QuickSort sort a given array $A[1..n]$ using divide and conquer:

$$\text{quickSort}(A, l, r)$$
\begin{align*}
&\text{if } l < r \\
&\text{partition}(A, l, r) \\
&\text{quickSort}(A, l, (l + r) / 2) \\
&\text{quickSort}(A, (l + r) / 2 + 1, r) \\
&+ \text{if} \{\text{elements \leq A[l]} \text{ will precede} \} \\
&\{\text{elements} > A[l] \}. \\
\end{align*}

$$\text{partition}(A, l, r)$$
\begin{align*}
&\text{let } B[1..r] \text{ be a new array} \\
&j = l; k = r \\
&\text{for } i = l \text{ to } r \\
&\text{if } A[i] \leq A[l] \\
&B[j] = A[i]; \text{ } j = j + 1 \\
&\text{else} \\
&B[k] = A[i]; \text{ } k = k - 1 \\
&\text{copy } B[1..r] \text{ to } A[l..r] \\
&\text{return } j - 1 \\
\end{align*}

**Best Case:**

Running time proportional to the number of comparisons:

\begin{align*}
T(1) &= T(0) + T(0) + 1 \\
T(n) &= T(n) + 0
\end{align*}

In the worst case:

\begin{align*}
T(n) &\leq T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n \\
&\text{by the master theorem.}
\end{align*}

- \text{Claim: } T(n) \geq n \log_2 n \\
- \text{Proof by induction: case is not clear.}

For $n > 2$, we have

\begin{align*}
T(n) &\geq q \log_2 q + (n-q) \log_2(n-q) + q + 1 \\
\end{align*}

Since $x \log_2 x \geq 2 - \frac{n}{2} \cdot \log_2 n + n = n \log_2 n$

\begin{align*}
&\text{if } q \leq \frac{n}{2} \text{ or } q > \frac{n}{2} \\
&\Rightarrow T(n) = \Omega(n \log_2 n)
\end{align*}
For an amount of money \( X \), let \( f(X) \) be the minimum number of coins that make change for \( X \):

\[
f(X) = \min \left\{ k : \exists i_1, \ldots, i_k \ni \sum_{i=1}^{k} v(i) = X \right\}.
\]

Observe that \( f(0) = 0 \) and for \( X > 0 \) we have

\[
f(X) = \min \left\{ f(X - v(i)) + 1 : X \geq v(i), \ i \in \{1, \ldots, n\} \right\}.
\]

Name \( (i_1, \ldots, i_k) \) is optimal for \( X \) then \((i_1, \ldots, i_k)\) is optimal for \( X - v(i_k) \). Then \( i_k \) is the optimal substructure property.

A dynamic programming algorithm:

1. Initialize arrays \( p[1 \ldots C] \) and \( v[1 \ldots C] \); \( f[0] \leftarrow 0 \)

2. for \( X \leq 1 \) to \( C \)
   
   best \( \leftarrow \infty \)
   
   for \( i = 1 \) to \( n \)
   
   \( y \leftarrow X - v(i) \) + 1
   
   if \( y \geq 0 \) and \( f[y] \leq \text{best} \)
   
   \( \text{best} \leftarrow f[y] + 1 \)
   
   \( f[X] \leftarrow \text{best} \)

   // Store the coin type

2. while \( C > 0 \)

   2. point \( p[C] \)

   \( C \leftarrow C - v[p[C]] \)

Running time is clearly \( O(nC) \).
3. \textsc{Set-Cover} \\
Instance: collection \( C \) of subsets of \( S \), integer \( k \).
Query: Does there exist members \( S_1, \ldots, S_k \in C \) \( s.t. \ \cup_{i=1}^{k} S_i = S \)?

In \( \text{NP} \): For any yes-instance, and only for them, there exist a certificate \((S_1, \ldots, S_k)\) from which the inclusion \( S_1 \cup \cdots \cup S_k = S \) can be tested in polynomial time.

\( \text{NP} \)-hardness by reduction from \textsc{Vertex-Cover}.

Reduction function:
\[
\phi : (V, E), k \rightarrow (S, C), k,
\]
where \( S = E \) and \( C = \{ E_{v} : v \in V \} \).

Clearly \( \phi (V, E), k \) can be computed in polynomial time.

Equivalence:
\((S, C), k \) is a \( \text{yes} \)-instance iff
\[
\forall v_i, v_j \in V \ \land \ E_{v_i} \cup \cdots \cup E_{v_j} = E \ \iff \ \phi ((V, E), k) \text{ is a yes-instance of } \textsc{Vertex-Cover}.
\]

4. For \( 1 \leq i < j < k \leq n \) define
\[
X_{ijk} = \begin{cases} 
0 & \text{otherwise}
\end{cases} \quad \text{(2-inversion)}
\]

The expected number of 2-inversions of \( A \) is
\[
E \left( \sum_{i<j<k} X_{ijk} \right) = \sum_{i<j<k} E(X_{ijk}) = \sum_{i<j<k} P(X_{ijk} = 1) = \frac{1}{3} \cdot \sum_{i<j<k} \frac{1}{2} = \frac{(n-1)(n-2)}{12} + 2
\]

Since all the 3! permutations between \( A[i], A[j], A[k] \) are equally likely, and only one of them yields a 3-inversion.

\[
\frac{1}{3} \cdot \frac{n!}{2} = \frac{n!}{3} = \frac{n(n-1)(n-2)}{3} + 2
\]
Let $I$ be a maximum-size independent set of $G$.

\[ |I| \geq 1/2 |V| \]  \hspace{1cm} (\#)

Assumption implies $|I| \geq 1/2 |V| \geq 2/3 |V|$ (\#)

Let $S$ be the set of vertices of $G$ not matched by $M$.

1) $S$ is an independent set, since $M$ is a maximal matching.

2) Let $T$ be a minimum vertex-cover of $G$.

\[ |V| - |S| \leq 2 |T|, \]  since $T$ must contain at least one endpoint of each edge in $M$, i.e., at least $1/2$ of the vertices in $V \setminus S$.

3) Because the complement of an independent set is a vertex cover, and vice versa, we have

\[ |I| + |T| = |V|. \]

**Calculation:**

\[ |V| - |S| \leq 2 |T| \]

\[ \Rightarrow \quad |S| \geq 2 |T| - |V| \]

\[ \geq 2 |I| - \frac{3}{2} |I| \]

\[ = |I| / 2 \]

\[ |S| \text{ is a } 2\text{-approximation of } |I|. \]