Exercise 7.2: Consider the two-state Markov chain with the following transition matrix.

\[
P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}.
\]

Find a simple expression for \(P^t_{0,0}\).

For \(P\) to be a transition matrix we must have \(0 \leq p \leq 1\). If \(p = 1\), then \(P\) is an identity matrix and \(P^t_{0,0} = 1\) for all \(t\). If \(p = 0\), then \(P\) is an indentity matrix for even \(t\) and \(P^t_{0,0} = 1\) for odd \(t\) and \(P^t_{0,0} = 0\) for odd \(t\).

Assume then that \(0 < p < 1\). By conditioning on the last transition we get that

\[
P^t_{0,0} = P^{-1}_{0,0}P_{0,0} + P^{-1}_{0,1}P_{1,0} = P^t_{0,0}p + (1 - P^t_{0,0})(1 - p)
\]

Let \(x_t = P^t_{0,0}\). From the above we then get a recurrence relation

\[
x_t = x_{t-1}p + (1 - x_{t-1})(1 - p) = (2p - 1)x_{t-1} + (1 - p).
\]

We also know that \(x_1 = P_{0,0} = 1\). We apply the above relation few times see the general form:

\[
x_t = (1 - p) + (2p - 1)x_{t-1} = (1 - p) + (2p - 1)(1 - p) + (2p - 1)^2x_{t-2} = (1 - p) + (2p - 1)(1 - p) + (2p - 1)^2(1 - p) + (2p - 1)^3x_{t-3}.
\]

The closed form solution is therefore

\[
x_t = (1 - p) \sum_{k=0}^{t-2} (2p - 1)^k + (2p - 1)^{t-1}x_1 = (1 - p) \frac{1 - (2p - 1)^{t-1}}{1 - (2p - 1)} + (2p - 1)^{t-1}p = \frac{1}{2}(1 + (2p - 1)^t).
\]

It is easy to check this result by induction. (Note that the above formula also holds when \(p = 0\) or \(p = 1\), but the derivation is valid only for \(0 < p < 1\). For \(0 < p < 1\) we also get that \(x_t \to 1/2\) in the limit when \(t \to \infty\).

Exercise 7.5: Prove that if one state in a communicating class is transient (respectively, recurrent) then all states in that class are transient (respectively, recurrent).

Let \(i\) and \(j\) be two distinct states in the same communicating class and let \(i\) be recurrent. We show that also \(j\) must then be recurrent. Once in \(i\), with a nonzero probability the chain will move to \(j\) because it is in the same communicating class. But since \(i\) is recurrent, the chain will at some point return back to \(i\). Thus, once in \(j\), the chain always ends up in \(i\) and therefore \(j\) is recurrent if and only if from \(i\) the chain always gets back to \(j\).

Assume then that the chain has left state \(j\) and is now in state \(i\). Let \(E_n\) denote the event that the chain will visit state \(j\) before returning back to \(i\) for the \((n + 1)\)th time. From the Markov property it follows that \(E_n\)s are independent and identically probable and clearly \(\Pr(E_n) = p > 0\). The probability that the chain never visits state \(j\) again is then \(\Pr(\bigcap_{n=1}^\infty E_n) = \prod_{n=1}^\infty (1 - \Pr(E_n)) = (1 - p)^\infty = 0\). But this means that the chain will eventually get back to state \(j\) and thus state \(j\) must be recurrent. Therefore, if one state in a communicating class is recurrent then all states in that class are recurrent.

The version of the claim for transiency follows trivially.
3. **Exercise 7.12:** Let $X_n$ be the sum of $n$ independent rolls of a fair die. Show that, for any $k \geq 2$,

$$\lim_{n \to \infty} \Pr(X_n \text{ is divisible by } k) = \frac{1}{k}.$$  

We construct a Markov chain with $k$ states $0, 1, \ldots, k - 1$ and transition probability matrix $P$. From state $i$ one can move to states $i + 1 \pmod{k}, i + 2 \pmod{k}, \ldots, i + 6 \pmod{k}$, each of these 6 transitions having probability $1/6$ and the choices made according to the results of the die rolls. Denote by $Y_n$ the state of the chain after $n$ steps. Then $Y_n = X_n \pmod{k}$ and thus $k|X_n$ if and only if $Y_n = 0$. Clearly the chain is finite, irreducible and aperiodic.

Consider the distribution $\tilde{\pi} = (1/k, 1/k, \ldots, 1/k) \in [0, 1]^k$. Then for all $j$ we have

$$(\tilde{\pi}P)_j = \sum_{i=0}^{k-1} \pi_i P_{i,j} = \frac{1}{k} \sum_{i=0}^{k-1} P_{i,j} = \pi_j \sum_{i=0}^{k-1} P_{i,j} = \pi_j,$$

where the last equality holds since there are exactly 6 states $i$ with nonzero $P_{i,j}$, namely states $j - 1 \pmod{k}, j - 2 \pmod{k}, \ldots, j - 6 \pmod{k}$, and each has transition probability $1/6$ of moving to $i$. So $\tilde{\pi}P = \pi$ and thus $\pi$ is the stationary distribution of the chain. Now, according to Theorem 7.7 we have

$$\lim_{n \to \infty} \Pr(Y_n \text{ is divisible by } k) = \lim_{n \to \infty} \Pr(Y_n = 0) = \lim_{n \to \infty} P^n_{0,0} = \pi_0 = \frac{1}{k}.$$  

In the above proof we observed that $\sum_{i=0}^{k-1} P_{i,j}$, that is, the sum of the elements in a column, is 1. A matrix is said to be **doubly stochastic** if all entries are nonnegative and if row and column sums are 1. The result of the exercise can be generalized: every finite, irreducible and aperiodic Markov chains with doubly stochastic transition matrix has uniform stationary distribution.

4. **Exercise 7.13:** Consider a finite Markov chain on $n$ states with stationary distribution $\pi$ and transition probabilities $P_{i,j}$. Imagine starting the chain at time 0 and running it for $m$ steps, obtaining the sequence of states $X_0, X_1, \ldots, X_m$. Consider the states in reverse order, $X_m, X_{m-1}, \ldots, X_0$.  

**a)** Argue that given $X_{k+1}$, the state $X_k$ is independent of $X_{k+2}, X_{k+3}, \ldots, X_m$. Thus the reverse sequence is Markovian.

We need to show that also in the reverse sequence a state is independent of all preceding states given the previous state, that is, that for all $k$

$$\Pr(X_k = a_k \mid X_{k+1} = a_{k+1}, \ldots, X_m = a_m) = \Pr(X_k = a_k \mid X_{k+1} = a_{k+1}).$$

Denote $"X_j = a_j"$ shortly by "$j$" so that the left hand side of the equation becomes $\Pr(k \mid k+1, \ldots, m)$. Now, by using the definition of conditional probability we get

$$\Pr(k \mid k+1, \ldots, m) = \frac{\Pr(k, \ldots, m)}{\Pr(k+1, \ldots, m)} = \frac{\Pr(k, k+1) \Pr(k+2, \ldots, m \mid k, k+1)}{\Pr(k+1) \Pr(k+2, \ldots, m \mid k+1)}.$$  

But since for the original chain is Markovian, we know that $\Pr(k+2, \ldots, m \mid k, k+1) = \Pr(k+2, \ldots, m \mid k+1)$. Therefore

$$\Pr(k \mid k+1, \ldots, m) = \frac{\Pr(k+1)}{\Pr(k+1)} = \Pr(k \mid k+1),$$

which completes the proof.

**b)** Argue that for the reverse sequence, the transition probabilities $Q_{i,j}$ are given by

$$Q_{i,j} = \frac{\pi_j P_{j,i}}{\pi_i}.$$
Exercise 7.19:

Consider the gambler’s ruin problem, where a player plays until they lose $l_1$ dollars or win $l_2$ dollars. Prove that the expected number of games played is $l_1 l_2$.

Let $X_{a,b}^k$ be the number of games to be still played when the player has already played $k$ games and the player and the opponent have currently $a$ and $b$ coins respectively. We are then interested in the expectation of $X_{l_1,l_2}^0$.

In each game, with probability $1/2$ the player wins and with probability $1/2$ the player loses, so for $a, b > 0$ we have $X_{a,b}^k = 1 + \frac{1}{2} X_{a+1,b-1}^{k+1} + \frac{1}{2} X_{a-1,b+1}^{k+1}$ and by the linearity of expectations $E[X_{a,b}^k] = 1 + \frac{1}{2} E[X_{a+1,b-1}^{k+1}] + \frac{1}{2} E[X_{a-1,b+1}^{k+1}]$. Since the results of previous games do not affect the future games, $E[X_{a,b}^k] = E[X_{a,b}^0]$ for all $k$. Thus, using $h_{a,b}$ as a shorthand for $E[X_{a,b}^0]$, for $a, b > 0$ we have

\[
h_{a,b} = 1 + \frac{1}{2} h_{a+1,b-1} + \frac{1}{2} h_{a-1,b+1}.
\]

If $a = 0$ or $b = 0$, then no more games are played and therefore for all $a$ and $b$ we have

\[
h_{0,b} = h_{a,0} = 0.
\]

When starting from $a = l_1$ and $b = l_2$ we thus get the following $l_1 + l_2 + 1$ equations:

\[
h_{l_1+l_2,0} = 0;
\]

\[
h_{a,b} = 1 + \frac{1}{2} h_{a+1,b-1} + \frac{1}{2} h_{a-1,b+1}, \quad a + b = l_1 + l_2 \text{ and } a, b > 0;
\]

\[
h_{0,l_1+l_2} = 0.
\]

Since this system of equations has $l_1 + l_2 + 1$ linearly independent equations and $l_1 + l_2 + 1$ variables (of form $h_{a,b}$), there is a unique solution. We show that this solution is $h_{a,b} = ab$. Clearly $h_{a,b} = ab$ satisfies the first and the last equation. By plugging $h_{a,b} = ab$ into the other $l_1 + l_2 - 1$ equations we get

\[
1 + \frac{1}{2} h_{a+1,b-1} + \frac{1}{2} h_{a-1,b+1} = 1 + \frac{1}{2} (a + 1)(b - 1) + \frac{1}{2} (a - 1)(b + 1) = ab = h_{a,b},
\]

so also the middle equations are satisfied. Hence, $h_{a,b} = ab$ must be the unique solution of the system and therefore $E[X_{l_1,l_2}^0] = h_{l_1,l_2} = l_1 l_2$.

Exercise 7.18:

a) Consider a random walk on the 2-dimensional integer lattice, where each point has four neighbors (up, down, left, and right). Is each state transient, null recurrent, or positive recurrent? Give an argument.

Let $P_{i,j}^t$ denote the probability of moving from state $i$ to state $j$ in $t$ steps. First we prove the following lemma (see Exercise 7.15).

Lemma 1. The sum $\sum_{t=1}^{\infty} P_{i,i}^t$ is unbounded if and only if state $i$ is recurrent.
Proof: Let $X$ be the number of times the chain visits state $i$ after starting at $i$. If state $i$ is transient, then the probability of returning back to $i$ is $p = \sum_{t=1}^{\infty} P_{t,i} < 1$, variable $X$ is geometrically distributed with parameter $p$ and thus $E[X] = 1/p$ is finite. On the other hand, if $E[X]$ is finite, then by Markov’s inequality the probability of visiting $i$ more than $2E[X]$ times is at most $1/2$. Hence, $p$ must be less than 1 and state $i$ is transient. Therefore the state is transient if and only if $E[X]$ is finite.

Let $X_i$ be an indicator variable for the event that the chain is at state $i$ after $t$ steps when starting at $i$. The probability of this event is $P_{t,i}$. Now $X = \sum_{t=1}^{\infty} X_i$ and by linearity of expectation $E[X] = \sum_{t=1}^{\infty} E[X_i] = \sum_{i=1}^{\infty} P_{t,i}$. This completes the proof. \qed

Therefore we are interested in the sum $\sum_{t=1}^{\infty} P_{t,i}$. Clearly $P_{t,i} = 0$ for odd $t$, since if we color the lattice with a checkerboard pattern, then after an odd number of steps the chain is at a state with different color than the starting state. Thus we only need to consider probabilities $P_{2t}^{2t}$.

Furthermore we make the observation that while the chain with transition matrix $P$ is periodic, the chain with transition matrix $P^2$ is aperiodic. Thus, if the states are recurrent, we can use Theorem 7.11 from the book to see whether the states are null recurrent or positive recurrent. (Clearly the chain $P$ is positive recurrent if and only if the chain $P^2$ is positive recurrent.)

Let $X_t$ be the position after $t$ steps and let $X_t^+$ and $X_t^-$ be the orthogonal projections of $X_t$ on diagonals $y = +x$ and $y = -x$ respectively. Now, when the chain $X_t$ takes one step, both $X_t^+$ and $X_t^-$ move a length of $1/\sqrt{2}$ along the respective diagonals. Thus one step breaks down into two substeps, one in both diagonal directions. It is easy to see that the directions of these two moves are chosen independently and uniformly at random. Thus, $X_t^+$ and $X_t^-$ are independent and identically distributed random walks on the 1-dimensional lattice. Since $X_0 = X_0$ and only if $X_t^+ = X_0^+$ and $X_t^- = X_0^-$, we have that

$$P_{2t}^{2t} = \Pr(X_{2t} = i \mid X_0 = i) = \Pr(X_{2t}^+ = i \mid X_0^+ = i)\Pr(X_{2t}^- = i \mid X_0^- = i) = \Pr(X_{2t}^+ = i \mid X_0^+ = i)^2.$$

Above $\Pr(X_{2t}^+ = i \mid X_0^+ = i)$ is the probability that on a walk on the 1-dimensional lattice we are at the starting point after $2t$ steps. This only happens if we take exactly $t$ steps left and $t$ steps right. There are $t$ ways to do this and total of $2^t$ ways to choose the directions arbitrarily, so the probability is $t^t/2^t$. Therefore

$$P_{2t}^{2t} = \left(\frac{2t}{t}ight)^2 2^{-4t}.$$  \hfill (1)

By using Lemma 7.3 (Stirling’s Formula) from the book, we get that

$$\left(\frac{2t}{t}\right)^t \geq \frac{\sqrt{2\pi(2t)}}{4\sqrt{2\pi} \sqrt{t}} \left(\frac{2t}{e}\right)^t \left(\frac{e}{2t}\right)^t = \frac{c_1 2^{2t}}{\sqrt{t}}$$  \hfill (2)

for a constant $c_1$ and similarly that

$$\left(\frac{2t}{t}\right)^t \leq \frac{c_2 2^{2t}}{\sqrt{t}}$$  \hfill (3)

for another constant $c_2$.

From (1) and (2) we obtain

$$\sum_{t=1}^{n} P_{2t}^{2t} \geq \sum_{t=1}^{n} \left(\frac{c_1 2^{2t}}{\sqrt{t}}\right)^2 2^{-4t} = c_1^2 \sum_{t=1}^{n} \frac{1}{t} \xrightarrow{n \to \infty} \infty$$

Thus $\sum_{t=1}^{\infty} P_{2t}^{2t} = \infty$ and by Lemma 1 the chain is recurrent. From (1) and (3) we get that

$$P_{2t}^{2t} \leq \left(\frac{c_2 2^{2t}}{\sqrt{t}}\right)^2 2^{-4t} = \frac{c_2^2}{t} \xrightarrow{n \to \infty} 0.$$

Therefore, $P_{2t}^{2t} \to 0$ as $n \to \infty$ and by Theorem 7.11 the chain must be null recurrent.

b) Answer the problem in (a) for the 3-dimensional integer lattice.
Again we have $P_{i,j}^t = 0$ for odd $t$. To get back to the starting position after $2t$ steps we have to take $a$ up, $a$ down, $b$ left, $b$ right, $c$ forward and $c$ backward steps, where $a, b, c \in \mathbb{N}$ and $a + a + b + b + c + c = 2t$. The number of ways to do this is

$$\sum_{a,b,c \geq 0, \ a+b+c=t} \binom{2t}{a, a, b, b, c, c} = \sum_{a,b,c \geq 0, \ a+b+c=t} \frac{(2t)!}{(a!b!c!)^2}$$

As there are $6^{2t}$ ways to choose the directions arbitrarily, the probability of getting back to the starting state is

$$P_{i,i}^{2t} = \sum_{a,b,c \geq 0, \ a+b+c=t} \frac{(2t)!}{(a!b!c!)^2} \left(\frac{1}{6}\right)^{2t}$$

For the cases where $t$ is divisible by 3, that is, $t = 3m$, we have that

$$a!b!c! \geq m!m!m! = ((t/3)!)^3$$

for all $a, b, c \geq 0$ such that $a + b + c = t$. Using this lower bound we get

$$P_{i,i}^{2t} \leq \sum_{a,b,c \geq 0, \ a+b+c=t} \frac{(2t)!}{((t/3)!)^3(a!b!c!)} \left(\frac{1}{6}\right)^{2t}$$

$$= \frac{(2t)!}{t!((t/3)!)^3} \left(\frac{1}{2}\right)^{2t} \left(\frac{1}{3}\right)^t \sum_{a,b,c \geq 0, \ a+b+c=t} \frac{t!}{a!b!c!} \left(\frac{1}{3}\right)^t.$$

But

$$\sum_{a,b,c \geq 0, \ a+b+c=t} \frac{t!}{a!b!c!} \left(\frac{1}{3}\right)^t = 1,$$

since the left hand side is the total probability of all the ways of placing $t$ balls randomly into three bins (A single term in the sum is the probability of placing exactly $a$, $b$ and $c$ balls into first, second and third bin respectively, and the sum enumerates all possible combinations of $a$, $b$ and $c$.) Therefore, using the above fact and then Lemma 7.3 (Stirling’s Formula) we obtain

$$P_{i,i}^{2t} \leq \frac{(2t)!}{t!((t/3)!)^3} \left(\frac{1}{2}\right)^{2t} \left(\frac{1}{3}\right)^t$$

$$\leq \frac{\sqrt{2\pi(2t)}}{16\sqrt{2\pi t} (\sqrt{2\pi t}/3)^3} \left(\frac{2t}{e}\right)^{2t} \left(\frac{e}{t/3}\right)^t \left(\frac{1}{2}\right)^{2t} \left(\frac{1}{3}\right)^t$$

$$= \frac{c_3}{t^{3/2}},$$

where $c_3$ is a constant.

For the cases where $t$ is not divisible by 3, that is, $t = 3m - 1$ or $t = 3m - 2$, we know that

$$P_{i,i}^{3m} = \Pr(X_{3m} = i \mid X_0 = i)$$

$$\geq \Pr(X_{3m} = i, X_{3m-2} = i \mid X_0 = i)$$

$$= \Pr(X_{3m} = i \mid X_{3m-2} = i) \Pr(X_{3m-2} = i \mid X_0 = i)$$

$$= \left(\frac{1}{6}\right) P_{i,i}^{3m-2}$$

and similarly

$$P_{i,i}^{3m} \geq \Pr(X_{3m} = i \mid X_{3m-2} = i) \Pr(X_{3m-2} = i \mid X_{3m-4} = i) \Pr(X_{3m-4} = i \mid X_0 = i)$$

$$=(1/6)^2 P_{i,i}^{3m-4}.$$
As a result we get that the sum
\[ \sum_{t=1}^{\infty} P_{ij}^t \leq \sum_{m=1}^{\infty} (6^2 + 6 + 1) P_{ii}^{2m} \leq \sum_{m=1}^{\infty} \frac{(6^2 + 6 + 1)c_3}{3^{3/2}m^{3/2}} = \frac{(6^2 + 6 + 1)c_3}{3^{3/2}} \sum_{m=1}^{\infty} \frac{1}{m^{3/2}} \]
converges since \( \sum_{m=1}^{\infty} \frac{1}{m^{3/2}} \) is known to converge, and thus by Lemma 1 each state is transient.

7. **Exercise 7.26:** Let \( n \) equidistant points be marked on a circle. Without loss of generality, we think of the points as being labeled clockwise from 0 to \( n-1 \). Initially, a wolf begins at 0 and there is one sheep at each of the remaining \( n-1 \) points. The wolf takes a random walk on the circle. For each step, it moves with probability \( \frac{1}{2} \) to one neighboring point and with probability \( \frac{1}{2} \) to the other neighboring point. At the first visit to a point, the wolf eats a sheep if there is still one there. Which sheep is most likely to be the last eaten.

Let \( i \neq 0 \) and denote by \( p_i \) the probability that the sheep at \( i \)th point is eaten last. This is the same as the probability that \( i \) is the last point reached by the wolf. If \( n = 2 \) then there is only one sheep and trivially \( p_1 = 1 \). Assume then that \( n \geq 3 \).

If the point \( i \) is visited last, then the wolf must have reached both \( i-1 \) and \( i+1 \) before that (here and later \( n \) is same as 0). On the other hand, if the wolf has visited both of these, then it must have also visited all the other points too except \( i \). In addition, to be able to visit \( i \), the wolf must first visit at least one of the points \( i-1 \) and \( i+1 \). Denote by \( p_i^- \) the probability that the wolf visits point \( i-1 \) before point \( i+1 \) and denote by \( p_i^+ \) the opposite event. Clearly \( p_i^- + p_i^+ = 1 \).

First consider the case that the wolf visits \( i-1 \) before \( i+1 \). When reaching \( i-1 \), it has thus not visited \( i+1 \) or \( i \) yet. We are interested in the probability that, when the wolf continues, it reaches \( i+1 \) before \( i \). The situation corresponds to the Gambler’s Ruin problem where the first player has 1 dollar and the second player has \( n-2 \) dollars. The probability of reaching \( i+1 \) before \( i \) is the same as the probability that the first player wins, which is \( 1/(1+(n-2)) = 1/(n-1) \).

The case where the wolf visits \( i+1 \) before \( i-1 \) is identical. Thus the total probability of visiting \( i \) last is
\[ p_i = p_i^- \cdot \frac{1}{1-n} + p_i^+ \cdot \frac{1}{1-n} = \frac{1}{1-n}. \]

Since this is the same for all points \( i \), every sheep is equally likely to be eaten last.