1. **Exercise 8.23:** In a tandem queue, customers arrive to an $M/M/1$ queue according to a Poisson process of rate $\lambda$ with service times independent and exponentially distributed with parameter $\mu_1$. After completing service at this first queue, the customers proceed immediately to a second queue, also being served by a single server, where service times are independent and exponentially distributed with parameter $\mu_2$. Find the stationary distribution of this system. (*Hint:* Try to generalize the form of the stationary distribution for a single queue.)

2. **Exercise 10.3:** Show that the following alternative definition is equivalent to the definition of an FPRAS given in the chapter: A fully polynomial randomized approximation scheme (FPRAS) for a problem is a randomized algorithm for which, given an input $x$ and any parameter $\epsilon > 0$, the algorithm outputs an $(\epsilon, 1/4)$-approximation in time that is polynomial in $1/\epsilon$ and the size of the input $x$. (*Hint:* To boost the probability of success from $3/4$ to $1 - \delta$, consider the median of several independent runs of the algorithm. Why is the median a better choice than the mean?)

3. **Exercise 10.5:**
   a) Let $S_1, S_2, \ldots, S_m$ be subsets of a finite universe $U$. We know $|S_i|$ for $1 \leq i \leq m$. We wish to obtain an $(\epsilon, \delta)$-approximation to the size of the set
   \[ S = \bigcup_{i=1}^{m} S_i. \]
   We have available a procedure that can, in one step, choose an element uniformly at random from a set $S_i$. Also, given an element $x \in U$, we can determine the number of sets $S_i$ for which $x \in S_i$. We call this number $c(x)$. Define $p_i$ to be
   \[ p_i = \frac{|S_i|}{\sum_{j=1}^{m} |S_j|}. \]
   The $j$th trial consists of the following steps. We choose a set $S_j$, where the probability of each set $S_j$ being chosen is $p_i$, and then we choose an element $x_j$ uniformly at random from $S_j$. In each trial the random choices are independent of all other trials. After $t$ trials, we estimate $|S|$ by
   \[ \left( 1 - \frac{1}{t} \sum_{j=1}^{t} \frac{1}{c(x_j)} \right) \left( \frac{m}{\sum_{i=1}^{m} |S_i|} \right). \]
   Determine – as a function of $m$, $\epsilon$ and $\delta$ – the number of trials needed to obtain an $(\epsilon, \delta)$-approximation to $|S|$.
   b) Explain how to use your results from part (a) to obtain an alternative approximation algorithm for counting the number of solutions to a DNF problem.

4. **Exercise 10.6:** The problem of counting the number of solutions to a knapsack instance can be defined as follows: Given items with sizes $a_1, a_2, \ldots, a_n > 0$ and an integer $b > 0$, find the number of vectors $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$ such that $\sum_{i=1}^{n} a_i x_i \leq b$. The number $b$ can be thought of as the size of a knapsack, and the $x_i$ denote whether or not each item is put into the knapsack. Counting solutions correspond to counting the number of different sets of items that can be placed in the knapsack without exceeding its capacity.
   a) A naïve way of counting the number of solutions to this problem is to repeatedly choose $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$ uniformly at random, and return the $2^n$ times the fraction of samples that yield valid solutions. Argue why this is not a good strategy in general; in particular, argue that it will work poorly when each $a_i$ is 1 and $b = \sqrt{n}$. 

   **Submit solutions at mikko.koivisto@cs.helsinki.fi by May 3, 2016, noon**
b) Consider a Markov chain $X_0, X_1, \ldots$ on vectors $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$. Suppose $X_j$ is $(x_1, x_2, \ldots, x_n)$. At each step, the Markov chain chooses $i \in [1, n]$ uniformly at random. If $x_i = 1$ then $X_{j+1}$ is obtained from $X_j$ by setting $x_i$ to 0. If $x_i = 0$, then $X_{j+1}$ is obtained from $X_j$ by setting $x_i$ to 1 if doing so maintains the restriction $\sum_{i=1}^{n} a_i x_i \leq b$. Otherwise $X_{j+1} = X_j$. Argue that this Markov chain has a uniform stationary distribution whenever $\sum_{i=1}^{n} a_i > b$. Be sure to argue that the chain is irreducible and aperiodic.

c) Argue that, if we have an FPAUS for the knapsack problem, then we can derive an FPRAS for the problem. To set the problem up properly, assume without loss of generality that $a_1 \leq a_2 \leq \cdots \leq a_n$. Let $b_0 = 0$ and $b_1 = \sum_{j=1}^{n} a_j$. Let $\Omega(b_1)$ be the set of vectors $(x_1, x_2, \ldots, x_n) \in \{0, 1\}^n$ that satisfy $\sum_{i=1}^{n} a_i x_i \leq b$. Let $k$ be the smallest integer such that $b_k \geq b$. Consider the equation

\[
|\Omega(b)| = \frac{|\Omega(b)|}{|\Omega(b_{k-1})|} \times \frac{|\Omega(b_{k-1})|}{|\Omega(b_{k-2})|} \times \cdots \times \frac{|\Omega(b_1)|}{|\Omega(b_0)|} \times |\Omega(b_0)|.
\]

You will need to argue that $|\Omega(b_{i-1})|/|\Omega(b_i)|$ is not too small. Specifically, argue that $|\Omega(b_i)| \leq (n + 1)|\Omega(b_{i-1})|$.

5. **Exercise 10.12**: The following generalization of the Metropolis algorithm is due to Hastings. Suppose that we have a Markov chain on a state space $\Omega$ given by the transition matrix $Q$ and that we want to construct a Markov chain on this state space with a stationary distribution $\pi_x = b(x)/B$, where for all $x \in \Omega$, $b(x) > 0$ and $B = \sum_{x \in \Omega} b(x)$ is finite. Define a new Markov chain as follows. When $X_n = x$, generate a random variable $Y$ with $\Pr(Y = y) = Q_{x,y}$. Notice that $Y$ can be generated by simulating one step of the original Markov chain. Set $X_{n+1}$ to $Y$ with probability

\[
\min \left( \frac{\pi_x Q_{x,y}}{\pi_y Q_{y,x}}, 1 \right),
\]

and otherwise set $X_{n+1}$ to $X_n$. Argue that if this chain is aperiodic and irreducible, then it is also time reversible and has a stationary distribution given by the $\pi_x$.

6. **Exercise 11.11**: Show that the Markov chain for sampling all independent sets of size exactly $k \leq n/3(\Delta + 1)$ in a graph with $n$ nodes and maximum degree $\Delta$, as defined in Section 11.2.3, is ergodic and has a uniform stationary distribution.

7. **Exercise 11.14**: Consider the following variation on shuffling for a deck of $n$ cards. At each step, two specific cards are chosen uniformly at random from the deck, and their positions are exchanged. (It is possible both choices give the same card, in which case no change occurs.)

a) Argue that the following is an equivalent process: at each step, a specific card is chosen uniformly at random from the deck, and a position from $[1, n]$ is chosen uniformly at random; then the card at position $i$ exchanges positions with the specific card chosen.

b) Consider the coupling where the two choices of card an position are the same for both copies of the chain. Let $X_i$ be the number of cards whose positions are different in the two copies of the chain. Show that $X_i$ is nonincreasing over time.

c) Show that

\[
\Pr(X_{i+1} \leq X_i - 1 \mid X_i > 0) \geq \left( \frac{X_i}{n} \right)^2.
\]

d) Argue that the expected time until $X_i$ is 0 is $O(n^2)$, regardless of the starting state of the two chains.