Laskarit

Tulevat jakoon (webiin) toivottavasti keskiviikkona. Jatkossa pyrimme aiemmin.

Palautus *ennen* maanantain luennon alkua, Johanna Patokosken postilokeroon, hallintokäytävällä 2. kerroksessa, lähellä huonetta C229.

(Johannan huone on C221)

Laskareista lisäpisteitä tenttiin

Navier-Stokes yhtälö

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{V} + \vec{F}$$

On olemassa yleisempiäkin muotoja, meille riittää tämä. Oletukset:

- Newtonin neste, μ vakio
- kokoonpuristumaton, ρ vakio

(μ = dynaaminen viskositeetti, $\nu = \frac{\mu}{\rho}$ kinemaattinen viskositeetti)

Puretaan termit auki

$$\vec{V} = (u, v, w)$$

$$\frac{\partial \vec{V}}{\partial t} = \frac{\partial}{\partial t}(u, v, w)$$

$$= \left(\frac{\partial}{\partial t}u, \frac{\partial}{\partial t}v, \frac{\partial}{\partial t}w\right)$$

Helppoa!

$(\vec{V} \cdot \nabla)\vec{V}$?

$$\vec{V} = (u, v, w)$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\vec{V} \cdot \nabla = (u, v, w) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$(\vec{V} \cdot \nabla)\vec{V} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u, v, w)$$

$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z},\ u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z},\ u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\underline{\underline{\frac{1}{\rho} \nabla p}}$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\frac{1}{\rho} \nabla p = \left(\frac{1}{\rho} \frac{\partial p}{\partial x}, \frac{1}{\rho} \frac{\partial p}{\partial y}, \frac{1}{\rho} \frac{\partial p}{\partial z}\right)$$

(p skalaari, ∇p vektori)

$$\underline{\underline{\nu \nabla^2 \vec{V}}}$$

$$\nabla^2 = \nabla \cdot \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nu \nabla^2 \vec{V} = \nu \nabla^2 (u, v, w)$$

$$= \left(\nu\left(\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u + \frac{\partial^2}{\partial z^2} u\right), \nu\left(\frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v + \frac{\partial^2}{\partial z^2} v\right), \nu\left(\frac{\partial^2}{\partial x^2} w + \frac{\partial^2}{\partial y^2} w + \frac{\partial^2}{\partial z^2} w\right) \right)$$

NS purettuna komponentteihin

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Stationaarinen (ei riipu ajasta)

$\frac{\partial}{\partial t}$ - termit $= 0$

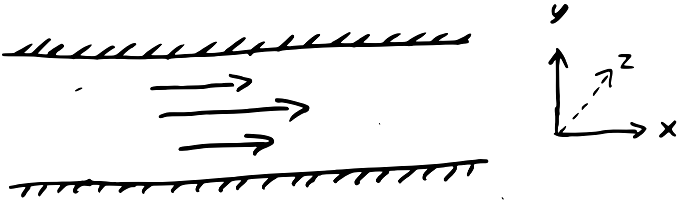
Vakio jonkin akselin suunnassa?

x-akselin? $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial w}{\partial x} = 0$

Ei virtaa johonkin suuntaan?

u tai v tai w $= 0$

Virtaus 2 levyn välissä (Couette-virtaus)



$$\frac{\partial}{\partial t} = 0 \,, \quad \frac{\partial}{\partial z} = 0 \,, \quad w = 0 \,, \quad v = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} \qquad = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \right)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} \qquad = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \qquad \right)$$

$$u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\quad + \frac{\partial^2 u}{\partial y^2} \quad \right)$$

Saman voisi tehdä virtaukselle putkessa (Hagen-Poiseulle-virtaus), mutta pitäisi kirjoittaa NS yhtälö sylinterikoordinaatistossa, eli siis muistella ∇:n muoto sylinterikoordinaatistossa.

Jos olisimme huippuyliopisto, antaisin tämän laskariksi.

Notaatiosta

∇ voisi periaatteessa olla myös gradientti-operaattori n-ulotteisessa avaruudessa.

Esim. $n=4$, $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}\right)$

Mutta jos tällä kurssilla on 4-muuttujan funktio, vaikka $\vec{V}(x,y,z,t)$, ei kuitenkaan haluta nablata t:n suhteen.

Siis $\nabla = \nabla_3 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

I han vaan huomantus:

Jos merkitään

$$\hat{u} = \frac{u}{|\vec{V}|}, \quad \hat{v} = \frac{v}{|\vec{V}|}, \quad \hat{w} = \frac{w}{|\vec{V}|}, \quad \hat{V} = \frac{\vec{V}}{|\vec{V}|}$$

$\hat{\mathbf{V}} = (\hat{u}, \hat{v}, \hat{w})$ on $\vec{V}$:n suuntainen yksikkövektori

$$(\vec{V} \cdot \nabla)\vec{V} = (|\vec{V}|\hat{V} \cdot \nabla)\vec{V} = |\vec{V}|\underbrace{(\hat{V} \cdot \nabla)\vec{V}}$$

$$\hat{V} \cdot \nabla = \left(\hat{u}\frac{\partial}{\partial x} + \hat{v}\frac{\partial}{\partial y} + \hat{w}\frac{\partial}{\partial z}\right)$$

derivaatta suuntaan $(\hat{u}, \hat{v}, \hat{w})$

$$\frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\frac{1}{\rho} \nabla p + \nu \nabla^2 V$$

$$\rho \frac{\partial V}{\partial t} + (V \cdot \nabla) \rho V = -\nabla p + \mu \nabla^2 V$$

" $m\dot{a} \cdot \frac{d}{dx} m v = F \quad F_\mu$ "

melkein $F = ma$

Kohteen etäisyys ajan funktiona

$s(t)$ yksikkö: m

nopeus

$v(t) = \frac{\partial}{\partial t} s(t)$ yksikkö: $\frac{m}{s}$

Derivointi lisää yksikön nimittäjään sen yksikön jonka suhteen derivoidaan

$$NS: \boxed{\frac{\partial V}{\partial t} + \underbrace{(V \cdot \nabla) V}_{\text{inertia}} = -\frac{1}{\rho}\nabla p + \nu \underbrace{\nabla^2 V}_{\text{sisäinen kitka}}}$$

$$\frac{(V \cdot \nabla) V}{\nu \nabla^2 V} \sim \frac{U \cdot U/L}{\nu\, U/L^2} = \frac{UL}{\nu} = Re$$

ks. monisteen taulukko 1

Jatkossa jätetään viskositeettitermi pois, sillä se on pieni (rajakerroksessa)

Turbulenssia yritetään kuvata aikakeski-
arvojen avulla. Aikakeskiarvoistamme
NS-yhtälön.

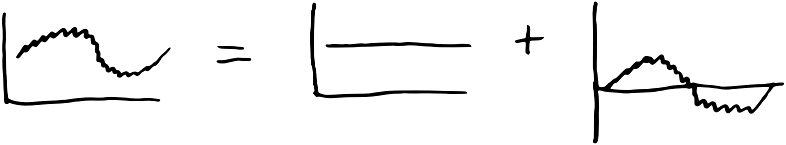
Tätä kutsutaan Reynolds-keskiarvoistamiseksi.

Sitä varten termeille tehdään
Reynolds-hajotelma.

Reynoldsin hajotelma

$$f(t) = \bar{f} + f'(t)$$

aikakeskiarvo   hetkellinen poikkeama



1 2 3 1 3 = 2 2 2 2 2 + -1 0 1 -1 1

Diskreetti aikasarja:

$$f_1, f_2, f_3, \ldots, f_n$$

$$\bar{f} = \frac{1}{n}\sum_{i=1}^{n} f_i \qquad f_i' = f_i - \bar{f}$$

Jatkuva funktio:

$$f(t), \quad 0 \leq t \leq T$$

$$\bar{f} = \frac{1}{T}\int_0^T f(t)\,dt \qquad f'(t) = f(t) - \bar{f}$$

Reynoldsin sääntöjä

$\overline{c} = c$ (kun c = vakio)

seuraus:

$\overline{\overline{f}} = \overline{f}$ (tätä ihan tosi tarvitaan)

$\overline{f+g} = \overline{f} + \overline{g}$

$$\overline{f'} = \overline{f - \overline{f}} \qquad \text{(määritelmästä)}$$

$$= \overline{f} - \overline{\overline{f}} = \overline{f} - \overline{f} = 0$$

$$\overline{c \cdot f} = c \cdot \overline{f} \qquad (c = \text{vakio})$$

joten:

$$\overline{\overline{f} \cdot g} = \overline{f} \cdot \overline{g} \quad \text{, ja vielä:}$$

$$\overline{\overline{f} \cdot g'} = 0 \quad \text{koska } \overline{g'} = 0$$

ja kolme tärkeintä:

1) $\overline{f'g'} \neq 0$ (yleensä)

$f = 1\ 2\ 3\ 4\ 5 = 3\ 3\ 3\ 3\ 3 + -2\ -1\ 0\ 1\ 2$

$g = 1\ 0\ 0\ 1\ 3 = 1\ 1\ 1\ 1\ 1 + 0\ -1\ -1\ 0\ 2$

$f'g' = 0\ 1\ 0\ 0\ 4$

$\overline{f'g'} = \overline{01004} = 1 \neq 0$

$f = f(x,t)$ tai $f(x,y,z,t)$

2) $\overline{\frac{\partial}{\partial x} f} = \frac{\partial}{\partial x} \overline{f}$

$= \frac{1}{T} \int_0^T \frac{\partial}{\partial x} f(x,t)\, dt = \frac{\partial}{\partial x}\left(\frac{1}{T} \int_0^T f(x,t)\, dt \right)$

$= \frac{\partial}{\partial x} \overline{f}$

Samoin kuin saatiin $\overline{\frac{\partial}{\partial x} f} = \frac{\partial}{\partial x} \overline{f}$

saadaan myös:

$$\overline{\nabla f} = \nabla \overline{f}$$

$$\overline{\nabla f'} = \nabla \overline{f'} = \nabla 0 = 0$$

ja jos f on vektori:

$$\overline{\nabla \cdot f} = \nabla \cdot \overline{f}$$

$$\overline{\nabla \cdot f'} = 0$$

3)

$$\overline{f\,g} = \overline{(\bar{f} + f')(\bar{g} + g')}$$

$$= \overline{\bar{f}\bar{g} + f'\bar{g} + \bar{f}g' + f'g'}$$

$$= \overline{\bar{f}\bar{g}} + \overline{f'\bar{g}} + \overline{\bar{f}g'} + \overline{f'g'}$$

$$= \bar{f}\bar{g} + 0 + 0 + \overline{f'g'}$$

$$= \bar{f}\bar{g} + \overline{f'g'}$$

koska $\overline{f'g'} \neq 0$ (yleensä)

niin myös $\overline{f'f'} \neq 0$ (yleensä)

$\overline{f'f'}$ f:n varianssi

$\overline{f'g'}$ f:n ja g:n kovarianssi

NS (ilman viskositeettitermiä)

$$\frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\frac{1}{\rho} \nabla p$$

Seuraavaksi: sijoitetaan

$$V = \overline{V} + V' = (\overline{u} + u', \overline{v} + v', \overline{w} + w')$$

ja katsotaan mitä saadaan ...