Design and Analysis of Algorithms
Exam / example solutions
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1.

Quicksort sorts a given array $A[1..n]$ using divide and conquer:

```
quicksort(A, l, r)                       // call first with l=1, r=n.
  if l < r
    j ← partition(A, l, r)                // elements ≤ A[l] will precede
    quicksort(A, l, j)                    // elements > A[l].
    quicksort(A, j+1, r)
```

+6

```
partition(A, l, r)
  let B[l..r] be a new array
  j ← l ; k ← r
  for i ← l to r
      if A[i] ≤ A[l]                      // A[l] is the pivot.
            B[j] ← A[i]; j ← j+1
      else
            B[k] ← A[i]; k ← k-1
  copy B[l..r] to A[l..r]
  return j-1
```

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Best case analysis

Running time proportional to the number of comparisons:

$$\begin{cases} T(n) = T(q) + T(n-q) + n & \text{for some } q \\ T(0) = T(1) = 0 \end{cases}$$ +2

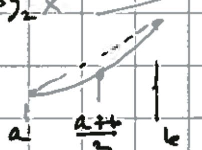
In the best case $T(n) \le T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil) + n$

a) Thus $T(n) = O(n \log n)$ by the master thm. +2

b) Claim: $T(n) \ge n \log_2 n$. Proof by induction: Case $n \le 1$ clear.
For $n \ge 2$ we $T(n) \ge q \log_2 q + (n-q) \log_2 (n-q) + n$ +2

$$\ge 2 \cdot \frac{n}{2} \cdot \log_2 \frac{n}{2} + n = n \log_2 n.$$

Since $x \log_2 x$ is convex



a) & b) $\Rightarrow$ $T(n) = \Theta(n \log n)$

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2

For an amount of money X, let $f(X)$ be the minimum number k of coins that make change for X;

+3 $f(X) = \min\{k : \exists i_1,\dots,i_k \text{ s.t. } v(i_1)+\dots+v(i_k) = X\}$.

Observe that $f(0) = 0$ and for $X > 0$ we have

+3 $f(X) = \min\{f(X - v(i)) + 1 : X \geq v(i), i \in \{1,\dots,n\}\}$.

Namely, if $(i_1,\dots,i_k)$ is optimal for X, then $(i_1,\dots,i_{k-1})$ is optimal for $X - v(i_k)$. Thus is the

+2 optimal substructure property.

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A dynamic programming algorithm:

Introduce arrays $f[1..C]$ and $p[1..C]$, $f[0] \leftarrow 0$

```
for X ← 1 to C
        best ← ∞
        for i ← 1 to n
                y ← X − v(i)
                if y ≥ 0 and f[y] +1 < best
                        best ← f[y] + 1
                        p[X] ← i                // store the coin type
        f[X] ← best

while C > 0
        print p[C]
        C ← C − v(p[C])
```

+2 (loop), +2 (output)

Running time is clearly $\Theta(n \cdot C)$

3.

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SET-COVER

Instance: collection $\mathcal{C}$ of subsets of S, integer k.

Question: Does there exist members $S_1, \ldots, S_k \in \mathcal{C}$

+2 s.t. $S_1 \cup \cdots \cup S_k = S$?

In NP: For any yes-instance, and only for them, there exist a certificate $(S_1, \ldots, S_k)$ for which

+1 the condition $S_1 \cup \cdots \cup S_k = S$ can be tested in polynomial time.

NP-hardness by reduction from VERTEX COVER.

Reduction function

+1 $$f : (V, E), k \mapsto (S, \mathcal{C}), k,$$

where $S := E$ and $\mathcal{C} := \{E_v : v \in V\}$

E_v: the set of edges in E that have v as an endpoint

+1 Clearly $f(V, E, k)$ can be computed in polynomial time.

Equivalence: $(S, \mathcal{C}), k$ is a yes-instance iff

+1 $\exists v_1, \ldots, v_k \in V$ s.t. $E_{v_1} \cup \cdots \cup E_{v_k} = E$ iff $(V, E), k$ is a yes-instance of VERTEX-COVER.

4.

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For $1 \le i < j < k \le n$ define

+3 $$X_{ijk} = \begin{cases} 1 & \text{if } A[i] > A[j] > A[k] \quad \text{(3-inversion)} \\ 0 & \text{otherwise} \end{cases}$$

The expected number of 3-inversions of A is

+2

$$E\Big(\sum_{i<j<k} X_{ijk}\Big) = \sum_{i<j<k} E(X_{ijk}) = \sum_{i<j<k} P\{X_{ijk} = 1\}$$

$$= \sum_{i<j<k} 1/3! = \binom{n}{3} / 3! = \frac{n(n-1)(n-2)}{36}$$

+2 +2

+3 Since all the $3!$ permutations between $\{i, j, k\}$ and $\{A[i], A[j], A[k]\}$ are equally likely, and only one of them yields a 3-inversion.

5\.

Let I be a maximum-size independent set of G.

+2

Assumption implies $|I| \geq |V| \cdot 2/3$ $(*)$

Let S be the set of vertices of G not matched by M.

1) S is an independent set, since M is a maximal matching.

+3

2) Let T be a minimum vertex-cover of G.
Now $|V \setminus S| \leq 2|T|$, since T must contain at least one endpoint of each edge in M, i.e., at least 1/2 of the vertices in $V \setminus S$.

+2

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3) Because the complement of an independent set is a vertex cover, and vice versa, we have $|I| + |T| = |V|$.

+2

Calculation:

$$|V| - |S| \overset{2)}{\leq} 2|T|$$

$$\Leftrightarrow \quad |S| \overset{3)}{\geq} -2(|V| - |I|) + |V|$$

$$= 2|I| - |V|$$

+3

$$\overset{(*)}{\geq} 2|I| - \frac{3}{2}|I|$$

$$= |I|/2$$

$\therefore$ $|S|$ is a 2-approximation of $|I|$.