(N+1)a_0 + \sum_{l=0}^{N}a_l P_{N-l}$$

Instability of S-ALOHA 4/5

$$P_N(1-a_0) \;=\; \pi(N)b_0(N)a_1 + \pi(N+1)b_1(N+1)a_0 + \sum_{l=1}^{N} a_l P_{N-l}$$

and therefore

$$P_N(1-a_0) \;\leq\; \pi(N)b_0(N)a_1 + \pi(N++1)b_1(N+1)a_0 + P_{N-1}(1-a_0)$$

from which it follows that Since P_N increases with N and $\sum_{l=1}^{N} a_l \leq \sum_{l=0}^{\infty} a_l \;=\; 1 \,-\, a_0$ we have,

$$\sum_{l=1}^{N} a_l P_{N-l} \leq P_{N-1}(1 \,-\, a_0)$$

Instability of S-ALOHA 5/5

$$\frac{\pi(N+1)}{\pi(N)} \geq \frac{1 - a_0 - b_0(N)a_1}{b_1(N+1)a_0}$$

Using the expression for retransmission probability $b_i(k)$ we get

$$\frac{\pi(N+1)}{\pi(N)} \geq \frac{1 - a_0) - (1-\nu)^N a_1}{(N+1)\nu(1-\nu)^N a_0}$$

- For values of $\nu \in (0,1)$ the RHS of the inequality eventually becomes infinite and this contradicts the equality $\sum_{N=1}^{\infty} \pi(N) = 1$ and the inequalities $\pi(N) > 0$ that π should satisfy as the stationary distribution of an irreducible Markov Chain
- From the analysis we conclude that S-ALOHA is unstable for Bernoulli retransmission policy

Summary

- The Markov chain model for S-ALOHA plays a crucial role in the stability analysis of the protocol
- The Markov chain analysis of S-ALOHA is unstable for Bernoulli retransmission policy
- Stabilized S-ALOHA protocols such as Rivest's Pseudo-Bayesian algorithm are described in Chapter 4, Multiaccess Communication of the book Data Networks by Bertsekas and Gallager