You are allowed to bring with you one 2-sided paper sheet of notes. Calculators, phones, or other smart devices are not allowed.

Please solve all the four problems given below. Show your work—partial credits will be given. Be clear and neat. Use either English, Swedish, or Finnish. You have two and a half hours in total—plan your time usage! Include your name and identity number, and the name of the course in every paper sheet you return—you need not use a separate sheet for each problem. Number the problems clearly.

Note: To give an algorithm or scheme means not only describing it but also proving that it has the claimed properties (correctness, running time, approximation guarantee, etc.).

1. (max 12 points)
Which of the following claims are correct? A correct answer yields 2 points, an incorrect answer -2 points, and no answer yields 0 points. (You need not justify your answers.)

(a) Every 2-state Markov chain has a unique stationary distribution.
(b) In a finite Markov chain no state is null-recurrent.
(c) Let $X$ and $Y$ be the minima of 5 independent $\text{Unif}[0,1]$ and $\text{Expon}(1)$ random variables, respectively. Then the expected values of $X$ and $Y$ are the same.
(d) A randomized algorithm gives a $(0,0)$-approximation for the value $V$ if the output $X$ equals $V$ with probability 1.
(e) The idea in the coupling method is to show that two copies of a Markov chain reach the same state after some number of steps with probability 1.
(f) The variation distance between two distributions on a countable state space is zero only if the two distributions are the same.

2. (max 12 points)
Prove that the Markov chain corresponding to a random walk on an undirected, non-bipartite graph that consists of one component is time reversible.

3. (max 12 points)
Consider a complete graph on $n$ vertices. Each edge is assigned a weight chosen independently and uniformly at random from the real interval $[0,1]$. Show that the expected weight of the minimum spanning tree of this graph is at least $1 - 1/(1 + \binom{n}{2})$.

4. (max 12 points)
A matching of a graph is a set of edges without common vertices. Give a fully polynomial randomized approximation scheme for counting the matchings of a given graph. You may assume the availability of a polynomial-time algorithm that generates a given number of independent samples from the uniform distribution over the matchings of a given graph.

A scheme that requires polynomially many samples in total yields 10 points (at maximum). To get full points, you need to show that your scheme requires only $O(m^3 \log m)$ samples, where $m$ is the number of edges in the input graph. (In fact, $O(m^2)$ samples suffice, even if one only has a fully polynomial approximately uniform sampler.)