582746 Modelling and Analysis in Bioinformatics

Lecture 3: Global Network Models

20.9.2016
Outline

Examples of biological networks

Global properties of networks
  Distance measures
  Degree measures
  Local clusters

Network Models
  Erdös-Renyi Model
  Watts-Strogatz Model
  Barabasi-Albert Model

More Network Properties

Statistical Testing of Network Properties
Outline

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More Network Properties

Statistical Testing of Network Properties
Protein-protein interaction network

- Vertices are proteins
- The proteins are connected if they interact with each other.
Metabolic network

- Vertices are metabolites, i.e. chemical compounds
- Edges describe how the cell can transform a metabolite into another
Gene regulatory network

- Vertices are genes
- Genes are linked if one regulates the other

Source: Shen-Orr SS, Milo R, Mangan S, Alon U. 2002
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Statistical Testing of Network Properties
Distance $d_{ij}$ is the length of the shortest path between vertices $n_i$ and $n_j$, i.e. the minimal number of edges one needs to traverse to get from $n_i$ to $n_j$.

The shortest path may not be unique, but the length of the shortest path is unique.

In directed network, we may have $d_{ij} \neq d_{ji}$.

If there is no path between $n_i$ and $n_j$, we have $d_{ij} = \infty$.

Ignoring weights, above we have $d_{12} = 2$, $d_{13} = 1, d_{14} = 2, \ldots$
Diameter and average path length

- The diameter $d_m = \max(d_{ij})$ is the maximal distance between any two nodes (= the longest shortest path)
- Average or characteristic path length
  \[ d = \langle d_{ij} \rangle = \frac{1}{N^2} \sum_{i=1}^{N_V} \sum_{j=1}^{N_V} d_{ij} \]

Ignoring weights, above $d_m = 3$, $d \approx 1.57$
Efficiency

- Efficiency, or average inverse path length: \( d_{\text{eff}} = \langle 1/d_{ij} \rangle \)
- Useful when average path length is infinite (disconnected network)
- Fully connected network has efficiency \( d_{\text{eff}} = 1 \), graph with no edges has \( d_{\text{eff}} = 0 \)

Ignoring weights, above \( d_{\text{eff}} \approx 0.73 \)
Weighted graphs

- If the edges in the graph have associated weights $w_{ij}$, it is natural to define distances based on the weights:
  - $d_{ij}$ as the sum of weights in the minimum weight path between $n_i$ and $n_j$
  - Maximum and average path length as well as efficiency naturally generalize by changing the distance measure to the weighted version

With weights, $d_{12} = 40$ (red path), $d \approx 24.2$, $d_m = 50 = d_{25}$, $d_{eff} = 0.06$
Finding shortest paths in graphs is part of classical algorithm theory, two efficient algorithms

- **Dijkstra’s algorithm**: given a vertex find shortest paths to all other vertices, basic implementation runs in $O(N_V^2)$ time, can be implemented faster for sparse graphs.
- **Floyd-Warshall algorithm**: find shortest paths for all pairs of vertices in the graph in $O(N_V^3)$ time; outputs a distance matrix $(d_{ij})_{i,j=1}^{N_V}$ in same time.
- Both work with weighted formulations.
Shortest path distances in empirical networks

Path length analysis of many networks that occur in nature reveals the small-world property

- **Metabolite graphs:** average path length $d \approx 3 \ (N_V \approx 10^3 - 10^4)$
- **WWW:** links chains between two web documents $d \approx 16 \ (N_V > 10^9)$
- **Erdős number:** shortest co-author chain to Paul Erdős, $d = 4.65 \ (N_V \approx 4 \times 10^5)$
- ...
Node degree

- Degree $k_i$ of vertex $n_i$ is the number of edges adjacent to a vertex.
- In a network without self-loops and without multiple edges between any pair of edges: degree = number of neighbours.
- In directed networks: in-degree is the number of incoming edges and out-degree is the number of outgoing edges.
Degree distribution

- Given a fixed set of vertices, \( p(k) \) denotes the probability that a randomly chosen vertex has degree \( k \).
- (Empirical) degree distribution is the list of probabilities (or relative frequencies) \( p(k), k = 0 \ldots N_V \).
- Analysis of the degree distribution is an important means to characterize networks.
Degree distribution

Fitting the empirical degree distribution to a theoretical distribution given by a mathematical law is an important tool for network analysis.

- Regular lattice: $p(k) \approx 1$, where $k$ is a constant.
- Scale free network: $p(k) \propto k^{-\gamma}$
- Random network: 
  $$p(k) \propto \binom{N_V - 1}{k} p^k (1 - p)^{N_V - 1 - k}$$
Degree distribution

The degree distributions of scale-free network and random network look markedly different:

- Scale free network: \( p(k) \propto k^{-\gamma} \) (power law, heavy tail)
- Random network: \( p(k) \propto \binom{N-1}{k} p^k (1-p)^{N-1-k} \) (binomial, light tail)
Fitting degree distributions

- Typically the fitting of the empirical distribution is based on the histogram of observations for $p(k)$.
- This is prone to errors in the region of high degree nodes due to low number of observations.
- Binning can help: divide the range of $k$ into intervals and put all observations in the interval into a common bin.
- Cumulative degree distribution $p_c(k) = \sum_{l=k}^{\infty} p(l)$, the likelihood that a given node has degree at least $k$, is more reliable and does not require binning.
Degree correlations and assortative mixing

*Degree correlation* is a statistic that reveals additional information of the connection patterns of the nodes

- **Assortative networks**: high correlation between the degrees of adjacent nodes; highly connected nodes mostly connect to other highly connected nodes
- **Disassortative networks**: highly connected nodes mostly connect to low degree nodes
- **Assortativity index** $-1 \leq r \leq 1$: Pearson correlation coefficient of degrees of adjacent nodes, $r > 0$ assortative, $r < 0$ disassortative
Social networks are typically assortative, technological and biological networks tend to be disassortative

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<tr>
<th>Group</th>
<th>Network</th>
<th>Type</th>
<th>Size $n$</th>
<th>Assortativity $r$</th>
<th>Error $\sigma_r$</th>
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<tr>
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<td>Freshwater food web</td>
<td>directed</td>
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<td>$-0.326$</td>
<td>0.031</td>
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</tbody>
</table>

Clustering coefficient

- Clustering coefficient measures the probability that two vertices with a common neighbor are connected.
- Let $E_i$ denote the number of edges between the neighbors of $v_i$, and $E_{max} = k_i(k_i - 1)/2$ the theoretical maximum. Clustering coefficient for vertex $n_i$ is now
  \[ C_i = \frac{E_i}{E_{max}} = \frac{2E_i}{k_i(k_i - 1)} \]
- Clustering coefficient for the whole graph is obtained by averaging over the vertices.

http://www.nd.edu/~swuchty/Download/Wuc.pdf, p. 24, Fig. 2
Clustering coefficient in natural networks

- Natural networks often have relatively high clustering coefficient indicating local clustering within the network.
- Negative correlation between the degree and the clustering coefficient has also been observed;
  - Low degree nodes lie in local clusters, while the neighbors of high degree nodes are less often connected.
  - Indicates modular network structure.

Example: PPIs in Mouse and Human:

(https://bccs.bristol.ac.uk/toProgramme/project/2008/Angela_Onslow_S08/)
Matching index

To be functionally related, two vertices do not need to be connected, examples:

- Two transcription factor proteins regulating the same gene
- Two metabolite molecules taking part in similar reactions

\[ MI(v, v') = \frac{2}{5} = 0.4 \]

Zamora-Lopez. Frontiers in Neuroinformatics 4, 2010
Matching index

Matching index measures the amount of neighbors the two nodes share:

\[ MI_{ij} = \frac{Shared_{ij}}{k_i + k_j - Shared_{ij}} \]

Similarity in terms of perceiving the neighborhood similarly

\[ MI(v, v') = \frac{2}{5} = 0.4 \]

Zamora-Lopez. Frontiers in Neuroinformatics 4, 2010
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More Network Properties

Statistical Testing of Network Properties
Models of complex networks

- Theoretical models of networks are needed as a basis for comparison to determine the significance of global properties or non-trivial substructures of natural networks.
- We will look at three specific models:
  - Erdös-Renyi Model
  - Watts-Strogatz Model
  - Barabasi-Albert model
Erdös-Renyi Model

- ER network consists of $N_V$ vertices
- Edge is drawn between a pair of nodes randomly with probability $p$
- Degree distribution of the ER model is binomial:
  \[ p(k) \propto \binom{N_V - 1}{k} p^k (1 - p)^{N_V - 1 - k} \]
- Degree distribution can be approximated by Poisson distribution for large graphs
Significant body of theoretical research exists for the ER model, e.g.

- For $N_V p < 1$ the network almost surely has no large connected components.
- For $N_V p \approx 1$ the network will almost surely have one large connected component.
- For $N_V p > \log N_V$ the network will almost surely be connected.
- ER network has the small-world property when $p > 1/N_V$ with average path length scaling as $l \sim \log N_V$.
- No local clustering, expected clustering coefficient $C = p = \langle k \rangle / N_V$ for all nodes.
Watts-Strogatz model

1. Arrange vertices in a ring structure
2. Connect each vertex to $K$ closest neighbours
3. With probability $p_{rew}$, rewire each edge by detaching from one end and attaching to a randomly chosen vertex.

After steps (1-2) there is local clustering, step (3) lowers average path length by creating shortcuts.

Watts-Strogatz model

- Even for low rewiring probability \((p_{rew} \ll 1)\) the average path length goes down rapidly
- Small average path length and local clustering is retained for intermediate \(p_{rew}\)
- When \(p_{rew} \to 1\), we get ER model, i.e. local clustering is destroyed
- Degree distribution is similar to ER graph: homogeneous and peaked around \(k = K\)

Barabasi-Albert model

- Start with an initial small connected network of $N_0$ vertices
- Iteratively add new vertices and connect the new vertex to $m \leq N_0$ vertices
- Draw the nodes that will be connected to the new vertex with probability proportional to their degree (preferential attachment):

$$\rho(n_i) = \frac{k_i}{\sum_j k_j}$$

(BA graph from http://melihsozdinler.blogspot.com/)
Barabasi-Albert model

- Unlike ER or WS model, Barabasi-Albert model explain the inhomogeneous degree distribution observed in natural graphs.
- With enough iterations, the degree distribution of the BA model is scale-free, with $p(k) \sim k^{-3}$.
- Average path length in BA networks has been found to be smaller than in ER and WS models.
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More Network Properties

Statistical Testing of Network Properties
Robustness and Attack tolerance

- Robustness against perturbations (mutations, environment changes) is a preferable property for biological networks.
- Networks analysis is interested in preservation of network topology under perturbations (usually: removals of vertices or edges).
Robustness and Attack tolerance

Both ER networks and scale-free networks (such as BA model) are robust towards random deletions of nodes and connections
  - A random mutation is likely to hit a low degree node in BA model

Scale-free networks are not robust towards intentional attacks
  - Removal of a set of highly connected nodes may collapse the global structure
  - "Robust, yet fragile"
Many natural networks are observed to possess modular structure with densely connected functional clusters of nodes that are sparsely connected to other nodes.

Also, hierarchical organization of network structure can be observed.

The random network models discussed above, do not directly explain these phenomena.

(Zhao et al. BMC Bioinformatics 2006, 7:386)
Modularity and hierarchical organization

- Barabasi and Albert model has been later extended to that direction
  - Based on replicating basic modules and wiring them to the central module of rest of the network
  - Recursive application leads to hierarchical organization
  - Deterministic rather than random procedure
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Statistical Testing of Network Properties
Statistical testing of network properties

How to determine if an observed property of the network is significant or if it occurred just by chance?
- Set up a null hypothesis
- Test if the observed property is consistent with the null hypothesis
Statistical testing of network properties: Example

- Suppose that we have observed a clustering coefficient $C$ for a given network. Is the network highly clustered?
- Null hypothesis: The clustering coefficient is consistent with a network of the same size and degree distribution.
- Create an ensemble of random networks with same size and degree distribution and compute the clustering coefficient of each network.
- Reject the null hypothesis if the probability of a network with clustering coefficient of at least $C$ is low enough.
- If the null hypothesis can be rejected, we can conclude that the network is highly clustered as compared to the null model.
What next?

- Thursday 10-12: Study group on global network models
- Thursday 12-14: Exercise session