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Sets as Graphs

Candidate:
Alexandru Ioan Tomescu

Supervisors:
Alberto Policriti
Eugenio G. Omodeo

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Author's e-mail: alexandru.tomescu@uniud.it

Author's address:

Dipartimento di Matematica e Informatica
Università degli Studi di Udine
Via delle Scienze, 206
33100 Udine
Italia

To my parents

Abstract

In set theory and formal logic, a *set* is generally an object containing nothing but other sets as elements. Not only sets enable uniformity in the formalization of the whole of mathematics, but their ease-of-use and conciseness are employed to represent information in some computer languages. Given the intrinsic nesting property of sets, it is natural to represent them as directed graphs: vertices will stand for sets, while the arc relation will mimic the membership relation. This switch of perspective is important: from a computational point of view, this led to many decidability results, while from a logical point of view, this allowed for natural extensions of the concept of set, such as that of *hyperset*.

Interpreting a set as a directed graph gives rise to many combinatorial, structural and computational questions, having as unifying goal that of a transfer of results and techniques across the two areas. Here, we study sets under the spotlight of combinatorial enumeration, canonical encodings by numbers, random generation, digraph immersions as well-quasi-orders. We also tackle the decidability problem for the celebrated Bernays-Schönfinkel-Ramsey class of first-order formulae, over hypersets, motivated by a recent decidability result for standard sets.

This thesis is also devoted to an investigation on the underlying structure of sets; ultimately, by studying the undirected graphs underlying sets, which we call *set graphs*, we study which graphs can be 'implicitly' represented by sets. We elucidate the complexity status of the recognition problem for set graphs, we give characterizations in terms of hereditary graph classes, and put forth polynomial algorithms for certain graph classes. The set interpretation of a graph also leads to simpler proofs of two classical results on claw-free graphs. We have taken advantage of their set-theoretic flavor to formalize them with moderate effort in the set-based proof-checker Referee; these formal proofs are presented in full in an Appendix.

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Contents

Introduction

Set theory was initially proposed as a study of infinite sets, its birthplace being a series of papers published between 1874 and 1884 by Georg Cantor. In these, Cantor proved the uncountability of real numbers, introduced cardinal and ordinal numbers and formulated the celebrated Continuum Hypothesis. During the so-called "foundational crisis in mathematics", one of the representative contradictions appearing was Bertrand Russell's 1901 paradox concerning the existence of a set of all sets. Actually, it is the naturalness involved in the spontaneous concept of set that, unless properly tamed, leads to Russell's and to other similar antinomies. David Hilbert acknowledged that, on the one hand, set theory had pointed out the necessity to perfect logical theory, and that, on the other hand, set theory itself, once established axiomatically, can lie at the foundations of mathematics. The axiomatization of set theory which has now become standard is the one presented by Ernst Zermelo in 1908, with later emendations and additions due to Hermann Weyl, to Abraham Fraenkel and Thoralf Skolem (1922/1923), and to John von Neumann (1925). This theory is commonly referred to as the Zermelo-Fraenkel set theory, or ZF.

The ZF axioms permit a bottom-up construction of the "universe of all sets": they basically assert the existence of an empty set and provide rules for constructing new sets out of existing ones. Under these axioms, a set can only have other sets for elements. Restricting one's consideration to such "pure" sets is not limiting in principle: in fact, ZF suffices for the formalization of the whole of mathematics.

The ease and conciseness with which sets can render complex mathematical or abstract objects have been the motivation behind the aim of representing information in a set theoretic manner. A new field, today called computable set theory, emerged as a long-term research project initiated by Jacob T. Schwartz in the 1970s with the intention of cross-fertilizing set theory and computer science. This has led, on the one hand, to set-based programming languages such as SETL [134], or the more recent {log} [48] and CLP($\mathcal{SET}$) [49]. On the other hand, it has uncovered decidable fragments of set theory together with decidability algorithms implementable on a computer [25, 116]. One emblematic example is the **M**ulti-**L**evel-**S**yllogistic with **S**ingleton fragment and its enaction into the proof-checker Referee/ÆtnaNova [96, 133]. Referee aims at assisting its users in the development of computer-verified proofs of mathematical facts, even one as profound as the Cauchy integral theorem on analytic functions, from the bare root of the ZF axioms.

The origins of graph theory can be traced back to the 1736 paper of Leonhard Euler on the Königsberg bridge problem. For many years, graph theory remained a sub-discipline of combinatorics, but this has no longer been the case during the last century. The rapidly growing number of applications of graphs as models of various theoretical and practical problems led to what is now known as modern graph theory. This has not been incidental, since, like sets, graphs are among the most natural mathematical structures. For this reason, it is difficult to give an immediate definition of a graph without falling into one of the two extremes: either a formal definition, or just a delegation to a synonym: network, map, list of adjacencies. Graphs have been usually considered finite, and it is in fact their finite combinatorics that gives rise to deep and difficult problems. Many problems ask what properties are enjoyed by graphs as a consequence of some structural property. For example, the even degree of every vertex of a graph and its connectedness guarantee the existence of an Eulerian tour (Euler, 1736), the lack of an odd cycle in a graph ensures that it is bipartite (König, 1936), the lack of an odd induced cycle of length at least five, and of the complement of such a cycle, ensures that a graph is perfect (Chudnovsky, Robertson, Seymour, Thomas, 2006), the fact that a graph has no induced subgraph isomorphic to the complete bipartite graph $K_{1,3}$ ensures that certain otherwise difficult computational problems are tractable, for example the independent set problem [90].

Graphs make no exception from most of mathematics, and their formal definition involves the concept of set: a graph is in fact a set of vertices paired with a set of (ordered) pairs of vertices. The axiomatic foundation of set theory becomes apparent when studying graphs endowed with infinitely many vertices. Nevertheless, graphs enter into play when devising a representation of sets. The nesting property of sets, emerging from their very first axiomatic beginnings, is the reason why they are better represented by a graph-based model than by flat Venn diagrams: a directed graph (or digraph) whose vertices correspond to sets and whose arcs mimic membership. Almost as old as sets themselves, this interpretation of a set has been around mostly for expository purposes. Only recently it has become crucial, on the one hand, due to the computable set theory community, and on the other hand thanks to the rise of hyperset theory [3, 11, 12, 56]. Born to model 'extraordinary' circular phenomena, these new sets have the peculiarity that the membership relation between them is no longer required to be well-founded. Regarding a set as a digraph is so rewarding in this more general context, that the axiom stating the existence of hypersets (the so-called Anti-Foundation Axiom) is expressed by means of digraphs [3].

We are undertaking here a study on graphs which have as defining property that of being the representations of sets (be they well- or non-well-founded). Contrary to the common practice of defining a concept in a precise formal set theoretic manner, and then forgetting this definition once some basic properties have been derived, we are proposing a slight shift of focus back to the hardwired structure of sets. Thus, we will be looking at sets under the spotlight of more modern computational and combinatorial issues, and will strive to find new connections between sets and (di)graphs. We will also seek methods for shifting results or techniques from one field to the other.

Let us now briefly define sets, together with their digraph-representation. Under the ZF axioms, each set is uniquely characterized by its elements (Extensionality Axiom), and the membership relation is well-founded (Foundation Axiom). The standard universe of ZF sets is von Neumann's cumulative hierarchy namely the class inductively defined over all ordinals α, as the union of $\mathbb{V}_\alpha$, where $\mathbb{V}_0 = \emptyset$, each level $\mathbb{V}_\alpha$ is

$$\mathbb{V}_\alpha = \bigcup_{\beta<\alpha} \mathcal{P}(\mathbb{V}_\beta)$$

and $\mathcal{P}(\cdot)$ stands for the power-set operator. For example, $\mathbb{V}_1 = \{\emptyset\}$, $\mathbb{V}_2 = \big\{\emptyset, \{\emptyset\}\big\}$, $\mathbb{V}_3 = \big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\big\}$.

To represent a set as a digraph, one must consider the collection of its elements, of the elements of its elements, and so on. This leads to the *transitive closure* of a set x, defined as being $\mathsf{TrCl}(x) = x \cup \bigcup_{y\in x} \mathsf{TrCl}(y)$. The *membership digraph* associated to x has $\mathsf{TrCl}(x)$ as the vertex set and the inverse of the membership relation as its arc relation:

$$(\mathsf{TrCl}(x), \{u \to v \mid u, v \in \mathsf{TrCl}(x), v \in u\}).$$

Clearly, any such digraph is acyclic—as $\in$ is well-founded—, and *extensional*, in the sense that different vertices have different sets of out-neighbors. It can be easily seen that there actually exists a bijection between membership digraphs and (unlabeled) extensional acyclic digraphs. This is given by the so-called *Mostowski's collapse* of an extensional acyclic digraph, which recursively associates with each vertex the set of sets associated with its out-neighbors.

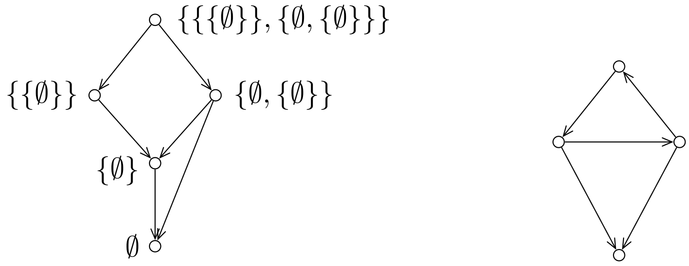


Figure 1: On the left, a membership digraph and the sets associated to its vertices by Mostowski's collapse; on the right, a hyper-extensional digraph owning a cycle.

If in the well-founded case extensionality is the criterion that prevents overcrowding the universe of sets, in a non-well-founded context extensionality must be strengthened to also take into account cyclic situations. This can be done only by inspecting the full structure of a set, not just its elements, or, its immediate out-neighbors. For his theory of hypersets, Aczel proposed the notion of *bisimulation* for such an irredundancy criterion. The notion of bisimulation is a broad-range concept in computer science, surfacing almost contemporarily in various fields: modal logic [145], concurrency theory [89,110], set theory [56], formal verification (see [37]). Under Aczel's Anti-Foundation Axiom, a set can be simply taken to be a digraph devoid of distinct bisimilar vertices (to be called *hyper-extensional* in the ongoing).

A method to characterize the structure of objects sharing common specified properties is also one of the oldest in mathematics, namely counting. Finding the number of distinct sets with n elements is not a well posed question, since the n elements of such a set can be arbitrarily complex. However, this problem becomes relevant when considering *transitive* sets, that is, sets whose elements are also subsets of them. A recurrence relation for transitive sets with n elements was given by Peddicord in 1962 [118], recurrence which also appears as sequence A001192 in Sloane's On-Line Encyclopedia of Integer Sequences [135]. Peddicord used Ackermann's encoding to map such sets to particular n-element vectors, which were subsequently counted. In light of the bijection between transitive sets and extensional acyclic digraphs, in Section 2.1 we deal with counting methods for sets inspired from the count of acyclic digraphs [63]. For example, we will show the following:

Theorem. ([124]) *The number $\widehat{e}_n^r$ of extensional acyclic digraphs on $n \geqslant 2$ unlabeled vertices, out of which $0 < r < n$ are vertices of maximum rank, is*

$$\widehat{e}_n^r = \sum_{k=1}^{n-r} \widehat{e}_{n-r}^k \binom{(2^k-1)2^{n-r-k}}{r}, \qquad \widehat{e}_1^1 = 1.$$

In the same paper [124] we conjectured the following result, which was later proved by Wagner.

Theorem. (Wagner [152]) *The proportion of labeled extensional acyclic digraphs among all labeled acyclic digraphs converges to the limit 0.326210....*

Finding the number of hyper-extensional digraphs with n vertices is still an open problem. Only few tentative steps have been made until now: for example, a brute-force approach was employed in [88] by generating all digraphs with at most 5 vertices and checking which of them are hyper-extensional, while [95] gave the complete lists of Finsler, Scott and Boffa transitive non-well-founded sets with at most 3 elements. Actually, we believe that a 'good' understanding of the structure of hyper-extensional digraphs would not only solve this problem, but would also provide useful insight in establishing whether there exists a linear-time algorithm to compute the maximum bisimulation over a digraph, or it would provide a method for random sampling hyper-extensional digraphs with n vertices.

Our contribution to this problem, guided by the fact that counting objects amounts to assigning consecutive natural numbers to them, lies in a generalization of Ackermann's order and numeric encoding $\mathbb{N}_A(a) = \sum_{b \in a} 2^{\mathbb{N}_A(b)}$ of hereditarily finite well-founded sets, HF, to hereditarily finite hypersets, $\overline{\mathsf{HF}}$ [42,43]. In Section 2.2 we use a splitting technique borrowed from algorithmics to define an order $\prec$ on $\overline{\mathsf{HF}}$, and then an encoding to dyadic rationals $\mathbb{Q}_2$ such that, like in the well-founded case, a simple reading of a binary code of $y \in \mathbb{Q}_2$ allows one to inductively determine the hyperset having y as code. This encoding is twofold: we first define a map from $\overline{\mathsf{HF}}$ to $\mathbb{Z}$ which assigns positions to hypersets:

$$\mathbb{Z}_A(a) = \begin{cases} |\{\, b : b \in \mathsf{HF} \wedge b \prec a \,\}| & \text{if } a \in \mathsf{HF}, \\ -|\{\, b : b \in \overline{\mathsf{HF}} \setminus \mathsf{HF} \wedge b \prec a \,\}| - 1 & \text{if } a \in \overline{\mathsf{HF}} \setminus \mathsf{HF}, \end{cases}$$

and then define a bijection $\mathbb{Q}_A$ from $\overline{\mathsf{HF}}$ to dyadic rationals

$$\mathbb{Q}_A(a) = \sum_{b \in a} 2^{\mathbb{Z}_A(b)}.$$

Since a set is a basic mathematical and computational object, one is also interested in uniformly sampling sets in order to perform tests and benchmarks, collect statistical data, (dis)prove conjectures, etc. In Section 2.3 we tackle the problem of generating at random transitive sets with n elements, or equivalently, extensional acyclic digraphs with n vertices. Notice that, in light of the above mentioned result of [152], sets can be sampled by running a sampler for acyclic digraphs and then checking whether the produced digraph is extensional. However, we will give a direct method tailored to (weakly) extensional acyclic digraphs [122] and obtained by adapting a Markov chain approach for generating labeled acyclic digraphs [68, 82, 83].

When enumerating finite objects whose size is no longer bounded by a fixed value, the sequences under consideration become infinite, and thus the existence of a well-quasi-order over such objects is an appropriate issue. For example, in a graph-theoretic setting, one usually looks for graph immersions that are also well-quasi-orders, so that no 'new' structures can be built up *ad infinitum.* The celebrated theorem of Robertson and Seymour states that the minor relation between graphs is a well-quasi-order on the class of all finite graphs. There are not many results in the case of digraphs. Recently, *strong immersion* between digraphs was proved to be a well-quasi-order on the set of all tournaments [36], while immersion between Eulerian digraphs was studied by Johnson (see [9, p. 517], [36]).

In [121] we identified a class of well-founded sets whose membership digraphs are well-quasi-ordered by strong immersion. This result was later refined in [123]: the defining property of these sets, namely *slimness*, was translated into a graph-theoretic property of a digraph. This required that for every vertex x and every out-neighbor of it, y, there exists another vertex of the digraph having precisely the same out-neighborhood as x, minus the element y. In order to generalize this result to hypersets, the acyclicity assumption was also dropped; consequently *channeled* digraphs were considered, 'channeled' meaning that from every vertex there is a directed path to a sink:

Theorem. ([123]) *For every $s \geqslant 1$, the collection $\mathcal{D}_s$ of channeled slim finite digraphs with at most s sources is well-quasi-ordered by strong immersion.*

Neither the slimness property, nor the bound on the number of sources can be dropped without losing the well-quasi-ordering property. This will be argued in Chapter 3.

For our next topic, which we develop in Chapter 4, we turn our attention to the structure underlying sets, more precisely, to the underlying graphs of the membership digraphs of hereditarily finite sets. These objects have not been studied before in the literature; therefore we could freely attach the name *set graphs* [86] to them. Being born to facilitate the formalization of the whole of mathematics, the structure underlying sets is also expected to be as rich as possible. This is in fact true: the collection of set graphs does not coincide with the class of finite graphs, but deciding whether a graph belongs to this class is an NP-complete problem:

Theorem. ([85]) *It is* NP*-complete to decide whether a graph admits an extensional acyclic orientation, a slim extensional acyclic orientation, or a hyper-extensional orientation, even when the input is restricted to bipartite graphs.*

This complexity result shows that it is unlikely that a good characterization of them exists. Instead, one can look for the largest hereditary (i.e., closed under taking induced subgraphs) class of graphs such that every connected member of it is a set graph. It turns out that this class is obtained by forbidding the smallest connected graph which is not a set graph, the *claw*, $K_{1,3}$.

Theorem. (**[86]**) *Let G be a connected claw-free graph and let $r \in V(G)$. G admits an extensional acyclic orientation whose sink is r if and only if r is not a cut vertex of G. Moreover, an extensional acyclic orientation of such a graph can be found in polynomial time.*

The connection between set graphs and claw-freeness is all but superficial [87]. On the one hand, there exists a largest hereditary class of graphs where being a set graph is equivalent to being claw-free. On the other hand, the claw-freeness condition can be generalized in two ways. First, by requiring that all claws of a graph be vertex-disjoint together with a further connectivity condition, another subclass of set graphs will be identified, in Section 4.3.2. Second, in Section 4.3.3 we show that if we forbid $K_{1,r+2}$, $r \geqslant 1$, instead of the claw $K_{1,3}$, a pseudo-extensionality property, which we call *r-extensionality*, can be guaranteed.

A practical motivation for the study of this problem lies in the area of identifying and separating codes. Let us say that a subset C of vertices of a digraph D is an *open-out-separating code* if the (open) out-neighborhoods of the vertices of D have pairwise distinct intersections with C. It is easy to see that a digraph D admits such a separating code if and only if D is extensional. In Section 4.2.4 we show that it is NP-hard to find an open-out-separating code of minimum size. To place this in historical context, notice that we are slightly deviating from the nomenclature introduced by [57], where the notion of separating code referred to *closed in*-neighborhoods.

Another motivation for this study is in the graph-theoretic expressive power of hereditarily finite sets. Taking as the vertex set of a set graph any of the transitive closures from which it originates, its edge relation need not be defined separately since it can be implicitly read from the membership relation among its vertices: two vertices are adjacent if and only if one is a member of the other. As just mentioned, transitive hereditarily finite sets do *express* connected claw-free graphs. By studying set graphs, we are studying which graphs can be represented in this alternative, economical way, useful for any computer language able to represent/manipulate ZF hereditarily finite sets, such as Referee.

This point of view led to a shorter proof of the fact that squares of connected claw-free graphs are vertex-pancyclic, which is covered in Chapter 5. This result is due to Matthews and Sumner [81], who showed that such graphs are Hamiltonian and then resorted to the general result that in squares of graphs being Hamiltonian is equivalent to being vertex-pancyclic [54]. Our short proof directly shows vertex-pancyclicity [144]. We also show that the same framework can be employed for proving another well-known result on claw-free graphs, namely that connected claw-free graphs of even order have a perfect matching [141]. For both of these short proofs, it actually suffices to work with an acyclic orientation with a unique sink. However, these results were spawned by our proof of the result that every connected claw-free graph is a set graph.

The set-theoretic insight behind these proofs shows that two properties of connected claw-free graphs can be even extended to a larger class of graphs:

Theorem. (**[144]**) *If G is a connected graph admitting an acyclic orientation with a unique sink that has none of the two digraphs depicted in Figure 2 as induced subdigraphs, then*

- *the square of G is vertex-pancyclic;*
- *if G has an even number of vertices, then G has a perfect matching.*

(a) orientation C_1 (b) orientation C_2

Figure 2: Two forbidden orientations of a claw that allow the generalization of two classical results on connected claw-free graphs.

A formalization of these two results [106] using the proof-checker Referee is presented in Section 5.2. Since Referee deals only with Zermelo-Fraenkel sets, representing a connected claw-free graph by a transitive 'claw-free' set turned out to require the least amount of formalistic effort. On the one hand, we avoid explicitly defining graphs, together with an entire armamentarium of graph-theoretic concepts that the original proofs required. On the other hand, we exploit Referee's built-in set manipulating operations to reflect with a minimum degree of encumbrance the two set-theoretic proofs. The complete formalization in Referee is given as Appendix B.

This is no isolate example of an implicit representation of graphs by sets, in view of the following more general representation theorem, where weak extensionality means that only non-sink vertices have pairwise distinct out-neighborhoods:

Theorem. **([86])** *Every graph admits a weakly extensional acyclic orientation.*

This entails that, given an undirected graph G, one can construct a bijection ϱ between G and a set x_G such that $\{u, v\}$ is an edge of G if and only if either $\varrho(u) \in \varrho(v)$ or $\varrho(v) \in \varrho(u)$ holds. Moreover, x_G is *almost transitive*, in the sense that for any $y \in x_G$, if $y \cap x_G \neq \emptyset$, then $y \subseteq x_G$ (recall that, if G is connected and claw-free, then a *transitive* set x_G always exists); more on this in Section 5.2.4.

Finally, we focus on infinite sets, with special emphasis on infinite hypersets. As we will see in due course, this study is motivated by the decidability problem for the class of all $\forall^*$ prenex formulae (the so-called Bernays-Schönfinkel-Ramsey class) involving membership and equality, over hypersets. Since the celebrated result of Ramsey [126], the logical variant of this class has stimulated a lot of research, among which we mention the recent [7].

A graph class defined by forbidden induced subgraphs can be characterized by a $\forall^*$-formula: the adjacencies between any tuple of vertices are required not to be the same as the adjacencies between the vertices of one of the forbidden subgraphs. Having a good understanding of the structure of graphs from such a class can guarantee, among others, tractability of otherwise NP-hard problems. The analogous problem of characterizing sets satisfying a fixed $\forall^*$-formula is certainly rewarding. For example, a $\forall^*$-formula can characterize the class of sets obtained by forbidding the two orientations of a claw depicted in Figure 2, which thus ensures the presence of the two mentioned graph-theoretic properties in sets. In connection with the decidability problem, we will consider however a more general question: If one considers the collection of *all* $\forall^*$-formulae, which are the sets that they can express? Since all hereditarily finite (hyper)sets can be characterized by a

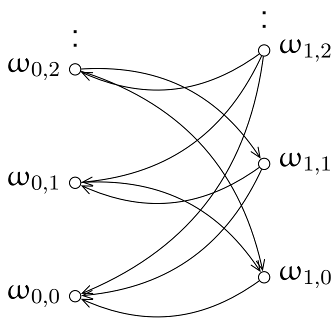


Figure 3: Sets $\omega_0 = \{\omega_{0,j} : j \in \omega\}$, $\omega_1 = \{\omega_{1,j} : j \in \omega\}$ such that $\iota\iota(\omega_0, \omega_1)$ holds.

$\forall^*$-formula, this question becomes intriguing when asking for the *infinite* (hyper)sets that can be thus characterized.

For example, one such infinite set, depicted in Figure 3, is characterized by the following $\forall\forall$-formula $\widetilde{\iota\iota}(a, b)$ obtained by the conjunction of the following conditions:

(i) $a \neq b \wedge a \notin b \wedge b \notin a \wedge a \cap b = \emptyset$

(ii) $\bigcup a \subseteq b \wedge \bigcup b \subseteq a$

(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$.

Theorem. ([105]) *Any two hypersets a, b for which $\widetilde{\iota\iota}(a, b)$ holds are infinite and well-founded.*

Since infinite sets cannot be expressed with less than two existential and two universal quantifiers (see Section 6.1.2), the above theorem shows that $\exists a\, \exists b\, \widetilde{\iota\iota}(a, b)$ is a syntactically simplest formulation of the Infinity Axiom, independent of whether the Foundation Axiom, or the Anti-Foundation Axiom, is assumed.

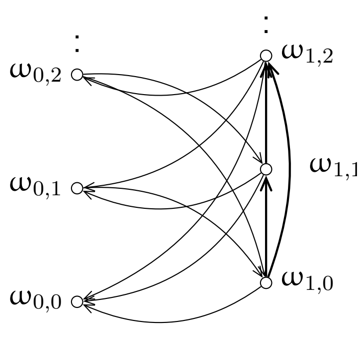


Figure 4: Hypersets $\omega_0 = \{\omega_{0,j} : j \in \omega\}$, $\omega_1 = \{\omega_{1,j} : j \in \omega\}$ such that $\underline{\iota\iota}_1(\omega_0, \omega_1)$ holds.

However, genuinely non-well-founded hypersets can be expressed by $\forall^*$-formulae and they can be quite involved, as testified by the $\forall\forall\forall$-formula $\underline{\iota\iota}_1(a, b)$ obtained by the conjunction of the following sub-formulae (a model of $\underline{\iota\iota}_1$ is depicted in Figure 4):

(i) $a \neq b \wedge a \notin b \wedge b \notin a$

(ii′) $\bigcup a \subseteq b \wedge \bigcup b \subseteq a \cup b \wedge (\forall y \in b)(y \notin y)$

(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$

(iv′) $(\forall x \in a)(\forall y_1, y_2 \in b)(y_1 \in y_2 \rightarrow y_2 \subseteq y_1)$

(v) $(\forall y_1, y_2 \in b)(y_1 = y_2 \vee y_1 \in y_2 \vee y_2 \in y_1)$.

Theorem. ([102]) *Any two hypersets* a, b *for which* $\underline{\omega}_1(a,b)$ *holds are infinite; moreover, they have membership cycles and infinite descending membership chains in their transitive closure.*

The graph-theoretic interpretation of a set in this context should be apparent. On the one hand, we are dealing with hypersets as hyper-extensional digraphs. On the other hand, our proofs that the sets satisfying such $\forall^*$-formulae are infinite proceed by contradiction, turning them into finitarily combinatorial arguments.

Chapter 6 presents an overview of formulae expressing infinite (non-)well-founded sets [100, 112–114], together with the more recent results regarding hypersets [102, 103, 105].

In conclusion, this thesis should argue that, under our premise "sets as graphs", a more systematic study of graph-theoretic problems on sets and of set-theoretic problems of graphs is fruitful. Some of the connections between the two fields have already surfaced in the literature, albeit implicitly, e.g., Peddicord's count of transitive sets, or the recent work on identifying and separating codes in graphs and digraphs.

We have mentioned, and we will illustrate in detail in the following chapters, multiple directions which are relevant for a deeper transfer of concepts. On the one hand, there is still more left to be done for sets under their representation as digraphs: further connections to acyclic digraphs (for example, a ranking/unranking method similar to the one for acyclic digraphs of [137], eventually inspired by Peddicord's bijection to n-element numeric vectors), or to separating codes (to our knowledge, our notion of "open-out-separating code" is new).

On the other hand, a flurry of new graph-theoretic questions can be asked for set graphs, starting from determining the complexity of set graph recognition for particular classes of graphs, to fixed-parameter algorithms [55], to connections between set graphs, together with their extensional acyclic orientations, and descriptive complexity theory. Moreover, set graph recognition can be also stated as an optimization problem, for example by asking for a weakly extensional acyclic orientation of a graph with a minimum number of sinks, or for an r-extensional acyclic orientation with minimum r. It is therefore interesting to study approximation algorithms [147] for such optimization variants. Given the correspondence between the acyclic orientations and the chromatic number of a graph, and in light of the above result stating that every graph admits a weakly extensional acyclic orientation, one can analogously introduce a notion of 'set chromatic number' of a graph, and study, for example, 'set perfect graphs'.

1

Basic Concepts

1.1 Well-founded sets

We are placing ourselves in the classical framework of the Zermelo-Fraenkel set theory (in short, ZF). We are thus assuming the Extensionality Axiom, EA,

$$\mathsf{EA} \quad \equiv \quad \forall\, u\, \forall\, v\, \Big(\forall\, w\big(w \in u \leftrightarrow w \in v\big) \to u = v\Big),$$

stating that two sets are equal if and only is they have the same *extension*, that is, the same elements. We are also assuming von Neumann's Foundation Axiom, FA

$$\mathsf{FA} \quad \equiv \quad \forall\, v\, \Big(v \neq \emptyset \;\to\; (\exists\, m \in v)(m \cap v = \emptyset)\Big),$$

stating that the membership relation between sets is (strict and) *well-founded*, i.e., it forms no cycles or infinite descending chains $v_0 \ni v_1 \ni v_2 \ni \cdots$.

The standard universe of ZF sets is von Neumann's *cumulative hierarchy* of well-founded sets $\mathbb{V}$, namely the class inductively defined over all ordinals α, as the union of $\mathbb{V}_\alpha$, where each level $\mathbb{V}_\alpha$ is

$\mathbb{V}_0 = \emptyset$,

$\mathbb{V}_\alpha = \mathcal{P}(\mathbb{V}_{\alpha-1})$, if α is a successor ordinal,

$\mathbb{V}_\alpha = \bigcup_{\beta<\alpha} \mathbb{V}_\beta$, if α is a limit ordinal,

so that, in particular, $\mathbb{V}_1 = \{\emptyset\}$, $\mathbb{V}_2 = \big\{\emptyset, \{\emptyset\}\big\}$, $\mathbb{V}_3 = \big\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\big\}$.

The well-foundedness of the membership relation allows us to associate with each set x an ordinal, $\mathsf{rank}(x)$, in the following recursive way

$$\mathsf{rank}(x) \quad =_{\text{Def}} \quad \begin{cases} 0 & \text{if } x = \emptyset, \\ \sup\{\, \mathsf{rank}(y) + 1 :\; y \in x \,\} & \text{otherwise.} \end{cases}$$

The rank function is a first approximation of the inner complexity of a set. For a full description of its structure, one has to consider the *transitive closure* of a set x, defined through the recursion

$$\mathsf{TrCl}(x) \quad =_{\text{Def}} \quad x \cup \textstyle\bigcup_{y\in x} \mathsf{TrCl}(y).$$

Among all sets, the ones that equal their own transitive closures are called *transitive* (or *full*). Such a set x can be equivalently characterized by the property $(\forall y \in x)(\forall z \in y)(z \in x)$.

Example 1.1.1 The set $\{\{\emptyset\}\}$ has rank 2, and its transitive closure is $\{\emptyset, \{\emptyset\}\}$. ■

An important part of this thesis will focus on *hereditarily finite* sets, that is, sets having a finite transitive closure (equivalently, a finite rank). The collection HF of hereditarily finite sets can be readily singled out from $\mathbb{V}$, since it suffices to consider only the levels $\mathbb{V}_\alpha$, where α is a finite ordinal.

Another equivalent wording of the property of a set x being hereditarily finite is that x can be obtained from $\emptyset$ by applying a finite number of times the adjunction operation $x = y \cup \{z\}$. This last characterization should be readily apparent in light of a specific one-to-one correspondence between hereditarily finite sets and natural numbers, discovered by Ackermann. This is recursively defined, for all $x \in \mathsf{HF}$, as

$$\mathbb{N}_A(x) \quad =_{\text{Def}} \quad \textstyle\sum_{y \in x} 2^{\mathbb{N}_A(y)},$$

where $\mathbb{N}_A(\emptyset) = 0$. For example, $\mathbb{N}_A(\{\emptyset\}) = 1$, $\mathbb{N}_A(\{\{\emptyset\}\}) = 2$, $\mathbb{N}_A(\{\emptyset, \{\emptyset\}\}) = 3$, $\mathbb{N}_A(\{\{\{\emptyset\}\}\}) = 4$. Among other virtues, Ackermann's bijection enables one to retrieve the full structure of a hereditarily finite set from its numeric encoding by means of a simple recursive routine:

$$\begin{gathered}\text{the binary representation of } \mathbb{N}_A(x) \text{ has a `1' in position } \mathbb{N}_A(y) \\ \text{if and only if } y \in x.\end{gathered} \tag{1.1.1}$$

If a set x satisfies $x = y \cup \{z\}$ and $z \notin y$, then $\mathbb{N}_A(x) = \mathbb{N}_A(y) + 2^{\mathbb{N}_A(z)}$ holds, which corresponds to the addition of a '1' on bit $\mathbb{N}_A(z)$ of $\mathbb{N}_A(x)$. Via the encoding $\mathbb{N}_A$, the standard order $<$ on natural numbers induces an order, called the *Ackermann order*, on hereditarily finite sets:

$$x \prec y \quad \leftrightarrow_{\text{Def}} \quad \mathbb{N}_A(x) < \mathbb{N}_A(y).$$

Ackermann's order is compatible with rank comparison, and is, actually, fully described by the anti-lexicography criterion

$$x \prec y \quad \Leftrightarrow \quad \max\nolimits_\prec(x \setminus y) \prec \max\nolimits_\prec(y \setminus x), \tag{1.1.2}$$

where by convention, $\max_\prec \emptyset \prec z$ holds for any non-empty set z.

Given a set V, we write $[V]^2$ for the set of all 2-element subsets of V. We say that the elements of a partition P of V are its *blocks*.

1.2 Graphs and digraphs

Graphs. A *graph* (or *undirected graph*) is a pair $G = (V(G), E(G))$, where $V(G)$ is a set and $E(G) \subseteq [V(G)]^2$. To avoid notational ambiguities, we always assume that $V(G)$ and $E(G)$ are disjoint. The set $V(G)$ is called the set of *vertices* of G, while $E(G)$ is the set of its *edges*. We write uv as a shorthand for an edge $\{u, v\} \in E(G)$. If $e = uv$ is an edge, u and v are called its *end vertices* (or *end points*); we say that e is *incident* to u and to v, and that u and v are *adjacent*, or *neighbors*.

Given a vertex $v \in V(G)$, the *neighborhood* of v is the set of neighbors of v in G, $N_G(v) =_{\text{Def}} \{u \in V(G) \mid vu \in E(G)\}$. The *degree* of v is $d_G(v) =_{\text{Def}} |N_G(v)|$. A vertex of degree 1 is called a *leaf*. We may omit the subscript G when this is clear from the context.

Containment relations and graph operations. If H is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ we say that H is a *subgraph* of G. If $W \subseteq V(G)$, $G[W]$ is the graph $(W, \{e \in E(G) \mid e \subseteq W\})$, called the subgraph of G *induced* by W. If $H = G[V(H)]$, then H is an *induced subgraph* of G. We write $G - W$ for the graph $G[V(G) \setminus W]$. When $W = \{v\}$, we simply write $G - v$, instead of $G - \{v\}$. If $F \subseteq [V(G)]^2$, we write $G + F$ for the graph $(V(G), E(G) \cup F)$; analogously, $G - F$ is the graph $(V(G), E(G) \setminus F)$.

Given graphs G_1 and G_2, we say that G_1 and G_2 are *isomorphic* if there exists a bijection $f : V(G_1) \to V(G_2)$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$. In this case, f is called an *isomorphism* (between G_1 and G_2). A *graph property* (or *class*) is a set of graphs closed under isomorphisms. A graph class is *hereditary* if it is closed under taking induced subgraphs.

Given graphs H and G, we say that G is H-free if no induced subgraph of G is isomorphic to H. Graph H is also called a *forbidden induced subgraph.* The family of H-free graphs is a hereditary graph class, for any graph H.

For example, the *net* is a *claw*-free graph, while $K_{2,3}$ is not claw-free (see Figure 1.1).

Connectivity. A *path* is a graph $P = (V, E)$ of the form

$$V = \{v_1, v_2, \ldots, v_k\}, \qquad E = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\},$$

where $k \geqslant 1$ and the v_i's are pairwise distinct. We say that P *connects* (or *joins*, or *is between*) vertices v_1 and v_k, and that v_1 and v_k are its *end vertices.* The length of P is the number of its edges, that is, $k - 1$. We will often designate a path by either one of the natural orders of its vertices and write, e.g., $P = (v_1, v_2, \ldots, v_k)$. We say that a path P *is in* a graph G if P is a subgraph of G. We usually denote paths on k vertices with P_k.

Given a graph G, the *distance* in G between vertices $u, v \in V(G)$ is the shortest length of a path in G between u and v; if no such path exists, then the distance is taken to be ∞. A graph is *connected* if there is a path between any two distinct vertices of G. A maximal connected subgraph of G is called a *(connected) component* of G. If C is a connected component of G, we will sometimes write C when we actually mean $V(C)$. We say that a vertex $v \in V(G)$ is a *cut vertex* of G if $G - v$ has more connected components than G.

Basic graph properties. A graph with a finite vertex set is said to be *finite.* If $P_k = (v_1, v_2, \ldots, v_k)$ is a path and $k \geqslant 3$, then the graph $C_k = P_k + \{v_kv_1\}$ is called a *cycle.* A connected graph having no cycle as subgraph is called a *tree.* A graph G such that $uv \in E(G)$ holds for every distinct $u, v \in V(G)$ is said to be *complete.* A complete graph on n vertices is denoted as K_n. If $C \subseteq V(G)$, then C is called a *clique* if $G[C]$ is complete; if on the opposite $G[C]$ has no edges, then C is an *independent* set.

A graph G is said to be *multipartite* if there is a partition of $V(G)$ whose blocks are independent sets. If this partition has k blocks, then the graph is called *k-partite.* A multipartite graph G is *complete* if any two vertices belonging to different blocks of the partition of $V(G)$ are adjacent. A 2-partite graph is called *bipartite.* A complete bipartite graph whose partition of the vertices has blocks of sizes n and m, respectively, is denoted as $K_{n,m}$.

A path P in a graph G is *Hamiltonian* if $V(P) = V(G)$. Analogously, a cycle C in G is *Hamiltonian* if $V(C) = V(G)$; if G has a Hamiltonian cycle, G is said to be *Hamiltonian.*

Given a graph G, a subset $M \subseteq E(G)$ is called a *matching* if no edges of M share an end vertex. A matching M is said to be *perfect* if every vertex of G is an end vertex of an edge of M.

Example 1.2.1 Under the definitions $[V]^0 = \{\emptyset\}$, $[V]^1 = \{\{v\} \mid v \in V\}$, ($[V]^2$ as before), the graph

$$G^V_{\subsetneq} = \big([V]^0 \cup [V]^1 \cup [V]^2, \{\{u, v\} \mid v \in [V]^1 \cup [V]^2 \wedge u \subseteq v \wedge v \setminus u \in [V]^1\}\big),$$

is 3-partite, $[V]^0$, $[V]^1$, $[V]^2$ being the blocks of the partition.

The set of all *ordered pairs* of elements of a set V can be thus defined, slightly deviating from Kuratowski's definition, as the set of all edges of $G^V_{\subsetneq}$. Indeed, (v, v) will stand for the edge $\{\emptyset, \{v\}\}$, whereas (u, v) corresponds to $\{\{u\}, \{u, v\}\}$. ■

Special graphs. Some small graphs, depicted below, will turn out useful in our endeavor.

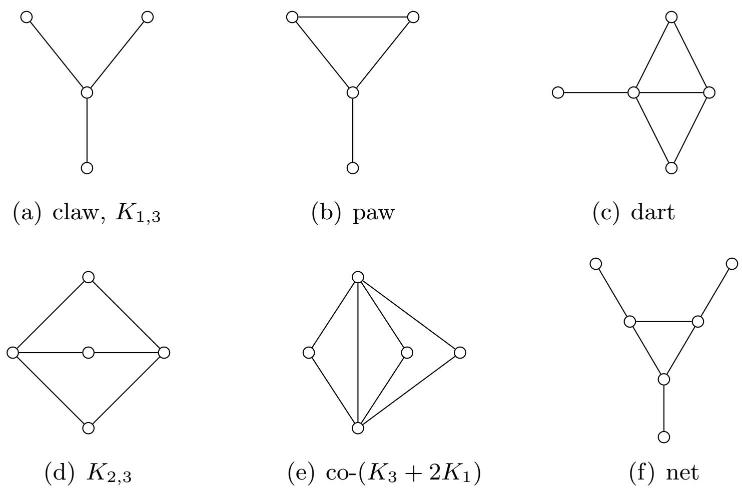


Figure 1.1: Some small graphs.

Digraphs. A *digraph* (or *directed graph*) is a pair $D = (V(D), E(D))$, where $V(D)$ is a set and $E(D)$ consists of ordered pairs of elements of $V(D)$ (we always assume that $V(D)$ and $E(D)$ are disjoint). The set $V(D)$ is called the set of *vertices* of D, while $E(D)$ is the set of its *arcs*. We write uv, or $u \to v$, as a shorthand for an arc $(u, v) \in E(D)$. Given an arc $e = uv$, we say the e *is from* u to v, that v is an *out-neighbor* of u, and that u is an *in-neighbor* of v. Notice that, in slight deviation from [10], we allow *self-loops* in a digraph, that is, arcs of the form $(v, v) \in E(D)$.

Given a vertex $v \in V(D)$, the *out-neighborhood* of v is the set of out-neighbors of v in D, $N^+_D(v) =_{\text{Def}} \{u \in V(D) \mid v \to u \in E(D)\}$. The *in-neighborhood* of v is the set $N^-_D(v) =_{\text{Def}} \{u \in V(D) \mid u \to v \in E(D)\}$. If the *in-degree* of $v \in V(D)$ $d^-_D(v) =_{\text{Def}} |N^-_D(v)|$ is 0, we say that v is a source in D, while v is a sink in D if its *out-degree* $d^+_D(v) =_{\text{Def}} |N^+_D(v)|$ is 0. We may skip the subscript D when this is clear from the context.

Containment relations and digraph operations. If H is a digraph with $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$ we say that H is a *subdigraph* of D. If $W \subseteq V(D)$, $D[W]$ is the digraph $(W, \{uv \in E(D) \mid \{u, v\} \subseteq W\})$, called the subdigraph of D *induced* by W. If $H = D[V(H)]$, then H is an *induced subdigraph* of D. We write $D - W$ for the digraph $D[V(G) \setminus W]$. When $W = \{v\}$, we simply write $D - v$, instead of $D - \{v\}$. If $F \subseteq V(D) \times V(D)$, we write $D + F$ for the digraph $(V(D), E(D) \cup F)$; analogously, $D - F$ is the digraph $(V(D), E(D) \setminus F)$.

Given digraphs D_1 and D_2, we say that D_1 and D_2 are *isomorphic* if there exists a bijection $f : V(D_1) \to V(D_2)$ such that $uv \in E(D_1)$ if and only if $f(u)f(v) \in E(D_2)$. In this case, f is called an *isomorphism* (between D_1 and D_2). The isomorphism relation between digraphs is an equivalence relation; any class of this relation will be referred to as an *unlabeled digraph*.

Connectivity. A *(directed) path* is a digraph $P = (V, E)$ of the form

$$V = \{v_1, v_2, \ldots, v_k\}, \qquad E = \{v_1 \to v_2, v_2 \to v_3, \ldots, v_{k-1} \to v_k\},$$

where $k \geqslant 1$ and the v_i's are pairwise distinct. We say that P *is from* vertex v_1 to v_k, and that v_1 and v_k are its *end vertices*. We also say that v_k is reachable from v_1. The length of P is the number of its arcs, that is, $k - 1$. We will often refer to a path by the natural order of its vertices and write $P = (v_1, v_2, \ldots, v_k)$. We say that a path P *is in* a digraph D if P is a subdigraph of D. We usually denote directed paths on k vertices with P_k.

Given a digraph D and $v \in V(D)$, we denote by $N^*_D(v)$ the set of vertices of D to which there is a directed path from v. A digraph D is *strongly connected* if for any $u, v \in V(D)$ there is a directed path from u to v and a directed path from v to u. A maximal strongly connected subdigraph of D is called a *(strongly connected) component* of D. For a strongly connected component C we will sometimes write C when we actually mean $V(C)$.

Basic digraph properties. A digraph with a finite vertex set is said to be *finite*. If $P_k = (v_1, v_2, \ldots, v_k)$ is a directed path and $k \geqslant 3$, then the digraph $C_k = P_k + \{v_k \to v_1\}$ is called a *(directed) cycle*. A digraph having no directed cycle as subdigraph is called *acyclic*. Note that every acyclic digraph has at least a sink and at least a source. The set of sources of an acyclic digraph D is denoted by $O(D)$.

The acyclicity of a digraph D can be equivalently characterized by requiring that $E(D)$ be a well-founded relation, i.e., that every non-empty subset W of vertices of D has a vertex s such that $N^+(s) \cap W = \emptyset$. This last interpretation has the advantage that it is extendable to digraphs having an infinite vertex set. When referring to a digraph as *acyclic*, we will leave as understood that its vertex set is finite. Otherwise, we will use the term *well-founded*.

Orientations. A digraph D is an *orientation* of a graph G if $V(D) = V(G)$, $|E(D)| = |E(G)|$, and $\{u, v\} \in E(G)$ for every $(u, v) \in E(D)$. Graph G is said to be the *underlying* graph of D. A digraph is said to be *connected* (or *weakly connected*) if its underlying graph is connected. An orientation of a complete graph is said to be a *tournament*.

1.3 Sets as digraphs

Pictures of sets. Even though for representing (di)graphs one uses sets, as flat collections of vertices and edges/arcs, digraphs enter into play in capturing the nested membership structure of a set. For this, one has to consider the digraph whose vertices are the elements of the transitive closure of x, together with x, and whose arcs correspond to the inverse of the membership relation. Formally, given a well-founded set x, call the *picture* of x the digraph

$$(\mathsf{TrCl}(\{x\}), \{u \to v \mid u, v \in \mathsf{TrCl}(\{x\}), v \in u\}).$$

A picture of a well-founded set x is a well-founded digraph, since the membership relation $\in$ is well-founded; it has precisely one source, the set x, and precisely one sink, the set $\emptyset$. Moreover, the out-neighborhoods of its vertices are pairwise distinct, since they arise from pairwise distinct sets. Let us pinpoint this notion by the following definition:

Definition 1.3.1 *A digraph D is said to be* extensional *if for any distinct $u, v \in V(D)$, it holds that $N^+(u) \neq N^+(v)$.*

Note that extensionality guarantees that a well-founded digraph has precisely one sink.

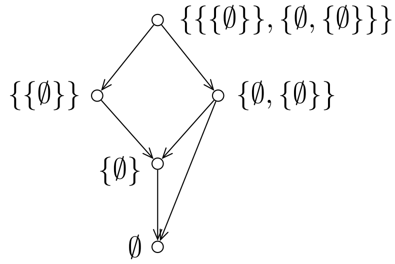


Figure 1.2: The picture of the set $x = \{\{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

A simple, but important, observation establishing the bridge between sets and digraphs is that the collection of extensional well-founded digraphs having precisely one source coincides with that of pictures of well-founded sets.

Lemma 1.3.2 (Mostowski's collapsing lemma) *Pictures of well-founded sets are in one-to-one correspondence with unlabeled extensional well-founded digraphs endowed with a unique source.*

Proof. The direct implication follows by the above observation. For the converse, let D be an extensional well-founded digraph with precisely one source. Define f recursively as

$$f(u) = \{\, f(v) \mid u \to v \in E(D)\,\},$$

where $f(u) = \emptyset$, if u is the sink of D. Since D is extensional, then $f : V(D) \to \{f(v) \mid v \in V(D)\}$ is a bijection. This shows that D is isomorphic to the picture of the set assigned by f to the source of D. ■

Membership digraphs. Trying to obtain a full description of a well-founded set x, we have included the vertex x in its picture, which entailed the requirement that the resulting digraphs have precisely one source. On the one hand, from a set theoretic perspective, often one is more interested in the inner structure of a well-founded set, that is, in its transitive closure. On the other hand, from a graph theoretic point of view, the assumption of a unique source is irrelevant. Therefore, we will speak of the *membership digraph* of a well-founded set x, to be denoted $D(x)$, when referring to the digraph

$$D(x) =_{\mathrm{Def}} (\mathsf{TrCl}(x), \{u \to v \mid u, v \in \mathsf{TrCl}(x), v \in u\}).$$

Since a transitive set owns as members all the elements of its transitive closure, we can state a slightly modified version of Lemma 1.3.2.

Lemma 1.3.3 *Pictures of transitive well-founded sets are in one-to-one correspondence with unlabeled extensional well-founded digraphs.*

The rank of a hereditarily finite well-founded set x admits a straightforward graph theoretic interpretation: the length of the longest directed path in the picture of x, from x to $\emptyset$. More generally, we define the *rank* of a vertex v in an acyclic digraph D as the length of the longest directed path from v to a sink of D.

Given an extensional acyclic digraph D, an Ackermann-like linear order $\prec$ can be inductively defined on its vertices by

$$u \prec v \quad \leftrightarrow_{\mathrm{Def}} \quad \max_{\prec}(N^+(u) \setminus N^+(v)) \prec \max_{\prec}(N^+(v) \setminus N^+(u)).$$

where by convention, $\max_{\prec} \emptyset \prec w$ holds for any vertex $w \in V(D)$. A linear-time algorithm to compute Ackermann's order on an extensional acyclic digraph appears in [47, Sec. 4]

Flavors of extensionality. Our graph-theoretic approach will consider various extensional criteria, which we summarize here. The first notion allows an arbitrary number of sinks in an acyclic digraph, while requiring that extensionality be maintained for non-sink vertices; this originates from set theories with atoms.

Definition 1.3.4 *A digraph D is said to be* weakly extensional *if for any distinct $u, v \in V(D)$, if $N^+(u) \neq \emptyset$, then $N^+(u) \neq N^+(v)$.*

Another way to generalize extensionality is by allowing non-sink vertices with the same out-neighborhood, but permitting at most $r \geqslant 1$ of them to share precisely the same set of out-neighbors.

Definition 1.3.5 *Given an acyclic digraph D and $A \subseteq V(D)$, we say that A is an r-*collision *of D if $|A| \geqslant r$ and for any $u, v \in A$ we have $N^+(u) = N^+(v)$. We say that D is r-extensional if no $(r+1)$-collision exists.*

In the following definition we capture instead 'minimal' extensional digraphs.

Definition 1.3.6 *An extensional digraph D is said to be*

- slim*, if for any $e \in E(D)$, the digraph $D - \{e\}$ ceases to be extensional;*
- dependent*, if for any arc $(u, v) \in E(D)$, the digraph $D - \{(u, v)\} + \{(v, u)\}$ is not extensional;*
- irredundant*, if for any vertex $v \in V(D)$, the digraph $D - v$ is not extensional.*

Observe that a slim extensional digraph is both dependent and irredundant.

1.4 Bisimulation, hyper-extensionality and hypersets

Bisimulation and the stable partitioning problem. The following notion of bisimulation is a broad-range concept appearing in many fields of theoretical computer science.

Definition 1.4.1 *A* bisimulation *over a digraph D is a binary relation $B \subseteq V(D) \times V(D)$ such that $x\,B\,y$ implies that*

i) for every x' such that $x \to x'$ holds, there exists a vertex y' such that $y \to y'$ and $x'\,B\,y'$; and

ii) for every y' such that $y \to y'$ holds, there exists a vertex x' such that $x \to x'$ and $x'\,B\,y'$.

One is usually interested in deciding whether two distinct vertices u, v of a digraph D are *bisimilar*, in the sense that there exists a bisimulation B over D such that uBv. It is therefore convenient to compute the *maximum bisimulation* over D, that is, the union of all bisimulation relations over D—the maximum bisimulation is an equivalence relation, hence it induces a partition of the vertices of a digraph.

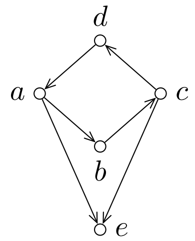


Figure 1.3: A digraph having $[e][b,d][a,c]$ as the maximum bisimulation.

This is usually done by a reformulation of this problem as the *stable partitioning problem.* To state this correspondence, we need the following definition.

Definition 1.4.2 *Let E be a relation on a set V, E^{-1} its inverse relation, and P a partition of V. The partition P is said to be* stable *with respect to E if for each pair B_1, B_2 of blocks of P, either $B_1 \subseteq E^{-1}[B_2]$, or $B_1 \cap E^{-1}[B_2] = \emptyset$.*

Otherwise stated, stability requires that a block B_1 of P be included in the pre-image $E^{-1}[B_2] =_{\text{Def}} \{\, x \in \bigcup P \mid (\exists\, y \in B_2)(x\,E\,y) \,\}$ of a block B_2 whenever B_1 intersects that preimage. Trivially, any partition whose blocks are singletons is stable.

Let us say that a partition P is *coarser* than a partition P' if every block of P' is included in a block of P (then we also say that P' is *finer* than P). The maximum bisimulation over a digraph D coincides with the relation induced by the coarsest partition of $V(D)$, stable with respect to $E(D)$.

Hypersets. If the membership relation between sets is no longer required to be well-founded, then it is legitimate to think that peculiar set theoretic equations such as

$$x = \{x\}, \text{ or } y = \{z\} \text{ and } z = \{y\},$$

admit set-solutions. Aczel's Anti-Foundation Axiom, AFA, ensures that, on the one hand, sets satisfying these three equations exist, and that, on the other hand, they all equal *the* set

$$\Omega = \{\Omega\}.$$

Given the close kinship between (pictures of) sets and digraph, we introduce Aczel's *hypersets*—the usual name associated to the elements of a model of ZF in which FA is withdrawn and EA is superseded by AFA—in terms of digraphs. We say that a pair (D, v), where D is a digraph and $v \in V(D)$ is a distinguished vertex of D, called its *point*, is *accessible*, if there is a directed path from v to any other vertex of D; a well-founded digraph is accessible if and only if it has a unique source.

Definition 1.4.3 *A digraph D is said to be* hyper-extensional *if every bisimulation over D is contained in the identity relation, that is, if for every bisimulation B over D, it holds that $x = y$ whenever $x\,B\,y$.*

Equivalently stated, a digraph D is hyper-extensional if the maximum bisimulation over D is the identity relation. This reduction criterion is the one used by AFA to prevent overcrowding the universe of non-well-founded sets.

Definition 1.4.4 *A hyper-extensional accessible pointed unlabeled digraph is called a* hyperset.

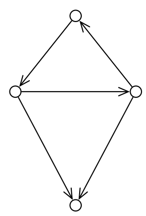


Figure 1.4: An accessible hyper-extensional digraph; any of its non-sink vertices can be its point. Notice that the symmetry of the 4-cycle (a, b, c, d) of the digraph in Figure 1.3 is now broken.

When referring to hypersets, we continue employing set theoretic notation, but the semantics will be the one introduced by the above definition. For example, the transitive closure of a hyperset $(h, h^\star)$ is, depending on whether or not we include its point $h^\star$ in it:

$$\begin{aligned} \mathsf{TrCl}(h) &=_{\text{Def}} \{(h[N_h^*(v)], v) \mid v \in V(h) \setminus \{h^\star\}\} \\ \mathsf{TrCl}(\{h\}) &=_{\text{Def}} \{(h[N_h^*(v)], v) \mid v \in V(h)\} \end{aligned}$$

In particular, given hypersets $(h_1, h_1^\star)$ and $(h_2, h_2^\star)$, we use the set theoretic notation $h_1 \in h_2$ if there exists an out-neighbor h of h_2^* and there exists an isomorphism f between $h_2[N_{h_2}^*(h)]$ and h_1, and $f(h) = h_1^\star$. Sometimes, for a hyperset $(h, h^\star)$, we will write h when referring to its point $h^\star$. We also say that a hyperset is *hereditarily finite* if its vertex set is finite; the collection of hereditarily finite hypersets will be denoted with $\overline{\mathsf{HF}}$.

As already done for the well-founded framework, we will sometimes drop the accessibility requirement, and speak only of a 'hyper-extensional digraph'. The following lemma, shows, on the one hand, that hypersets generalize well-founded sets, and, on the other hand, that they maintain a property reminiscent of well-foundedness: they have a (unique) sink and from every vertex there is a directed path to that sink.

Lemma 1.4.5 *Let D be a hyper-extensional orientation of a graph. The following hold:*

i) *D is extensional;*

ii) *D has a (unique) sink;*

iii) *there is a directed path from every $v \in V(D)$ to the sink of D;*

iv) *every extensional well-founded digraph is hyper-extensional.*

Proof. To see that *i*) holds, it suffices to observe that if distinct $u, v \in V(D)$ have $N^+(u) = N^+(v)$, then the equivalence relation that puts u and v in the same class and keeps every other vertex in a singleton class, that is, the equivalence relation induced by the partition $\{\{u, v\}\} \cup \{\{w\} \mid w \in V(D) \setminus \{u, v\}\}$, is a non-trivial bisimulation over D.

If *ii*) does not hold, then D is due to have at least two vertices. Hence, the universal relation, that is, the equivalence relation that puts all vertices of D in the same equivalence class, is a non-trivial bisimulation over D.

To show *iii*), take, for a contradiction, a vertex $v \in V(D)$ so that there is no directed path from v to a sink of D. Let C be the set of all vertices u such that there is a directed path from v to u. By assumption on v, each vertex in C has at least one out-neighbor, and all such out-neighbors are in C. Since $N^+(v) \neq \emptyset$, we have that $|C| \geqslant 2$. The equivalence relation induced by the partition $\{C\} \cup \{\{w\} \mid w \in V(D) \setminus C\}$ is a non-trivial bisimulation over D, contradicting the hyper-extensionality of D.

As far as *iv*) is concerned, suppose that D is an extensional acyclic digraph that admits a non-trivial bisimulation B, and let x_0, y_0 be two distinct vertices of D so that $x_0 B y_0$. By the extensionality of D, $N^+(x_0) \neq N^+(y_0)$. Therefore, we may assume w.l.o.g. that there exists an $x_1 \in N^+(x_0) \setminus N^+(y_0)$. Since $x_0 B y_0$, there exists $y_1 \in N^+(y_0)$, thus $y_1 \neq x_1$, so that $x_1 B y_1$. The above procedure can be repeated indefinitely, which contradicts the fact that D has no infinite descending directed paths. ■

A digraph satisfying condition *iii*) of Lemma 1.4.5 will be called *channeled*. The following lemma shows that hyper-extensional digraphs, even though not simple, are almost channeled.

Lemma 1.4.6 *If D is a hyper-extensional digraph, then at most one vertex $v \in V(D)$ with $N^+(v) = \{v\}$ exists.*

Proof. The claim readily follows, since the equivalence relation induced by the partition $\{\{w\} \mid w \in V(D) \wedge N^+(w) \neq \{w\}\} \cup \{\{v \in V(D) \mid N^+(v) = \{v\}\}\}$ is a bisimulation over D. ■

Historical remarks

A first axiomatization of set theory was proposed by Zermelo in 1908, with the goal of avoiding logical traps that originate from the careless use of the intuitive notion of set [153]. This axiomatization was subsequently improved by Skolem, Fraenkel, and also by von Neumann, who, in 1925, introduced the Foundation Axiom [149,150]. The concept of rank appears first in Mirimanov [91], while Ackermann's encoding of hereditarily finite well-founded sets emerged in 1937 [2]. For more details on set theoretic concepts and axiomatics, refer to [25, 70, 78]. Our graph theoretic notation generally follows [10, 44]. The term 'picture of a set' was coined by Aczel [3]. Mostowski's collapsing lemma was introduced in [93]. Weakly extensional acyclic digraphs were introduced in [124], in connection with a counting problem, on which we report in Section 2.1.2. Slim extensional acyclic digraph identified a class of digraph well-quasi-ordered by the digraph immersion relation [121], and they were generalized to slim digraphs in [123]; Chapter 3 is devoted to slim digraphs. Dependent and irredundant extensional digraphs were briefly mentioned in [85], where it was argued that it is NP-complete to decide whether a graph admits such an orientation.

The notion of bisimulation emerged in various fields, almost contemporarily: modal logic [145], concurrency theory [89, 110], set theory [56], formal verification (cf. [37]). A notable application of the stable partitioning problem appeared in Hopcroft's work on minimizing the number of states in a given finite state deterministic automaton [67]. A linear-time algorithm for the stable partitioning problem, when the input relation corresponds to a single function, was given by Paige, Tarjan and Bonic in [108]. This problem was solved for the general case by Paige and Tarjan [107], with an algorithm of complexity $O(|E| \log |V|)$ (it is open whether this problem admits a linear-time algorithm). A linear-time algorithm for the case when the relation is well-founded was later given by Dovier, Piazza and Policriti in [47]; this approach also produces a more efficient algorithm for real-case instances than the one by Paige and Tarjan. The connection between stable partitioning and the maximum bisimulation was observed in [72, 73] (see also [47]). Refer to [62] for other applications of partition refinement techniques.

The Anti-Foundation Axiom was introduced by Aczel in [3]; Barwise and Moss [12, p. 5] indicate the paper by Forti and Honsell [56] as a precursor of Aczel's set theory. The term 'hyperset' was coined by [11]. In [3, 79], hyper-extensionality is called 'strong extensionality'. Our choice, motivated by the terminology of [11, 99], should emphasize the fact that a more involved irredundancy criterion is at work here, much more complex than, for example, the variants of extensionality considered at the end of Section 1.3.

Notational stipulations

We will use $\mathsf{ZF} - \mathsf{FA}$ for the set of axioms of Zermelo-Fraenkel deprived of the Foundation Axiom FA. Analogously, $\mathsf{ZF} - \mathsf{FA} + \mathsf{AFA}$ will denote Aczel's hyperset theory. In the last chapter we will also drop the Infinity Axiom from ZF and consequently employ ZF^- instead of ZF. Furthermore, the word 'set' will stand both for a flat collection of elements, and for an element of a model of the theory $\mathsf{ZF} - \mathsf{FA}$. When willing to emphasize whether the membership relation is well-founded or not, we will use 'set' for the theory ZF, 'non-well-founded set' for the theory $\mathsf{ZF} - \mathsf{FA}$, and 'hyperset' for the theory $\mathsf{ZF} - \mathsf{FA} + \mathsf{AFA}$.
We also use the shorthand (w.)e.a. digraph for a (weakly) extensional acyclic digraph.

2

Combinatorial Enumeration of Sets

Enumerative combinatorics is concerned with counting finite discrete objects sharing common specified properties. In the 1960s, the following question was addressed: Given n, how many sets S with n elements exist with the property that any element of S is also a subset of S? This, of course, is the problem of counting transitive sets of cardinality n. We will see in this chapter that the graph-theoretic interpretation of this problem offers an elegant and simpler solution than the one given five decades ago. In doing so, we count more general objects, that is, sets that can also have atoms, and bring to light connections with the count of acyclic digraphs.

The well-foundedness of the membership relation between sets plays a crucial role in this, since it allows one to do recursion. However, this is no longer the case when trying to enumerate hypersets. This problem remains (wide) open, and we believe that the ability to give a solution to it would have computational consequences for the maximum bisimulation problem. Instead, we show that a canonical linear order can be given on hypersets, so that it extends Ackermann's celebrated order on well-founded sets. We also show that hypersets can be mapped to numbers, with possible algorithmic implications.

As it is generally the case, with the increase of n, there is a combinatorial explosion in the number of these sets (and digraphs), so that an exhaustive generation of them is not possible. However, since one is usually interested in performing tests and benchmarks, collecting statistical data, (dis)proving conjectures, these sets can be sampled uniformly at random. In the last section of this chapter we adapt a Markov chain originally designed for acyclic digraphs to work for extensional ones.

2.1 Counting sets as extensional acyclic digraphs

In many cases, having an encoding of structurally rich objects into simpler domains not only reveals certain hidden properties, but also facilitates many problems. An emblematic example is the counting of labeled trees. Cayley's formula [27] stating that there are n^{n-2} trees with vertex set $\{1,\ldots,n\}$ was obtained by counting trees by their vertex degrees. However, there is a very simple and elegant encoding for such trees, called Prüfer code [125], which readily yields this result. Prüfer established a bijection between trees with vertex set $\{1,\ldots,n\}$ and vectors

$$(x_1,\ldots,x_{n-2}) \text{ where each } x_i \in \{1,\ldots,n\}, \tag{2.1.1}$$

obtained in the following algorithmic way: at the ith step ($1 \leqslant i \leqslant n-2$), the least remaining leaf is deleted, and x_i is set to be the *neighbor* of this leaf. For further details see [92], which enumerates many classes of trees.

Structurally alike to trees by some means, acyclic digraphs have a similar historical outline. They were counted in an algebraic manner by Stanley [136], who showed that the number of acyclic orientations of a labeled undirected graph equals the module of the value its chromatic polynomial takes in the point -1. Independently, Robinson obtained the count for unlabeled [130] and labeled [131] acyclic digraphs. Denoting by a_n the number of labeled acyclic digraphs with vertex set $\{1,\ldots,n\}$, his result states that

$$a_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} 2^{k(n-k)} a_{n-k}, \tag{2.1.2}$$

where a_0 stands for 1. Harary and Palmer counted labeled acyclic digraphs with a given number of sources [63, p. 19], while Gessel [59] counted them by sources, sinks and arcs. Since a_n has no closed form, its asymptotic behavior was studied in [17, 18].

Only recently, a Prüfer-like encoding of labeled acyclic digraphs emerged [137], which also yields a recurrence relation for a_n. Steinsky's bijection maps acyclic digraphs with vertex set $\{1,\ldots,n\}$ into the set of vectors

$$\begin{gathered}(X_1,\ldots,X_{n-1}), \text{ where each } X_i \subseteq \{1,\ldots,n\} \text{ and} \\ \text{for all } 1 \leqslant k \leqslant n-1, \left|\bigcup_{i=1}^{k} X_i\right| \leqslant k.\end{gathered} \tag{2.1.3}$$

Its algorithmic definition is analogous to the Prüfer code of a tree: at the ith step ($1 \leqslant i \leqslant n-1$), the least remaining *sink* is deleted, and X_{n-i} is set to be its *set* of *in-neighbors*. Since acyclic digraphs have many practical applications (e.g., in Bayesian networks [76], Information Visualization [143]), Steinsky also put forward a procedure to randomly generate them, by randomly generating the *code* of an acyclic digraph. The same problem of generating acyclic digraphs has also been approached by a Markov chain method in [83], and, for the weakly connected case, in [82].

It is not difficult to see that a code $(X_1,\ldots,X_{n-1})$ complying with (2.1.3) can also be interpreted as being obtained in the following way: at the ith step, the least remaining *source* is deleted, and X_{n-i} is set to be its set of *out-neighbors*. This vision is very similar to Peddicord's encoding of extensional acyclic digraphs [118], which we are about to introduce.

Bringing into play Ackermann's order, Peddicord's gave an encoding of transitive sets with n elements by a two-tier approach. At the outset, he restricted Ackermann's order to the elements of such a set, then he used this relative order to give an Ackermann-like encoding mapping its domain into the set of vectors

$$(x_0, \ldots, x_{n-1}), \text{ where } x_0 = 0 \text{ and } x_{i-1} < x_i < 2^i, \text{ for all } 1 \leqslant i \leqslant n-1. \quad (2.1.4)$$

Since transitive sets with n elements are in one-to-one correspondence with (unlabeled) extensional acyclic digraphs with n vertices, Peddicord's encoding of such a digraph D can be viewed as follows. The lexicographic interpretation (1.1.2) of Ackermann's order can be readily used to give a linear order $\prec^D$ on the vertices of any extensional acyclic digraph D. Using this, define a bijection $\pi_D : V(D) \to \{0, \ldots, n-1\}$ to assign *positions* to the vertices of D w.r.t. $\prec_A^D$, i.e.,

$$\pi(v) = |\{u \in V(D) : \ u \prec^D v\}|. \quad (2.1.5)$$

Then, at the ith step $(1 \leqslant i \leqslant n-1)$ of the algorithmic encoding procedure, remove the source v of D with $\pi(v)$ *maximum*, and set

$$x_{n-i} = \sum_{w \in N^+(v)} 2^{\pi(w)}, \quad (2.1.6)$$

that is, x_{n-i} equals the number whose binary expansion has a '1' in position $\pi(w)$ if and only if the vertex w is an out-neighbor of v. By counting how many encodings satisfying (2.1.4) exist, Peddicord gave the following recurrence for $\widehat{e}_n$, the number of unlabeled extensional acyclic digraphs with n vertices,

$$\widehat{e}_n = \binom{2^{n-1}-2}{n-2} - \sum_{k=1}^{n-3} \binom{2^{n-1}-2^{k+1}}{n-k-1} \widehat{e}_{k+1}, \quad (2.1.7)$$

where $\widehat{e}_1 = 1$ and $\widehat{e}_2 = 1$.

We also mention here a subclass of acyclic digraphs, namely labeled *essential* acyclic digraphs, which has been investigated in [138], in connection with Bayesian Networks. A digraph is said to be *essential* if for every arc $u \to v$, $N^-(u) \neq N^-(v) \setminus \{u\}$ holds. This set-theoretic flavor is further emphasized by that fact that the recurrence satisfied by a'_n, which denotes the number of labeled essential acyclic digraphs with vertex set $\{1, \ldots, n\}$, is quite similar to the one we will give in Section 2.1.1 for labeled extensional acyclic digraphs.

$$a'_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (2^{n-k} - n + k)^k a'_{n-k}, \quad (2.1.8)$$

Furthermore, the ratio between labeled acyclic digraphs and labeled essential acyclic digraphs is convergent, as $n \to \infty$ [139].

Since no direct counting recursion for extensional acyclic digraphs was considered before in the literature, our next section tackles this problem.

2.1.1 Counting labeled extensional acyclic digraphs

Our first approach is to count *labeled* e.a. digraphs, and then use the automorphism group of such a digraph to obtain the number of unlabeled e.a. digraphs.

Given two digraphs on $n \geqslant 1$ vertices bijectively labeled by numbers from the set $\{1, \ldots, n\}$, we say that they are *identical* if there exists an isomorphism preserving labels between them. Denote by e_n the number of e.a. digraphs on $n \geqslant 1$ vertices labeled by numbers from the set $\{1, \ldots, n\}$. Clearly, $e_1 = 1$ and in any acyclic digraph there is a vertex v with $d^-(v) = 0$.

For all $i \in \{1, \ldots, n\}$, let E_i stand for the set of all e.a. digraphs with n vertices labeled by numbers from the set $\{1, \ldots, n\}$ with the property that $d^-(i) = 0$. By the inclusion-exclusion principle, we have

$$e_n = |E_1 \cup E_2 \cup \cdots \cup E_n| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} |E_{i_1} \cap \cdots \cap E_{i_k}|.$$

Next, we will see that

$$\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} |E_{i_1} \cap \cdots \cap E_{i_k}| = \binom{n}{k} \big(2^{n-k} - n + k\big)_k e_{n-k},$$

where we have used the falling factorial notation $(x)_k = x(x-1)\ldots(x-k+1)$. Indeed, the number of e.a. digraphs with $n-k$ vertices labeled with numbers from the set $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ is equal to e_{n-k}. The vertices labeled with $i_1, \ldots, i_k$ have the property that $d^-(i_1) = \cdots = d^-(i_k) = 0$, hence they can be joined by arcs having a unique orientation only with the remaining $n-k$ vertices. As the resulting digraph must be extensional, no two of the k sources can have the same out-neighbors and, additionally, no source can have the same out-neighbors as a vertex among the remaining $n-k$. Hence, $|E_{i_1} \cap \cdots \cap E_{i_k}|$ is equal to $\big(2^{n-k} - (n-k)\big) \ldots \big(2^{n-k} - (n-k) - (k-1)\big) e_{n-k}$; as there exist $\binom{n}{k}$ ways to choose $i_1, \ldots, i_k$, we get the above expression.

In conclusion, $e_n = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} (2^{n-k} - n + k)_k e_{n-k}$. Note that the nonnull terms of this sum are for $2^{n-k} \geqslant n$, or, equivalently, for $k \leqslant n - \lceil \log_2 n \rceil$. We have the following:

Theorem 2.1.1 *The number e_n of labeled extensional acyclic digraphs on $n \geqslant 1$ vertices is*

$$e_n = \sum_{k=1}^{n - \lceil \log_2 n \rceil} (-1)^{k+1} \binom{n}{k} (2^{n-k} - n + k)_k e_{n-k}, \qquad e_0 = 1.$$

An automorphism of a digraph D is an isomorphism of D with itself, that is, a permutation on $V(D)$ that preserves adjacency and the orientation of arcs. It is well known that (under the operation of composition) the set of all automorphisms of a digraph G forms a group, denoted here by $\mathcal{A}ut(D)$, and referred to as the *automorphism group* of D. By bringing into play the automorphism group of a digraph D on n vertices, it is possible to determine the number of *distinct* labelings (i.e., not having the same set of arcs) of D from the same set of n labels.

Lemma 2.1.2 (see [13, Ch. 9]) *Let D be a digraph on n vertices. The number of distinct labelings of D is $n!/|\mathcal{A}ut(D)|$.*

Lemma 2.1.3 *Given an extensional acyclic digraph D, $\mathcal{A}ut(D) = \{id_D : V(D) \to V(D)\}$, where $id_D(v) = v$, for all $v \in V(D)$.*

Proof. Let $f \in \mathcal{A}ut(D)$ and suppose that in the writing of the permutation f on $V(D)$ as a product of disjoint cycles there is a cycle $c = (x_1, \ldots, x_r)$ of length $r \geqslant 2$. Therefore, $f(x_1) = x_2$, and, by the extensionality of D, $N^+(x_1) \neq N^+(x_2)$. As f is an automorphism, $|N^+(x_1)| = |N^+(x_2)|$, hence, $N^+(x_1) \neq \emptyset$ and there is a $y_1 \in N^+(x_1) \setminus N^+(x_2)$. If $f(y_1) = y_1$, then $(f(x_1), f(y_1)) = (x_2, y_1) \in E(D)$, contradicting $y_1 \notin N^+(x_2)$. Therefore, $f(y_1) \neq y_1$, and hence y_1 belongs to a cycle of the permutation f on $V(D)$ of length greater than or equal to 2. We can repeat the above procedure indefinitely, and, as the number of vertices of G is finite, we will reach a vertex already visited. We thus contradict the acyclicity of D. ■

By considering two digraphs on $n \geqslant 1$ unlabeled vertices *identical* if they are isomorphic, we obtain by Lemmas 2.1.2 and 2.1.3:

Theorem 2.1.4 *The number $\widehat{e}_n$ of extensional acyclic digraphs on $n \geqslant 1$ unlabeled vertices is*

$$\widehat{e}_n = \sum_{k=1}^{n-\lceil \log_2 n \rceil} (-1)^{k+1} \binom{2^{n-k} - n + k}{k} \widehat{e}_{n-k}, \qquad e_0 = 1.$$

2.1.2 Counting weakly extensional acyclic digraphs by sources, vertices of maximum rank, or arcs

Counting weakly extensional acyclic digraphs by sources

We consider now weakly extensional acyclic (*w.e.a.*, for short) digraphs, having their sinks labeled by distinct numbers from the set $\{0, \ldots, |A|\}$ (A to be intended as a finite set of *atoms*). Two such w.e.a. digraphs D_1 and D_2 are said to be *identical* if there is an isomorphism $f : V(D_1) \to V(D_2)$ such that for every sink $v \in V(D_1)$, the label of v in D_1 is the same as the label of $f(v)$ in D_2. As one can easily check, Lemma 2.1.3 can be generalized into the following:

Lemma 2.1.5 *If D is a w.e.a. digraph, then the only automorphism $f : V(D) \to V(D)$, with the property that for any sink v of D $f(v) = v$ holds, is the identity morphism.*

We proceed next by counting unlabeled digraphs directly, by a recursion on the number of sources. We will obtain a recurrence relation for the number of w.e.a. digraphs on $n \geqslant 1$ vertices, out of which $0 < s \leqslant n$ are sources and $0 < t \leqslant n$ are sinks labeled by numbers from the set $\{0, \ldots, |A|\}$, denoted here by $\widehat{w}_{n,s,t}$. Clearly, $\widehat{w}_{1,1,1} = |A| + 1$, and for all $n \geqslant 1$, $\widehat{w}_{n,s,t} = 0$, whenever $s = 0$ or $t = 0$, and $\widehat{w}_{n,k,n} = 0$, $\widehat{w}_{n,n,k} = 0$, for all $0 < k < n$. Moreover, $\widehat{w}_{n,s,t} = 0$ whenever $t > |A| + 1$. Note that if A is empty, $\widehat{w}_{n,s,1}$ is the number of e.a. digraphs with $n \geqslant 1$ vertices and $0 < s \leqslant n$ sources.

A w.e.a. digraph on $n \geqslant 2$ vertices, s sources and t sinks can be obtained from w.e.a. digraphs on $n - 1$ vertices by the addition of a source in several ways. First, a source can be added to a w.e.a. digraph on $n - 1$ vertices, $s - 1$ sources and $t - 1$ sinks, by putting no exiting arcs from it, such that it is a source and a sink at the same time. In all there are

$(|A| - t + 2)\,\widehat{w}_{n-1,s-1,t-1}$ ways to add this vertex, as it can be labeled in $|A| + 1 - (t - 1)$ possible ways.

Second, a source can be added to a digraph on $n - 1$ vertices, $s - 1$ sources, and t sinks. This will be connected to some of the $n - 1 - (s - 1)$ vertices which are not sources, such that in the resulting digraph its set of out-neighbors is nonnull and is different from the set of out-neighbors of the $(n - 1) - t$ vertices which are not sinks. There are $(2^{n-s} - n + t)\widehat{w}_{n-1,s-1,t}$ ways to add this new source. Note that when $2^{n-s} - n + t < 0$ the above expression would not make sense, unless $\widehat{w}_{n-1,s-1,t} = 0$. Indeed, if $2^{n-s} - n + t < 0$, then $n - t - 1 > 2^{n-s} - 1$; since the set of out-neighbors of each of the $n - t - 1$ non-sink vertices is one of the $2^{n-s} - 1$ nonnull subsets of the $n - s$ non-source vertices, the claim follows by the pigeonhole principle and extensionality.

Third, a new source can be added to a digraph on $n - 1$ vertices, t sinks and $s + k$ sources, for $k = 0, \ldots, n-s-1$, by connecting the new source with exactly $k+1$ preexistent sources. This new source can also have arcs towards the remaining $n-1-(s+k)$ vertices. In this case, the new source is not a sink and, since a preexistent source is among its elements, it will certainly have the set of out-neighbors different from any set of out-neighbors of the remaining $n - 1$ vertices. There are $\binom{s+k}{k+1}2^{n-1-(s+k)}\widehat{w}_{n-1,s+k,t}$ ways to add this vertex.

In the above process each w.e.a. digraph on n vertices, s sources, and t labeled sinks has been obtained *exactly* s times, by the addition of *each* one of its s sources to exactly *one* w.e.a. digraph on $n - 1$ vertices. In conclusion, the following theorem holds.

Theorem 2.1.6 *The number $\widehat{w}_{n,s,t}$ of w.e.a. digraphs on $n \geqslant 2$ vertices, out of which $0 < s < n$ are sources, and $0 < t < n$ are sinks labeled by distinct numbers from the set $\{0, \ldots, |A|\}$, is*

$$\widehat{w}_{n,s,t} = \frac{1}{s}\Big((|A| - t + 2)\,\widehat{w}_{n-1,s-1,t-1} + (2^{n-s} - n + t)\widehat{w}_{n-1,s-1,t} + \sum_{k=0}^{n-s-1}\binom{s+k}{k+1}2^{n-1-(s+k)}\widehat{w}_{n-1,s+k,t}\Big),$$

where $\widehat{w}_{1,1,1} = |A|+1$, and for all $n \geqslant 1$, $\widehat{w}_{n,s,t} = 0$, whenever $s = 0$ or $t = 0$ or $t > |A|+1$, and $\widehat{w}_{n,k,n} = 0$, $\widehat{w}_{n,n,k} = 0$, for all $0 < k < n$.

We now give an alternative expression for $\widehat{e}_n$ by summing up the number of all extensional digraphs with n vertices and s sources. Let $\widehat{e}_{n,s}$ be the number of e.a. digraphs on $n \geqslant 1$ unlabeled vertices, out of which s $(0 < s \leqslant n)$ are sources. Clearly, $\widehat{e}_{1,1} = 1$ and $\widehat{e}_{n,n} = 0$ for all $n \geqslant 2$. Since $\widehat{e}_{n,s} = \widehat{w}_{n,s,1}$ we have:

Corollary 2.1.7 *The number $\widehat{e}_{n,s}$ of extensional acyclic digraphs on $n \geqslant 2$ unlabeled vertices, out of which $0 < s < n$ are sources, is*

$$\widehat{e}_{n,s} = \frac{1}{s}\Big((2^{n-s} - (n-1))\widehat{e}_{n-1,s-1} + \sum_{k=0}^{n-s-1}\binom{s+k}{k+1}2^{n-1-(s+k)}\widehat{e}_{n-1,s+k}\Big),$$

where $\widehat{e}_{1,1} = 1$, and where we define $\widehat{e}_{n,0}$ as 0, for all $n \geqslant 1$.

By extensionality and the pigeonhole principle, $\widehat{e}_{n,s} \neq 0$ only for $2^{n-s} \geqslant n$, or equivalently, for $s \leqslant n - \lceil \log_2 n \rceil$. Hence, we have the following expression for $\widehat{e}_n$, for $n \geqslant 1$:

$$\widehat{e}_n = \sum_{s=1}^{n-\lceil \log_2 n \rceil} \widehat{w}_{n,s,1} = \sum_{s=1}^{n-\lceil \log_2 n \rceil} \widehat{e}_{n,s}.$$

Counting weakly extensional acyclic digraphs by vertices of maximum rank, or by arcs

Next, we employ the set-theoretic notion of *rank*, which in fact produces the simplest recursion. Note again that if a vertex has maximum rank, then it is a source, but the converse does not hold.

Let $\widehat{w}_n^r$ stand for the number of w.e.a. digraphs on $n \geqslant 1$ vertices, out of which $0 < r \leqslant n$ are vertices of maximum rank, and with an arbitrary number of sinks labeled by distinct numbers from the set $\{0, \ldots, |A|\}$.

Theorem 2.1.8 *The following recurrence relation holds for all $n \geqslant 2$ and all $0 < r < n$:*

$$\widehat{w}_n^r = \sum_{k=1}^{n-r} \widehat{w}_{n-r}^k \binom{(2^k - 1)2^{n-r-k}}{r},$$

where $\widehat{w}_n^n = \binom{|A|+1}{n}$, for all $n \geqslant 0$.

Proof. A w.e.a. digraph D_n^r on n vertices out of which r have maximum rank can be obtained by adding r new vertices to a w.e.a. digraph D_{n-r}^k on $n - r$ vertices out of which k have maximum rank ($1 \leqslant k \leqslant n - r$), such that in D_n^r only the new r vertices have maximum rank. There are $(2^k - 1)2^{n-r-k}$ total candidates for each set of out-neighbors of these r vertices, as from the set of k vertices of maximum rank of D_{n-r}^k at least one vertex must be chosen, while there is no restriction concerning the remaining $n - r - k$ vertices. Since the sets of out-neighbors of the new r vertices must be pairwise distinct, there are $\binom{(2^k-1)2^{n-r-k}}{r}$ ways of adding these vertices to D_{n-r}^k. ■

Corollary 2.1.9 *The number $\widehat{e}_n^r$ of extensional acyclic digraphs on $n \geqslant 2$ unlabeled vertices, out of which $0 < r < n$ are vertices of maximum rank, is*

$$\widehat{e}_n^r = \sum_{k=1}^{n-r} \widehat{e}_{n-r}^k \binom{(2^k - 1)2^{n-r-k}}{r}, \qquad \widehat{e}_1^1 = 1.$$

Extensional digraph can also be enumerated by arcs; however, the resulting recursions are quite involved. Denote by $z_{n,r,m}$ the number of w.e.a. digraphs on $n \geqslant 1$ vertices, out of which $1 \leqslant r \leqslant n$ are vertices of maximum rank, and having $0 \leqslant m \leqslant \binom{n}{2}$ arcs. We also let $a_{r,m}^{n,i}$ stand for the number of ways of adding $r \geqslant 1$ *new* vertices to a w.e.a. digraph on $n \geqslant 1$ vertices out of which i are vertices of maximum rank ($1 \leqslant i \leqslant n$), by connecting them with at least one of the i vertices of maximum rank, using a total of m arcs ($m \geqslant r$), such that the sets of out-neighbors of the r new vertices are pairwise distinct.

For all $n > 1$, $1 \leqslant r < n$, $0 \leqslant m \leqslant \binom{n}{2}$, we can write:

$$z_{n,r,m} = \sum_{k=0}^{m-r} \sum_{i=1}^{n-r} z_{n-r,i,m-r-k} a_{r,r+k}^{n-r,i},$$

where $z_{n,n,0} = \binom{|A|+1}{n}$, for $1 \leqslant n \leqslant |A|+1$; and for all $n \geqslant 1$, $1 \leqslant r \leqslant n$, $z_{n,r,m} = 0$ when $m > \binom{n}{2}$, and $z_{n,n,m} = 0$ when $m > 0$.

In order to find a recurrence relation for $a_{r,m}^{n,i}$, we shall denote by $a_{r,m,t,c}^{n,i}$ the number of ways of adding these new r vertices, with the restriction that the maximum cardinality of the sets of out-neighbors of the new vertices is $c \geqslant 1$ and that there are exactly t out of the r new vertices with this cardinality ($1 \leqslant t \leqslant r$).

Lemma 2.1.10 *The following recurrence relation holds for all $n \geqslant 1$, $1 \leqslant i \leqslant n$, $r \geqslant 1$, $m \geqslant 0$, $1 \leqslant t \leqslant r$, $1 \leqslant c \leqslant m$,*

$$a_{r,m,t,c}^{n,i} = \binom{\binom{n}{c} - \binom{n-i}{c}}{t} \sum_{d=1}^{c-1} \sum_{s=1}^{r-t} a_{r-t,m-tc,s,d}^{n,i},$$

where for all $n \geqslant 1$, $t \geqslant 1$, $c \geqslant 1$, $a_{t,tc,t,c}^{n,i} = \binom{\binom{n}{c}-\binom{n-i}{c}}{t}$; and $a_{r,m,t,c}^{n,i} = 0$ when $tc > m$ or $r = t$, but $m \neq tc$.

Proof. Since the t vertices of maximum cardinality c must be added along $r - t$ vertices of cardinality strictly less than c, their sets of out-neighbors will be different from the sets of the out-neighbors of the existing $r - t$ vertices. Therefore, we have to deal only with possible collisions among the sets of out-neighbors of cardinality c of the t new vertices. As they have to contain at least one of the i vertices of maximum rank and must have cardinality c, there are $\binom{n}{c} - \binom{n-i}{c}$ total candidates for each such set of out-neighbors. Since the sets of out-neighbors of the t new vertices must be pairwise distinct, there are $\binom{\binom{n}{c}-\binom{n-i}{c}}{t}$ ways of adding these vertices. ■

Since for all $n \geqslant 1$, $1 \leqslant i \leqslant n$, $r \geqslant 1$, $m \geqslant r$ it holds that $a_{r,m}^{n,i} = \sum_{c=1}^{m} \sum_{t=1}^{r} a_{r,m,t,c}^{n,i}$, we have:

Theorem 2.1.11 *For all $n > 1$, $1 \leqslant r < n$, $0 \leqslant m \leqslant \binom{n}{2}$, the following recurrence relation holds*

$$z_{n,r,m} = \sum_{k=0}^{m-r} \sum_{i=1}^{n-r} \sum_{c=1}^{r+k} \sum_{t=1}^{r} z_{n-r,i,m-r-k} a_{r,m,t,c}^{n-r,i},$$

where $z_{n,n,0} = \binom{|A|+1}{n}$, for $1 \leqslant n \leqslant |A|+1$; and for all $n \geqslant 1$, $1 \leqslant r \leqslant n$, $z_{n,r,m} = 0$ when $m > \binom{n}{2}$, and $z_{n,n,m} = 0$ when $m > 0$.

Numerical evaluations and asymptotic relations

Tables 2.1 and 2.2 contain some numerical evaluations for the recurrences given here. They suggest the following asymptotic relations, for $n \to \infty$:

1. $a_n \sim \alpha\, e_n$, where $\alpha \approx 3.06551$;
2. $\widehat{e}_n \sim \beta\, \widehat{e}_{n,1}$, where $\beta \approx 1.74106$;

n	$\widehat{e}_{n,1}$	$\widehat{e}_n^1$	$\widehat{w}_n^1, \lvert A\rvert = 1$	$\sum_{t=1}^{n} \widehat{w}_{n,1,t}$	$\widehat{w}_n^1$
1	1	1	2	1	1
2	1	1	2	2	2
3	2	2	7	9	15
4	8	8	28	84	132
5	68	76	284	1525	2335
6	1248	1504	5580	52954	79390
7	48640	62496	233600	3515169	5150593
8	3944336	5268272	19670856	448310168	648010744
9	655539168	897967376	3355273808	110518465641	158419173639
10	221111497856	307446110592	1148666852000	52956014818266	75557378403958

Table 2.1: Numerical evaluations for (w.)e.a. digraphs with a unique source and (w.)e.a. digraphs with a unique vertex of maximum rank. In the lower table we have chosen $|A| + 1 = n$.

n	a_n/e_n	$\widehat{e}_n/\widehat{e}_{n,1}$	$\widehat{e}_n/\widehat{e}_n^1$	$\widehat{w}_n^1/\widehat{e}_n^1$, $\lvert A\rvert = 1$	$\widehat{w}_n^1/\sum_{t=1}^{n} \widehat{w}_{n,1,t}$
5	2.772	1.294	1.157	3.736	1.531
6	2.914	1.443	1.198	3.710	1.449
7	2.988	1.554	1.209	3.737	1.465
8	3.026	1.628	1.218	3.733	1.445
9	3.046	1.674	1.222	3.736	1.433
10	3.055	1.702	1.224	3.736	1.426
15	3.065	1.739	1.226	3.736	1.419
20	3.065	1.740	1.226	3.736	1.419
25	3.065	1.741	1.226	3.736	1.419
30	3.065	1.741	1.226	3.736	1.419

Table 2.2: The ratio between some of the recurrences given in this section. In the last row we have chosen $|A| + 1 = n$.

3. $\widehat{e}_n \sim \gamma\, \widehat{e}_n^1$, where $\gamma \approx 1.22666$;
4. $\widehat{w}_n^1 \sim \delta\, \widehat{e}_n^1$, where $\delta \approx 3.73638$, if $|A| = 1$;
5. $\widehat{w}_n^1 \sim \varepsilon \sum_{t=1}^{n} \widehat{w}_{n,1,t}$, where $\varepsilon \approx 1.41935$, if $|A|$ is chosen as $n-1$.

Employing methods from generating functions and complex analysis, Stephan Wagner proved relations 1–4 in [152]. Wagner also noted that, since almost all acyclic digraphs have a trivial automorphism group (cf. [18]), relation 1 holds for unlabeled extensional acyclic digraphs as well. Moreover, the following two theorems were also proved.

Theorem 2.1.12 (Wagner [152]) *The maximum rank of a vertex in a random extensional acyclic digraph with n vertices is asymptotically normally distributed with mean $\sim 0.764334n$ and variance $\sim 0.145210n$.*

Theorem 2.1.13 (Wagner [152]) *The number of arcs in a random extensional acyclic digraph with n vertices is asymptotically normally distributed, with mean $\sim \frac{1}{2}\binom{n}{2}$ and variance $\sim \frac{1}{4}\binom{n}{2}$.*

2.2 An Ackermann-like enumeration of hypersets

Combinatorial enumeration of hereditarily finite hypersets is much less understood. There is as yet no counting recursion for the number of hyper-extensional digraphs with n vertices. A brute-force approach was employed in [88] with the help of a computer program that generated all digraphs with at most 5 vertices and counted how many of them are devoid of distinct bisimilar vertices. A similar problem was addressed in [95], which gave the complete lists of Finsler, Scott and Boffa transitive non-well-founded sets with at most 3 elements.

In this section, we are concerned instead with giving a 'natural' extensional of Ackermann's order to hereditarily finite hypersets. Traditional sets, HF, will retain in our bijection the same images as before; together with those images, which span all natural numbers, the images of hereditarily finite hypersets will span the set of all dyadic rationals. The choice of this numeric domain stems from the rationale that we want the membership relation to be readable, as before, from the binary representation of numbers. Our proposed extension will result from a natural move: we will construct both correspondences, Ackermann's and our own, via a splitting technique borrowed from algorithmics.

We can easily realize (cf. also [79]) that an extension of the Ackermann order to the entire $\overline{\mathsf{HF}}$ cannot be carried out naively on the basis on the property (1.1.2): to see this, consider the hypersets $a = \{b\},\ b = \{a, \emptyset\}$. Since $\emptyset \prec a$, we have $\max_{\prec}\{c : c \in a \setminus b\} = b$ and $\max_{\prec}\{c : c \in b \setminus a\} = a$. Property (1.1.2) then implies $a \prec b \Leftrightarrow b \prec a$, a contradiction.

A less naive attempt—which will, in fact, ultimately work—starts with the Ackermann function $\mathbb{N}_A$ (from which the Ackermann order can be defined), and tries to extend it to an encoding $\mathbb{Q}_A$ from $\overline{\mathsf{HF}}$ to a larger codomain $\mathbb{Y} \supset \mathbb{N}$. This must be done in such a way as to maintain the characteristic properties of $\mathbb{N}_A$, which we can state as follows:

- a "simple" recursive routine manipulating sets should allow one to get the code $y = \mathbb{Q}_A(a)$ from any given a;
- a "simple" reading of the code y should allow one to inductively determine the *extension* of the (unique) set a, such that $y = \mathbb{Q}_A(a)$.

We would also like to have a property corresponding to (1.1.1), which suggests what follows. As natural numbers have a twofold purpose in the Ackermann coding, being used both as *positions* inside a code $\mathbb{N}_A(a)$ and as the code itself, what happens if we split this purpose into two, by means of *two* functions, one assigning a position to each hereditarily finite hyperset and the other assigning a code to it? Since positions relative to natural numbers are already occupied by well-founded sets, it is natural to add *negative positions* to be used for hypersets. This will result in employing an extra function $\mathbb{Z}_A$ mapping into *integers* in place of natural numbers. By proceeding as outlined, we will obtain a *(dyadic) rational number* $\mathbb{Q}_A(a)$ as code of a, and we will aim at the following extension of property (1.1.1):

$$\text{the binary representation of } \mathbb{Q}_A(a) \text{ has a 1 in position } \mathbb{Z}_A(b) \text{ if and only if } b \in a, \qquad (2.2.1)$$

with both $\mathbb{Z}_A$ and $\mathbb{Q}_A$ extending $\mathbb{N}_A$.

In sight of defining the function $\mathbb{Z}_A$, which will be obtained via an extension to $\overline{\mathsf{HF}}$ of the Ackermann Order $\prec$ on HF, we now provide a concrete characterization of $\prec$, based on the so-called *splitting technique* for the stable partitioning problem (see also [79]).

2.2.1 A new look at the Ackermann order

We will work out an inductive characterization of Ackermann's order $\prec$ of HF, grounding it on the *splitting technique* devised in [107] for the stable partitioning problem, which was subsequently refined—to cite two among many—in [47] and [119]. In the ongoing, it will be used to impose an order on HF; then, in Section 2.2.2, it will be used to order $\overline{\mathsf{HF}}$ similarly.

For the stable partitioning problem, one is given a partition $\pi^\star$ along with a relation R on $\bigcup \pi^\star$, and must find the coarsest $\pi_\star$ of all partitions that refine $\pi^\star$ and are R-stable. Proceeding top-down, one can begin with $\pi = \pi^\star$ to then replace within π, as long as there are blocks p, q for which $p \cap R^{-1}[q]$ and $p \setminus R^{-1}[q]$ are nonnull, p by the latter two sets. If $\bigcup \pi^\star$ is finite, one will at last attain the desired $\pi_\star$ as value of π; more or less rapidly, depending on the order in which blocks are processed and split.

Within the stabilization process, the basic splitting action, namely replacing p by $p \cap R^{-1}[q]$ and $p \setminus R^{-1}[q]$, can be packaged together with many other actions of the same kind; for example (as proposed in [107]), one can trace all p's which can be split by the same q, and replace each of them by the resulting two blocks before seeking another q. Proceeding the other way around (as we will do), one can locate a p which is unstable relative to at least one q, and then supersede p inside π, in a single shot, by all equivalence classes into which p gets partitioned by the equivalence relation

$$x \sim_R y \ \leftrightarrow_{\mathrm{Def}} \forall\, q \in \pi \left(x \in R^{-1}[q] \Leftrightarrow y \in R^{-1}[q] \right) .$$

In the two cases which we will study, R will be $\ni$, while the initial partition $\pi^\star$ will first satisfy $\bigcup \pi^\star = \mathsf{HF}$ and then $\bigcup \pi^\star = \overline{\mathsf{HF}}$. Despite $\bigcup \pi^\star$ being infinite in either case, infinite repetition of the basic splitting action will end into something valuable. To set the ground for this on a simple preliminary example, suppose here that $\pi^\star = \{\mathsf{HF}\}$, let $\pi_0 = \pi^\star$, and then for $n = 0, 1, 2, \ldots$:

- observe that there is exactly one infinite block $p_n \in \pi_n$;
- observe that p_n is a culprit of the instability of π_n, as the sets

  $$\{\, x : \ x \in p_n \mid x \cap p_n \neq \emptyset \,\} \text{ and } \{\, x : \ x \in p_n \mid x \cap p_n = \emptyset \,\}$$

  are nonnull (actually, the former is infinite);
- put

  $$\pi_{n+1} = (\pi_n \setminus \{\, p_n \,\}) \cup \big\{ \{x : \ x \in p_n \mid x \cap p_n \neq \emptyset\}, \ \{x : \ x \in p_n \mid x \cap p_n = \emptyset\} \big\},$$

that is, we split the class p_n by using p_n itself as a splitter. At the conclusion, $\big\{ \{\, x : x \in p_n \mid x \cap p_n = \emptyset \,\} : \ n \in \mathbb{N} \big\}$ turns out to be the partition of HF whose blocks are the rank-equality classes. These blocks are all finite, but not singletons: an indication, since stable partitioning must give us the bisimilarity classes, that stability has not been attained as yet.

In what we are about to see, we resume work with the partition just found. We will sequence successive splitting actions fairly enough that the stable partition will result after denumerably many actions; along the way, we will impose an order on the singleton blocks.

An inductive definition of the Ackermann Order

Processing the collection HF will amount to defining a countable sequence $(\mathcal{X}^n)_{n\in\mathbb{N}}$ of ordered partitions $\mathcal{X}^n = \{ X^n_i : i \in \mathbb{N} \}$ of it. Each partition $\mathcal{X}^{n+1}$ will turn out to be an *ordered refinement* of $\mathcal{X}^n$, namely (for all $i, j, h, k \in \mathbb{N}$):

$$\exists k(X^{n+1}_i \subseteq X^n_k), \tag{2.2.2}$$

$$X^{n+1}_i \subseteq X^n_k \ \wedge\ X^{n+1}_j \subseteq X^n_h \ \wedge\ k > h \Rightarrow i > j\,. \tag{2.2.3}$$

That is, $\mathcal{X}^{n+1} \sqsubseteq \mathcal{X}^n$ and the ordering of the subblocks into which the blocks of $\mathcal{X}^n$ get split in the formation of $\mathcal{X}^{n+1}$ will be consistent with the preceding ordering.

For all n, we will maintain the invariant

$$\mathsf{Finite}\,(X^n_i) \wedge \big(x \in X^n_h \ \wedge \mathsf{rank}(y) < \mathsf{rank}(x) \ \wedge y \in X^n_k \quad\Rightarrow\quad h > k \big)\,, \tag{2.2.4}$$

implying that the blocks of $\mathcal{X}^n$ are finite and they, as well as their elements, are ordered in a way complying with rank comparison—hence complying, in this well-founded case, with membership. This is important because we want sets to be sorted *à la* Ackermann when, at the end of the process, the partition will be $\ni$-stable and blocks will be singletons. To meet (2.2.4) at the outset, we define $\mathcal{X}^0$ by putting

$$X^0_i = \{ x : x \in \mathsf{HF} \mid \mathsf{rank}(x) = i \} \quad \text{ for all } i \in \mathbb{N}.$$

Preliminary to defining $\mathcal{X}^{n+1}$, we consider the smallest index h such that the block X^n_h *can be split* in the sense that there exist $x, y \in X^n_h$, and some k, such that x shares elements with X^n_k whereas y does not. We also consider the equivalence relation $\sim_\ni$ on X^n_h given by

$$x \ \sim_\ni\ y \quad\Leftrightarrow\quad \forall k(X^n_k \cap x = \emptyset \leftrightarrow X^n_k \cap y = \emptyset).$$

Then we consider the partition induced by $\sim_\ni$ on X^n_h, ordered as follows: given two $\sim_\ni$-classes $Z', Z \subseteq X^n_h$, put Z' *before* Z if and only if, for $w \in Z'$ and $z \in Z$, the largest mismatch position k between w, z 'favors' z, i.e.

$$X^n_k \cap w = \emptyset \ \wedge\ X^n_k \cap z \neq \emptyset \ \wedge\ \forall j > k(X^n_j \cap w = \emptyset \leftrightarrow X^n_j \cap z = \emptyset).$$

It plainly ensues from the definition of $\sim_\ni$ that the mismatch position does not depend on the choice of w and z; hence this relationship imposes an order $Z_0, Z_1, \ldots, Z_m$ ($m \geqslant 1$) on the $\sim_\ni$-equivalence classes of X^n_h. On this ground we can put:

$$X^{n+1}_i \quad = \quad \begin{cases} X^n_i & \text{if } \ i < h, \\ Z_{i-h} & \text{if } \ h \leqslant i \leqslant h+m, \\ X^n_{i-m} & \text{if } \ h+m < i. \end{cases} \tag{2.2.5}$$

In the well-founded case at hand, an inductive argument on n shows that the smallest index h such that X^n_h can be split coincides with the smallest index h such that X^n_h is not a singleton; moreover, it turns out that the relation $\sim_\ni$ induces a partition of X^n_h into singleton blocks. These verifications are straightforward, and we leave them to the reader.

Properties (2.2.4), (2.2.2), and (2.2.3) hold throughout the construction and every element of HF will eventually belong to a singleton class. Given $n \in \mathbb{N}$ and $x \in \mathsf{HF}$, let $f(x,n) \in \mathbb{N}$ be such that

$$x \in X^n_{f(x,n)}.$$

Then one can easily prove that the full Ackermann order of Section 1.1 is the limit of the $\mathcal{X}^n$'s, that is:

$$x \prec y \quad \Leftrightarrow \quad \exists n\big(f(x,n) < f(y,n)\big).$$

The previous construction will be generalized next to hypersets, by producing a sequence $(\mathcal{Y}^n)_{n\in\mathbb{N}}$ of ordered partitions, whose limit linearly orders $\overline{\mathsf{HF}}$. For all $n \in \mathbb{N}$, the ordered partition $\mathcal{Y}^{n+1}$ will still be an ordered refinement of $\mathcal{Y}^n$, but we will not have the possibility to prove that $\mathcal{Y}^{n+1}$ results from splitting into *singleton* classes the first class of $\mathcal{Y}^n$ which is not a singleton: in spite of the close analogy between the constructions, the splitting process will behave differently in the case of hypersets.

2.2.2 An Ackermann order on hereditarily finite hypersets

Let us say that a linear order $\prec$ is *an Ackermann order* if it extends the Ackermann order of HF to a superset of HF. In order to get such an order on $\overline{\mathsf{HF}}$, we will mimic the splitting process just given for HF. In analogy with the above, we will build a sequence $(\mathcal{Y}^n)_{n\in\mathbb{N}}$ of ordered partitions $\mathcal{Y}^n = \{Y^n_i : i \in \mathbb{N}\}$ of $\overline{\mathsf{HF}}$, where each partition $\mathcal{Y}^{n+1}$ is an ordered refinement of $\mathcal{Y}^n$. The $\mathcal{Y}^n$'s are constructed inductively again, starting with an $\mathcal{Y}^0$ which, by way of first approximation, is taken arbitrarily; as we will see, a linear order on $\overline{\mathsf{HF}}$ will result as the limit of the sequence $(\mathcal{Y}^n)_{n\in\mathbb{N}}$ if all blocks in $\mathcal{Y}^0$ are finite. Moreover, one further restraint must be met by $\mathcal{Y}^0$ in order that this $\prec$ be an Ackermann order on $\overline{\mathsf{HF}}$.

The splitting procedure on $\overline{\mathsf{HF}}$

At step $n+1$, the ordered partition $\mathcal{Y}^{n+1}$ is defined as a refinement of $\mathcal{Y}^n$, in complete analogy with the splitting action exploited in the well-founded case. We say that a block Y^n_i *can be split* if it contains two inequivalent elements with respect to the relation $\sim_\ni$ defined by

$$x \ \sim_\ni \ y \quad \Leftrightarrow \quad \forall j\,(\,Y^n_j \cap x = \emptyset \leftrightarrow Y^n_j \cap y = \emptyset\,). \tag{2.2.6}$$

By considering the smallest number h such that Y^n_h can be split, and the partition of the block Y^n_h induced by $\sim_\ni$, we proceed exactly as before to sort the $\sim_\ni$-equivalence classes of Y^n_h as $Z_0, Z_1, \ldots, Z_m$ $(m \geqslant 1)$. Then we put:

$$Y^{n+1}_i \quad = \quad \begin{cases} Y^n_i & \text{if } \ i < h, \\ Z_{i-h} & \text{if } \ h \leqslant i \leqslant h+m, \\ Y^n_{i-m} & \text{if } \ h+m < i. \end{cases} \tag{2.2.7}$$

In sight of getting a linear order of $\overline{\mathsf{HF}}$, we define as before the dyadic relation

$$x \prec y \quad \Leftrightarrow \quad \exists n\big(\,f(x,n) < f(y,n)\,\big) \tag{2.2.8}$$

over $\overline{\mathsf{HF}}$ in terms of the function $f : \overline{\mathsf{HF}} \times \mathbb{N} \longrightarrow \mathbb{N}$ such that $x \in Y^n_{f(x,n)}$.

However, as we see in the following example, the relation $\prec$ is not necessarily a linear order.

Example 2.2.1 Suppose $\mathcal{Y}^0 = \{\overline{\mathsf{HF}}\}$, $x = \Omega$, and $y = \{\emptyset, \Omega\}$ for the unique hyperset Ω such that $\Omega = \{\Omega\}$. Then $f(x,2) < f(y,2)$ and hence $x \prec y$. As is easily proved by induction, for all n the class $Y^n_{f(x,n)}$ contains, besides x, the sequence $\emptyset^n, \emptyset^{n+1}, \emptyset^{n+2}, \ldots$ where $\emptyset^1 = \emptyset$ and $\emptyset^{n+1} = \{\emptyset^n\}$. It follows that $f(x,n)$ coincides with the smallest index h such that Y^n_h can be split. This implies that the non-singleton class $Y^n_{f(y,n)}$ is never split, and if $z \in Y^n_{f(y,n)} \setminus \{y\}$ then neither $z \prec y$ nor $y \prec z$ holds. ∎

We next give a necessary and sufficient condition for the relation $\prec$ defined in (2.2.8) to be a linear order on $\overline{\mathsf{HF}}$:

Lemma 2.2.2 *The relation $\prec$ is a linear order if and only if*

$$\forall x, y \in \overline{\mathsf{HF}} \Big(\forall n \left(f(x,n) = f(y,n) \right) \rightarrow \forall n \, \forall j \left(Y^n_j \cap x = \emptyset \leftrightarrow Y^n_j \cap y = \emptyset \right) \Big), \qquad (2.2.9)$$

i.e., iff any sets x, y in $\overline{\mathsf{HF}}$ that remain forever together in the same block never mismatch.

Proof. Condition (2.2.9) is clearly necessary in order that $\prec$ be a linear order.

Conversely, suppose (2.2.9) holds. Preliminary to proving that $\prec$ is a linear order, observe that $\prec$ is irreflexive and transitive; hence we must only prove that when $x \neq y$ holds there exists $n \in \mathbb{N}$ such that $f(x,n) \neq f(y,n)$. This in turn follows from the fact that the relation $\flat \subseteq \overline{\mathsf{HF}} \times \overline{\mathsf{HF}}$ defined by

$$x \flat y \Leftrightarrow \forall n \left(f(x,n) = f(y,n) \right)$$

is a bisimulation. To see this, suppose $x \flat y$ and $x' \in x$; then $x' \in Y^n_{f(x',n)} \cap x$ for all $n \in \mathbb{N}$. By (2.2.9) we obtain that also $Y^n_{f(x',n)} \cap y \neq \emptyset$ for all $n \in \mathbb{N}$. Since y is a finite set, from $Y^n_{f(x',n)} \cap y \neq \emptyset$ for all $n \in \mathbb{N}$ we deduce the existence of an element $y' \in y$ belonging to all classes $Y^n_{f(x',n)}$. This implies that $x' \flat y'$.

Likewise, $x \flat y$ and $y' \in y$ implies the existence of an $x' \in x$ such that $x' \flat y'$. ∎

One natural choice to achieve the condition expressed in Lemma (2.2.2) is to start the splitting process from a partition composed by finite sets, as the following Corollary shows.

Corollary 2.2.3 *If $\mathcal{Y}^0 = \{Y^0_i : i \in \mathbb{N}\}$, where every Y^0_i is finite, then $\prec$ linearly orders $\overline{\mathsf{HF}}$.*

Proof. We can prove that $\prec$ is a linear order by applying the above lemma, that is by proving that (2.2.9) holds. Assume x and y are such that there exists a stage n and a position j such that $Y^n_j \cap x = \emptyset \leftrightarrow Y^n_j \cap y \neq \emptyset$. If x and y belong to the same class Y^n_i, it follows from our hypothesis on $\mathcal{Y}^0$ that at stage n the number of elements belonging to classes preceding Y^n_i is finite. This is sufficient to guarantee that x and y will be eventually separated. ∎

Corollary (2.2.3) ensures the existence of infinitely many linear orders on $\overline{\mathsf{HF}}$ built up using the splitting procedure. Among them, we find an Ackermann order if the first partition $\mathcal{Y}^0$ is defined by resorting to a suitable notion of *rank*.

A rank notion for $\overline{\mathsf{HF}}$

We now define a notion of *rank* for the hereditarily finite hypersets $\overline{\mathsf{HF}}$. First, for $x, y \in \overline{\mathsf{HF}}$ we define a *x, y-path* of length $\ell \geqslant 0$ to be a sequence $x_0 = x, x_1, \ldots, x_\ell = y$ such that

1. $x_i \neq x_j$, for all $i, j \in \{0, \ldots, \ell\}$, $i \neq j$;
2. $x_{i+1} \in x_i$, for all $i \in \{0, \ldots, \ell - 1\}$.

Then, the *depth* $\overline{d}(x, y)$ of y relative to x is the maximum length of a x, y-path, if any.

Definition 2.2.4 *Let $x \in \overline{\mathsf{HF}}$. The* rank *of x is*

$$\mathsf{rank}(x) \ =_{Def} \max\left(\left\{\overline{d}(x, y) : \ y \in \mathsf{TrCl}(\{x\})\right\}\right).$$

Equivalently, the rank of a hyperset is the longest length of a directed path starting from its point. For example, the hypersets $\Omega = \{\Omega\}$ and $x = \{\Omega, \emptyset\}$ have rank 0 and 1, respectively.

The above definition of rank extends the classical one for well-founded sets and allows us to define a hierarchy of hereditarily finite hypersets. The rest of this section places a computable bound on the number of hereditarily finite hypersets of rank equal to n. The strategy is to place a bound $r(n)$ on the cardinality $\#\mathsf{TrCl}(\{x\})$, where $x \in \overline{\mathsf{HF}}$ and $\mathsf{rank}(x) = n$; indirectly, this gives us the desired bound, as the overall number of membership digraphs with a bounded number of vertices can be computed.

Given $x \in \overline{\mathsf{HF}}$ and $v \in \mathsf{TrCl}(\{x\})$, and a x, v-path π, we define

$$\begin{aligned}\mathsf{TrCl}(x, \pi, v) \ =_{\mathrm{Def}} \{v\} \quad \cup \quad \{\, z : \ z \in \mathsf{TrCl}(\{x\}) \mid & \text{ there exists a } v, z\text{-path whose} \\ & \text{components (other than } v\text{) do not belong to } \pi\}.\end{aligned}$$

For all $n \geqslant 0$ and $0 \leqslant k \leqslant n$, we let $r(n, k)$ be an upper bound for

$$\max_{\mathsf{rank}(x)=n,\ \pi \text{ a } x,v\text{-path of length } k} \#\mathsf{TrCl}(x, \pi, v).$$

Since $\mathsf{TrCl}(\{x\}) = \mathsf{TrCl}(x, [x], x)$,[1] the desired bound $r(n)$ is actually $r(n, 0)$. In order to express $r(n, k)$ recursively, we proceed by induction on $n - k$.

When $k = n$, consider $x, v \in \overline{\mathsf{HF}}$, $\mathsf{rank}(x) = n$, and a x, v-path π of length n. If there would exist a $z \in \mathsf{TrCl}(x, \pi, v) \setminus \{v\}$, then there would also exist a x, z-path of length at least $n + 1$, violating the rank of x. Hence $r(n, n)$ can be chosen as 1.

Assuming that we have $r(n, k+1)$, we look for $r(n, k)$. Let again $x, v \in \overline{\mathsf{HF}}$, $\mathsf{rank}(x) = n$, and π be a x, v-path of length k. Clearly,

$$\mathsf{TrCl}(x, \pi, v) = \{v\} \cup \bigcup_{w \in (\mathsf{TrCl}(x,\pi,v) \cap v) \setminus \{v\}} \mathsf{TrCl}(x, [\pi, w], w).$$

Since for all $w \in (\mathsf{TrCl}(x, \pi, v) \cap v) \setminus \{v\}$ the path $[\pi, w]$ has length $k + 1$,

$$\#\mathsf{TrCl}(x, \pi, v) \leqslant 1 + \sum_{w \in (\mathsf{TrCl}(x,\pi,v) \cap v) \setminus \{v\}} r(n, k + 1).$$

[1] For $x, y, z \in \overline{\mathsf{HF}}$, and π a x, y-path not containing z, we denote by $[\pi, z]$ the path π immediately followed by z. Moreover, we let $[x]$ stand for the unique x, x-path.

Hence, in order to calculate a bound for $r(n,k)$ we just need to know how many different elements w we find in $(\mathsf{TrCl}(x,\pi,v)\cap v)\setminus\{v\}$. Now, a hyperset w is always characterized by the isomorphic type of its membership digraph $(G_{\mathsf{TrCl}(\{w\})},w)$; if $w\in(\mathsf{TrCl}(x,\pi,v)\cap v)\setminus\{v\}$ and $z\in\mathsf{TrCl}(\{w\})$ then either $z\in\mathsf{TrCl}(x,[\pi,w],w)$, or $\exists z'\in\pi$ such that $z\in\mathsf{TrCl}(\{z'\})$. This implies that the elements $w\in(\mathsf{TrCl}(x,\pi,v)\cap v)\setminus\{v\}$ may be characterized by two factors:

1. the membership digraph of $\mathsf{TrCl}(x,[\pi,w],w)$ which has at most $r(n,k+1)$ vertices;
2. for each element $u\in\mathsf{TrCl}(x,[\pi,w],w)$, the set $u\cap\pi$, having at most $k+1$ vertices.

Since there are at most $2^{r^2(n,k+1)}$ ways to build membership digraphs with at most $r(n,k+1)$ vertices, and at most 2^{k+1} ways to add 'external' children in π to an element of $\mathsf{TrCl}(x,[\pi,w],w)$, we can set

$$r(n,k)\leqslant 1+r(n,k+1)2^{r^2(n,k+1)}2^{(k+1)r(n,k+1)}.$$

In conclusion we have, for all n:

Lemma 2.2.5 *There are finitely many hereditarily finite hypersets of rank n.*

Remark 2.2.6 Notice that hereditarily finite hypersets behave differently than arbitrary pointed finite digraphs with respect to Definition 2.2.4. If we define the rank of a pointed digraph as the maximum length of a directed path starting from the initial point, then it is not true that the number of pointed finite digraphs with rank smaller than n is finite: e.g. the *daisies* $G_n=(\{a_0,a_1,a_2,\dots,a_n\},E)$ with $E=\{(a_0,a_i):\ i\in\{1,\dots,n\}\}\cup\{(a_i,a_0):\ i\in\{1,\dots,n\}\}$ and initial point a_0 are pairwise non isomorphic and all have rank equal to one. ■

An Ackermann Order on $\overline{\mathsf{HF}}$

We start with the partition

$$\mathcal{Y}^0 \;=\; \{\,Y_i^0:\ i\in\mathbb{N}\,\}$$

where $Y_i^0=\{x:\ x\in\overline{\mathsf{HF}}\mid \mathsf{rank}(x)=i\}$, for all $i\geqslant 0$. Consider the splitting sequence $(\mathcal{Y}^n)_{n\in\mathbb{N}}$ defined as in Section 2.2.2. By Lemma (2.2.5), each class Y_i^0 contains a finite number of hypersets, and from Lemma (2.2.3) it follows that the order $\prec$, defined by

$$x\prec y\Leftrightarrow \exists n\in\mathbb{N}(f(x,n)<f(y,n)),$$

is a linear order on $\overline{\mathsf{HF}}$. Since the construction is a generalization of the splitting procedure on HF, and well-founded sets only contain well-founded sets, the order $\prec$ extends the Ackermann order on HF.

Example 2.2.7 We consider as an example the $\overline{\mathsf{HF}}$ hypersets depicted in Figure 2.1.

The hypersets a,b,c,d,e,f are all non-well-founded; a,d have rank equal to 3, while b,c,e,f have rank equal to 2. Hence a,d belong to Y_3^0, while b,c,e,f belong to Y_2^0. The splitting procedure goes as follows:

$$[\emptyset\dots]\dots[b,c,e,f\dots][a,d\dots]\dots$$

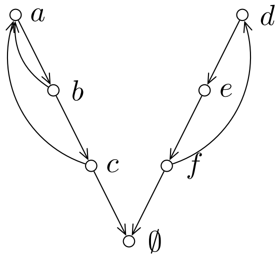


Figure 2.1: Hypersets $a = \{b\},\ b = \{c, a\},\ c = \{a, \emptyset\},\ d = \{e\},\ e = \{f\},\ f = \{d, \emptyset\}$

$$[\emptyset] \ldots [e \ldots][c, f \ldots][b \ldots][a, d \ldots] \ldots$$
$$[\emptyset] \ldots [e \ldots][c, f \ldots][b \ldots][d \ldots][a \ldots] \ldots$$
$$[\emptyset] \ldots [e \ldots][f \ldots][c \ldots][b \ldots][d \ldots][a \ldots] \ldots$$

Hence, the final order $\prec$ on a, b, c, d, e, f satisfies

$$\emptyset \prec e \prec f \prec c \prec b \prec d \prec a.$$

■

Remark 2.2.8 Notice that the extended Ackermann order $\prec$ resulting from the above construction is by no means unique. Arguing as in the preceding section, in fact, we see that the splitting process could have started with any partition $\mathcal{Y}^0 = \{Y_i^0 : i \in \mathbb{N}\}$ composed of finite sets Y_i^0 with $Y_i^0 \supseteq \{x : x \in \mathsf{HF} \mid \mathsf{rank}(x) = i\}$: the limit of the sequence $(\mathcal{Y}^n)_{n \in \mathbb{N}}$ would then have been an Ackermann order as well. ■

The above remark suggests that in the presence of hypersets, different notions of *rank* can be used to ground the splitting procedure.

Hereditarily finite hypersets as dyadic numbers

We are now ready to introduce the extension of the Ackermann function. First, we use the order $\prec$ on $\overline{\mathsf{HF}}$ just defined to give *positions* to hypersets in $\overline{\mathsf{HF}}$. As explained before, we need to use integer positions, since natural positions are already occupied by well-founded sets.

If $a \in \overline{\mathsf{HF}}$, define:

$$\mathbb{Z}_A(a) \quad = \quad \begin{cases} |\{\, b : b \in \mathsf{HF} \mid b \prec a \,\}| & \text{if } a \in \mathsf{HF}, \\ -|\{\, b : b \in \overline{\mathsf{HF}} \setminus \mathsf{HF} \mid b \prec a \,\}| - 1 & \text{if } a \in \overline{\mathsf{HF}} \setminus \mathsf{HF}. \end{cases}$$

Let $\mathbb{Q}_2$ be the set of all dyadic numbers, that is,

$$\mathbb{Q}_2 = \left\{ \frac{n}{2^m} : n, m \in \mathbb{N} \right\}.$$

Dyadic numbers are rational numbers having a binary expansion with a finite number of digits.

We define a bijection $\mathbb{Q}_A$ from $\overline{\mathsf{HF}}$ to dyadic numbers as follows:

$$\mathbb{Q}_A(a) \quad = \quad \Sigma_{b \in a} 2^{\mathbb{Z}_A(b)}.$$

All properties announced at the beginning are satisfied. In particular:

- $\mathbb{Q}_A : \overline{\mathsf{HF}} \to \mathbb{Q}_2$ extends the Ackermann function $\mathbb{N}_A : \mathsf{HF} \to \mathbb{N}$; that is, $\mathbb{Q}_A(x) = \mathbb{N}_A(x)$ holds when $x \in \mathsf{HF}$.

- A simple reading of the code $y \in \mathbb{Q}_2$ allows us to inductively determine the *extension* of the hyperset a such that $y = \mathbb{Q}_A(a)$. This is because from the digits of y we determine the positions $\mathbb{Z}_A(b)$ of all $b \in a$, and, since the bijection $\mathbb{Z}_A$ is effective, from $\mathbb{Z}_A(b)$ we are able to determine b.

- A simple recursive routine manipulating hypersets allows us to build the code $y = \mathbb{Q}_A(a)$ from any given hereditarily finite hyperset a, because if we know a we can compute $\mathbb{Z}_A(b)$ for all $b \in a$, and hence $\mathbb{Q}_A(a)$.

The twofold role of the Ackermann function $\mathbb{N}_A : \mathsf{HF} \to \mathbb{N}$, by which $\mathbb{N}_A(a)$ acts at the same time as code and as position for the hereditarily finite set a, must be played by distinct functions in the case of hypersets: for these, $\mathbb{Z}_A : \overline{\mathsf{HF}} \to \mathbb{Z}$ defines positions while $\mathbb{Q}_A : \overline{\mathsf{HF}} \to \mathbb{Q}_2$ assigns codes.

Mapping $\mathbb{Q}_2$ into $\overline{\mathsf{HF}}$

A final, natural, question can arise when considering the mapping $\mathbb{Q}_A$: is a mapping of *opposite direction*—namely, a mapping from $\mathbb{Q}_2$ into $\overline{\mathsf{HF}}$—definable?

Clearly, if no constraint is imposed, the answer is yes. However, if one requires some kinship with von Neumann's injection of $\mathbb{N}$ into HF, then the question becomes intriguing. In our opinion, a minimal requirement to impose on any $h : \mathbb{Q}_2 \to \overline{\mathsf{HF}}$ extending von Neumann's one, is the following:

$$\forall a \in \mathbb{Q}_2 \; \forall x \in h(a) \; \exists b \in \mathbb{Q}_2 \; \left(b \leqslant a \wedge x = h(b)\right). \tag{2.2.10}$$

It can be shown that a function h satisfying the above property cannot be defined. If such a function existed, fix an $x \in (0,1) \cap \mathbb{Q}_2$, and, among the $y \in (0,x) \cap \mathbb{Q}_2$, consider one with $\mathsf{TrCl}(\{h(y)\})$ of minimal cardinality and call it y_x. We claim that $h(y_x)$ must be either Ω or the solution Ω' to the equation $X = \{X, \emptyset\}$. This can be proved as follows. If $a \in h(y_x)$ and $a \neq \emptyset$, by (2.2.10) we find $z \in (0, y_x] \cap \mathbb{Q}_2$ with $h(z) = a$; moreover, by minimality of $|\mathsf{TrCl}(\{h(y_x)\})|$, we must have $\mathsf{TrCl}(\{h(z)\}) = \mathsf{TrCl}(\{h(y_x)\})$. Hence $h(y_x) \in \mathsf{TrCl}(\{h(z)\})$, which, by (2.2.10), implies $z = y_x$. Hence, all non empty elements of $h(y_x)$ must be equal to $h(y_x)$, which proves that either $h(y_x) = \Omega$ or $h(y_x) = \{h(y_x), \emptyset\}$.

From the claim we easily reach a contradiction by considering $x_1 = y_{1/2}, x_2 = y_{x_1}$ and $x_3 = y_{x_2}$, since we should have both $x_1 < x_2 < x_3$ and $h(x_i) \in \{\Omega, \Omega'\}$.

2.3 Random generation of sets

Since a set is such a basic mathematical object, one is interested in sampling uniformly at random sets for performing tests and benchmarks, collecting statistical data, (dis)proving conjectures, etc. We tackle here the problem of randomly generating transitive sets with n elements, or equivalently, extensional acyclic digraphs with n vertices. Notice that, in light of the result of [152] stating that the asymptotic ratio between labeled acyclic digraphs and labeled e.a. digraphs is constant, the latter class of digraphs can be sampled by running a sampler for acyclic digraphs and checking whether the produced digraph is extensional. However, we aim here for a direct method tailored to (w.)e.a. digraphs.

We will do this by adapting a Markov chain based procedure for generating acyclic digraphs, first introduced in [83], to our set-theoretic universe. This Markov chain algorithm was already modified in [82] to generate simply connected acyclic digraphs. The random generation of elements from a particular class of acyclic digraphs modeling Bayesian networks was proposed in [68]. Finally, the same approach was used in [26] to generate deterministic acyclic automata. Each of these examples can be seen as a less basic case than the one tackled here.

The idea behind this Markov chain technique is to start with an arbitrary (weakly) extensional acyclic digraph on n vertices and apply a certain number T of random local transformations which preserve weak extensionality and acyclicity. The uniformity of the resulting distribution is basically proved by showing that any w.e.a. digraph on n vertices can be thus transformed into any other w.e.a. digraph on n vertices. Like in the acyclic digraph case, we argue that the transformation rules are symmetric and always allow reaching a specific digraph among the collection of w.e.a. digraphs with n vertices. In our case, however, the most natural target digraph for this purpose turns out to be an acyclic tournament on n vertices, that is, the digraph whose interpretation in the universe of sets is von Neumann numeral of $n + 1$, the unique transitive set with n elements well-ordered by the membership relation.

We prove here only 'correctness' and defer to future work computational aspects such as estimations for the choice of T or an analysis of the mixing time of the Markov chain [77].

Definition 2.3.1 *A discrete time finite stochastic process is a sequence* $X = \{X_t \in \mathcal{S} : t \in \mathbb{N}\}$ *where* X_t *are random variables and* $\mathcal{S}$ *is finite. We say that* X *is a Markov chain if*

$$\Pr(X_{t+1} = s_{t+1} \mid X_t = s_t, \ldots, X_0 = s_0) = \Pr(X_{t+1} = s_{t+1} \mid X_t = s_t).$$

Moreover, a Markov chain X *is said to be homogenous if*

$$\Pr(X_{t+1} = s \mid X_t = s') = p_{ss'},\ \forall s, s' \in \mathcal{S},\ \forall t \in \mathbb{N}.$$

Definition 2.3.2 *A homogenous Markov chain over the state space* $\mathcal{S}$ *is said to be:*

- irreducible *iff* $\forall s, s' \in \mathcal{S}$, $\exists t \in \mathbb{N}$ *such that* $\Pr(X_t = s' \mid X_0 = s) > 0$*;*
- aperiodic *iff* $\forall s \in \mathcal{S}$, $\gcd\{t \in \mathbb{N} \mid \Pr(X_t = s \mid X_0 = s) > 0\} = 1$*;*
- symmetric *iff* $\forall s, s' \in \mathcal{S}$, $\Pr(X_{t+1} = s \mid X_t = s') = \Pr(X_{t+1} = s' \mid X_t = s)$*.*

A well-known result (see, e.g., [77]) states that any finite, irreducible, aperiodic and symmetric homogenous Markov chain converges toward the uniform distribution on every state of its space. Therefore, all the Markov chains presented here will be shown to satisfy these three properties.

Given $n \geqslant 1$, we denote by $\mathcal{W}_n$ the set of all w.e.a. digraphs on n vertices, labeled by distinct numbers from the set $\{1, \ldots, n\}$, while $\mathcal{W}_n^c$ denotes its subset of *weakly connected* digraphs. Analogously, $\mathcal{W}_{n,m}$ denotes the set of all w.e.a. digraphs on n vertices, labeled by distinct numbers from the set $\{1, \ldots, n\}$, and m arcs. We regard two labeled w.e.a. digraphs *identical* if they have the same set of arcs.

A Markov chain algorithm for generating weakly extensional acyclic digraphs

Let M be a Markov chain over $\mathcal{W}_n$, defined in Figure 2.2. Notice that for any $t \in \mathbb{N}$ and any two distinct states $s, s' \in \mathcal{W}_n$, $\Pr(X_{t+1} = s \mid X_t = s') > 0$ if and only if $\Pr(X_{t+1} = s' \mid X_t = s) > 0$. To be more precise, the probability of passing from a state $s \in \mathcal{W}_n$ to any other state $s' \neq s$ is either 0 or $1/n^2$, hence M is indeed symmetric. Since there exists $s \in \mathcal{W}_n$ such that the probability of remaining in s at any $t > 0$ is positive, it holds that if M turns out to be irreducible, then it will also be aperiodic.

Let X_t denote the state of the Markov chain at time t. Suppose a couple of integers (i, j) has been drawn uniformly at random from the set $\{1, \dots, n\} \times \{1, \dots, n\}$.

($\mathbf{T}_1$) **if** $(i, j) \in E(X_t)$ **and**
$X_t - (i, j)$ is w.e., **then** $X_{t+1} = X_t - (i, j)$
else $X_{t+1} = X_t$.
($\mathbf{T}_2$) **if** $(i, j) \notin E(X_t)$ **and**
$X_t + (i, j)$ is w.e.a., **then** $X_{t+1} = X_t + (i, j)$
else $X_{t+1} = X_t$.

Figure 2.2: A Markov chain algorithm for generating w.e.a. digraphs.

The initial state of this Markov chain and of the ones given in the next section can be taken to be a directed path $(n, n-1, \dots, 1)$. The acyclicity of a digraph on n vertices and m arcs can be established by a depth-first visit, in time $O(n + m)$. To test whether a digraph is (weakly) extensional, the algorithm in [47, Sec. 4] can be used, taking time $O(n + m)$.

Lemma 2.3.3 *Let D be a w.e.a. digraph with $E(D) \neq \emptyset$. There exists an arc $(u, v) \in E(D)$ such that the digraph $D - (u, v)$ is w.e.a.*

Proof. Observe first that there exists $u \in V(D)$ such that $\emptyset \neq N^+(u) \subseteq I(D)$. If this were not the case, then for all u in D with $N^+(u) \neq \emptyset$, there would exist a vertex u' in D with $N^+(u') \neq \emptyset$ such that $(u, u') \in E(D)$. Since the same property holds for u' as well, and as the number of vertices of D is finite, we can find a finite directed cycle $(u, u')(u', u'') \cdots$ in D, contradicting hence its acyclicity.

Let now $U(D)$ be the set of vertices of D with the above property, that is, $U(D) =_{\text{Def}} \{u \in V(D) \mid \emptyset \neq N^+(u) \subseteq I(D)\}$. Let $u_0 \in U(D)$ be a vertex of minimum out-degree, i.e., $d^+(u_0) = \min\{d^+(u) : u \in U(D)\}$. Since $N^+(u_0) \neq \emptyset$, let v_0 be an element of $N^+(u_0)$. The arc (u_0, v_0) can be removed and the resulting digraph remains w.e.a. Indeed, its removal can cause a collision only between the elements of $U(D)$. Since u_0 is among the vertices of minimal out-degree, in $D - (u_0, v_0)$ it will be the only vertex in $U(D - (u_0, v_0))$ with out-degree $d^+(u) - 1$, hence having its out-neighborhood different from that of any other vertex of $D - (u_0, v_0)$. ■

Theorem 2.3.4 (Irreducibility of M) *Let M be the Markov chain defined over the space $\mathcal{W}_n$ together with the transition rules $\mathbf{T}_1$ and $\mathbf{T}_2$. Given two digraphs D and H in $\mathcal{W}_n$, there exists in M a sequence of transitions $D = D_0 \to D_1 \to \cdots \to D_{p-1} \to D_p = H$,* *where $p \geqslant 1$ and $D_i \in \mathcal{W}_n$, for all $0 \leqslant i \leqslant p$. Such a sequence exists with length at most $n^2 - n$.*

Proof. Since M is symmetric, it suffices to show that there exists a sequence of transitions from any given w.e.a. digraph $D \in \mathcal{W}_n$ to a fixed element O in $\mathcal{W}_n$. For our purpose here, we will choose O to be the unique totally disconnected digraph, that is, having $E(O) = \emptyset$.

From Lemma 2.3.3, we get that there exists an arc $(u, v) \in E(D)$ such that $D - (u, v)$ is w.e.a. Using rule $(\mathbf{T}_1)$, we can step from D to $D-(u, v)$. Repeating the above argument a finite number of steps, we arrive at O. The number of transitions from D to O is at most $n(n-1)/2$, and this is obtained when D is a tournament. ■

A Markov chain algorithm for generating extensional acyclic digraphs

Instead of generating e.a. digraphs, we place ourselves in a more general setting, that of generating simply connected w.e.a. digraphs. Afterwards, we will argue that, with minor changes, the proposed Markov chain can generate e.a. digraphs.

Let M^c be the Markov chain over $\mathcal{W}_n^c$ whose transitions between states are given in Figure 2.3. Once again, the probability of passing from a state $s \in \mathcal{W}_n^c$ to any other state $s' \neq s$ is either 0 or $1/n^2$, implying that M^c is symmetric. Similarly, the aperiodicity of M^c is implied by its irreducibility, as there are digraphs in $\mathcal{W}_n^c$ for which there is a positive probability to remain in the same state, after a transition of M^c. Even if the two transition rules of M^c are not entirely specular, one can think of M^c as having three basic transitions: (1) removal of an arc, (2) reversal of an arc, (3) addition of an arc.

Let X_t denote the state of the Markov chain at time t. Suppose a couple of integers (i, j) has been drawn uniformly at random from the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$.

$(\mathbf{T}_1^c)$ **if** $(i, j) \in E(X_t)$ **then**
- (a) **if** $X_t - (i, j)$ is simply connected and w.e., **then** $X_{t+1} = X_t - (i, j)$, **else**
  - (b) **if** $X_t - (i, j) + (j, i)$ is w.e.a., **then** $X_{t+1} = X_t - (i, j) + (j, i)$,
  - (c) **else** $X_{t+1} = X_t$.

$(\mathbf{T}_2^c)$ **if** $(i, j) \notin E(X_t)$, **then**
- (a) **if** $X_t + (i, j)$ is w.e.a., **then** $X_{t+1} = X_t + (i, j)$,
- (b) **else** $X_{t+1} = X_t$.

Figure 2.3: A Markov chain algorithm for generating simply connected w.e.a. digraphs.

To show the irreducibility of the Markov chain M^c, it is useful to partition the vertices of an acyclic digraph according to the longest length of a directed path to the sinks of the digraph. Complying with standard set-theoretic notation, we give the following definition.

Definition 2.3.5 *Given an acyclic digraph D, the* rank *of a vertex $v \in V(D)$ is recursively defined as*

$$\mathsf{rank}(v) = 1 + \max\{\mathsf{rank}(u) : (v, u) \in E(D)\},$$

where $\mathsf{rank}(v) = 0$ *if v is a sink.*

Clearly, the following lemma holds.

Lemma 2.3.6 *Given an acyclic digraph D, if $v, u \in V(D)$ and $\mathsf{rank}(v) \neq \mathsf{rank}(u)$, then $N^+(v) \neq N^+(u)$ holds.*

Throughout the subsequent two proofs we employ the following notation: given a digraph D and a vertex v of D,

$$R(v) =_{\text{Def}} \{u \in V(D) \mid u \neq v \text{ and } \mathsf{rank}(u) \leqslant \mathsf{rank}(v)\}.$$

Theorem 2.3.7 (Irreducibility of M^c) *Let M^c be the Markov chain defined over the space $\mathcal{W}_n^c$ together with the transition rules $\mathbf{T}_1^c$ and $\mathbf{T}_2^c$. Given two digraphs D and H in $\mathcal{W}_n^c$, there exists in M^c a sequence of transitions $D = D_0 \to D_1 \to \cdots \to D_{p-1} \to D_p = H$, where $p \geqslant 1$ and $D_i \in \mathcal{W}_n^c$, for all $0 \leqslant i \leqslant p$. Such a sequence exists with length at most $(3n^2 - 7n + 4)/2$.*

Proof. As before, first we will show that there exists a sequence of transitions from any given w.e.a. digraph $D \in \mathcal{W}_n^c$ to an element $T(D)$ in $\mathcal{W}_n^c$, where $T(D)$ is an acyclic tournament, with the additional property that whenever $\mathsf{rank}(v) \geqslant \mathsf{rank}(u)$ in D, $\mathsf{rank}(v) \geqslant \mathsf{rank}(u)$ also holds in $T(D)$. Then, given any D and H in $\mathcal{W}_n^c$, we will show that there is a sequence of transitions from $T(D)$ to $T(H)$, completing hence the proof.

To show the former claim, we proceed as follows. Pick a vertex $v \in V(D)$, in decreasing order of rank (when more vertices of the same maximum rank exist, pick an arbitrary one). Apply rule ($\mathbf{T}_2^c$) and add arcs from v to all the vertices $u \in R(v) \setminus N^+(v)$, in decreasing order on the rank of the elements of $R(v) \setminus N^+(v)$. Note that this is possible, first of all, because the addition of an arc (v, u) does not create a cycle in the resulting digraph. Second, observe that the subdigraph of D induced by the vertices $V(D) \setminus R(v)$ is an acyclic tournament. Therefore, an arc addition would create a collision only between v and a vertex $u \in R(v)$. This is however not the case, since after the first addition of such an arc $\mathsf{rank}(v)$ becomes strictly greater than $\mathsf{rank}(u)$, for all $u \in R(v)$, and Lemma 2.3.6 guarantees the absence of collisions.

Denote by $T(D)$ the acyclic tournament obtained at the end of this process. Since for any vertex v we have added arcs only to those vertices of rank less than or equal to v, we also have that whenever $\mathsf{rank}(v) > \mathsf{rank}(u)$ in D, the same holds in $T(D)$.

Passing on to the latter point, observe that for any w.e.a. digraph D, since $T(D)$ is a tournament, there are no two distinct elements of the same rank in $T(D)$, and thus $\{\mathsf{rank}(v) : v \in V(T(D))\} = \{0, \ldots, n-1\}$. Hence, to each digraph $T(D)$ we can uniquely associate a linear order $\prec_{T(D)}$ on $V(D)$ defined in the following way: for all $u, v \in V(T(D))$

$$u \prec_{T(D)} v \text{ iff } \mathsf{rank}(u) < \mathsf{rank}(v) \text{ in } T(D).$$

We now show that given two orders $x_0 \prec_{T(D)} x_1 \prec_{T(D)} \cdots \prec_{T(D)} x_{n-1}$ and $y_0 \prec_{T(H)} y_1 \prec_{T(H)} \cdots \prec_{T(H)} y_{n-1}$, where $\{x_i : 0 \leqslant i \leqslant n-1\} = \{y_i : 0 \leqslant i \leqslant n-1\} = \{1, \ldots, n\}$, we can transform $T(D)$ in $T(H)$, applying rule ($\mathbf{T}_1^c$).

Observe first that for any two consecutive elements $x_i \prec_{T(D)} x_{i+1}$ ($0 \leqslant i < n-1$) it holds that $N^+(x_{i+1}) = N^+(x_i) \cup \{x_i\}$. Therefore, applying rule ($\mathbf{T}_1^c$) on $T(D)$, the arc (x_i, x_{i+1}) cannot be removed (by (a)), but can be reversed (by (b)). In the resulting acyclic tournament $T(D')$, x_i and x_{i+1} have swapped positions, i.e., $x_{i+1} \prec_{T(D')} x_i$. Starting from position $i = 0$ all the way to $i = n - 1$, apply the following procedure. If $y_i = x_j$, $(i < j \leqslant n - 1)$, then x_j will be brought to position i by iteratively reversing the arcs (x_j, x_{j-1}), (x_j, x_{j-2}), ...,(x_j, x_i).

The maximum number of transitions to pass from D to $T(D)$ is $\binom{n}{2} - (n-1) = (n^2 - 3n + 2)/2$, number obtained when the underlying graph of D is a tree, thus having $n - 1$ edges. To pass from $T(D)$ to $T(H)$, $\binom{n}{2}$ transitions are required at most, when all the arcs of $T(D)$ have to be reversed. Hence, to pass between two arbitrary D and H in $\mathcal{W}_n^c$, we need at most $(3n^2 - 7n + 4)/2$ transitions. ■

Let us denote by $\mathcal{E}_n$ the set of all e.a. digraphs with vertex set $\{1, \ldots, n\}$. The Markov chain illustrated in Figure 2.3 can be transformed into an irreducible, aperiodic and symmetric Markov chain, M^e, for the generation of digraphs from $\mathcal{E}_n$. The transitions between two states in M^e are given in Figure 2.4. Theorem 2.3.7 holds for M^e as well.

Let X_t denote the state of the Markov chain at time t. Suppose a couple of integers (i, j) has been drawn uniformly at random from the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$.

($\mathbf{T}_1^c$) **if** $(i, j) \in E(X_t)$ **then**
  (a) **if** $X_t - (i, j)$ is extensional, **then** $X_{t+1} = X_t - (i, j)$,
    else
    (b) **if** $X_t - (i, j) + (j, i)$ is e.a., **then** $X_{t+1} = X_t - (i, j) + (j, i)$,
    (c) **else** $X_{t+1} = X_t$.
($\mathbf{T}_2^c$) **if** $(i, j) \notin E(X_t)$, **then**
  (a) **if** $X_t + (i, j)$ is e.a., **then** $X_{t+1} = X_t + (i, j)$,
  (b) **else** $X_{t+1} = X_t$.

Figure 2.4: A Markov chain algorithm for generating e.a. digraphs.

A Markov chain algorithm for generating weakly extensional acyclic digraphs, with a specified number of arcs

A Markov chain M^a for generating w.e.a. digraphs with vertex set $\{1, \ldots, n\}$ and m arcs is given in Figure 2.5. The probability of passing from a state $s \in \mathcal{W}_{n,m}$ to any other state $s' \neq s$ is either 0 or $1/n^4$, implying that M^c is symmetric. As previously, for any state $s \in \mathcal{W}_{n,m}$ there is a positive probability to remain in s. Our next theorem shows that M^a is indeed irreducible. If $m < n - 1$, the initial state of the Markov chain can be a digraph whose arcs form a directed path of length m. Otherwise, the initial state can be a directed path $(n, n-1, \ldots, 1)$ together with $m - (n - 1)$ arbitrary arcs of the form (i, j), where $i > j$.

Theorem 2.3.8 (Irreducibility of M^a) *The Markov chain M^a is irreducible.*

Proof. We show that any digraph $D \in \mathcal{W}_{n,m}$ can be transformed, by transitions of M^a, into a digraph $K(D) \in \mathcal{W}_{n,m}$, satisfying the following three properties:

i) for all $v \in V(D)$ such that $\mathsf{rank}(v) > 1$ in $K(D)$, it holds that $N^+(v) = R(v)$ in $K(D)$;

Let X_t denote the state of M^a at time t. Suppose two pairs of integers (i_1, j_j) and (i_2, j_2) have been drawn uniformly at random and independently from the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$.

if $(i_1, j_1) \in E(X_t)$ **and** $(i_2, j_2) \notin E(X_t)$, **then**
  if $X_t - (i_1, j_1) + (i_2, j_2)$ is w.e.a., **then** $X_{t+1} = X_t - (i_1, j_1) + (i_2, j_2)$,
  else $X_{t+1} = X_t$.

Figure 2.5: A Markov chain algorithm for generating w.e.a. digraphs on n vertices and m arcs.

ii) there is only one $v \in V(D)$ such that $\mathsf{rank}(v) = 1$ in $K(D)$;

iii) for all $u, v \in V(D)$ such that $\mathsf{rank}(u) \geqslant \mathsf{rank}(v)$ in D, we have $\mathsf{rank}(u) \geqslant \mathsf{rank}(v)$ in $K(D)$.

To show this, we argue as in the proof of Theorem 2.3.7, paying particular attention to preserving m arcs at each intermediary step. Proceed in decreasing order on rank (arbitrarily choosing one vertex among more of the same rank): pick a vertex $v \in V(D)$ with $\mathsf{rank}(v) > 1$ and consider all the elements $u \in R(v) \setminus N^+(v)$, in decreasing order on rank. Moreover, by the proof of Lemma 2.3.3, there exists an arc (t, s) between a vertex $t \in V(D)$ of rank 1 and a sink s whose removal does not interfere with the weak extensionality of D. Swap arcs (v, u) and (t, s) by the transition of M^a.

This is possible, since, on the one hand, the addition of an arc (v, u) does not create a cycle in the resulting digraph. On the other hand, as before, the subdigraph of D induced by the vertices $V(D) \setminus R(v)$ is an acyclic tournament. Therefore, one such arc addition can create a collision only between v and a vertex $u \in R(v)$. This is not the case, since after the first addition of such an arc, $\mathsf{rank}(v)$ becomes strictly greater than $\mathsf{rank}(u)$, for all $u \in R(v)$, and Lemma 2.3.6 guarantees the absence of collisions.

If at the end of this process more than one vertex of rank 1 exists, denote by v_* a vertex of D whose out-neighborhood is inclusion-maximal among the vertices of rank 1. Repeatedly remove one outgoing arc from a vertex of rank 1 whose out-neighborhood is inclusion-minimal, and add an arc between v_* and a sink $s \notin N^+(v_*)$. The digraph obtained at the end of this process, which we denote by $K(D)$, satisfies i)–iii). Observe also that in $K(D)$ there are no two distinct vertices having the same positive rank.

It remains to show that, given two digraphs D and H in $\mathcal{W}_{n,m}$, there exists a sequence of transitions in M^a from $K(D)$ to $K(H)$. To any acyclic digraph K whose vertices of positive rank have pairwise distinct ranks we can associate a partial order $\prec_K$ in the following way: for all $u, v \in V(K)$,

$$u \prec_K v \text{ iff } \mathsf{rank}(u) < \mathsf{rank}(v) \text{ in } K.$$

For expository purposes, assume that we also order the sinks of K is an arbitrary way so that $\prec_K$ is a linear order on the vertices of K. Therefore, we have to show that we can transform any order $x_0 \prec_{K(D)} x_1 \prec_{K(D)} \cdots \prec_{K(D)} x_{n-1}$ into $y_0 \prec_{K(H)} y_1 \prec_{K(H)} \cdots \prec_{K(H)} y_{n-1}$, where $\{x_i : 0 \leqslant i \leqslant n-1\} = \{y_i : 0 \leqslant i \leqslant n-1\} = \{1, \ldots, n\}$.

Like in the proof of Theorem 2.3.7, given a digraph $K(D)$ satisfying i)–iii), we show that we can obtain, by transitions of M^a, a digraph $K(D')$, still satisfying i)–iii), and in which a given pair of consecutive elements $x_i \prec_{K(D)} x_{i+1}$, $0 \leqslant i < n-1$, have swapped positions. If such consecutive elements x_i and x_{i+1} are both sinks, then since their order has been imposed arbitrarily, they can be swapped without changing the digraph. Otherwise, we have to consider two cases.

If $\mathsf{rank}(x_{i+1}) > 1$, then $N^+(x_{i+1}) = N^+(x_i) \cup \{x_i\}$. The arc (x_{i+1}, x_i) can be reversed, by the application of the transition of M^a on the arcs (x_{i+1}, x_i) and (x_i, x_{i+1}). Indeed, the resulting digraph K' remains acyclic; K' is also w.e. since, on the one hand, vertices $x_i, x_{i+1}, \ldots, x_{n-1}$ induce an acyclic tournament in K', by conditions i) and ii). On the other hand, any non-sink x_j, $0 \leqslant j < i$, is an out-neighbor of both x_i and x_{i+1}. Moreover, if $\mathsf{rank}(x_{i+1})$ was equal to 2 in $K(D)$ (and hence $\mathsf{rank}(x_i) = 1$), then in K' we may have $N^+(x_i) \neq R(x_i)$. However, it suffices to swap arcs out-going from x_{i+1}, the unique the vertex of rank 1, to x_i. The digraph obtained after these transformations satisfies conditions i)–iii), thus is equal to some $K(D')$; the vertices of $K(D')$ have the same ranks as in $K(D)$, with the exception of x_i and x_{i+1} which have swapped ranks.

When however $\mathsf{rank}(x_{i+1}) = 1$, we have that $\mathsf{rank}(x_i) = 0$. Since x_{i+1} is the unique vertex of rank 1, there must be an arc from x_{i+1} to a sink s which can be removed in order to add the arc (x_i, x_{i+1}). After this first arc swap, continue changing all arcs (x_{i+1}, s) into (x_i, s). The resulting digraph K' satisfies condition i)–iii) and is equal to some $K(D')$; moreover, the vertices of $K(D')$ have the same ranks as in $K(D)$, with the exception of x_i and x_{i+1} which, as before, have swapped ranks.

In order to transform $K(D)$ into $K(H)$, start from position $i = n-1$ downward to $i = 0$, and proceed as follows: if $y_i = x_j$ $(0 \leqslant j < i)$, then bring x_j to position i by iteratively reversing the arcs (x_{j+1}, x_j), (x_{j+2}, x_j), $\ldots$, (x_i, x_j). Finally, change the out-going arcs of the unique vertex of rank 1 so that it has precisely the same out-neighborhood as it has in $K(H)$. ■

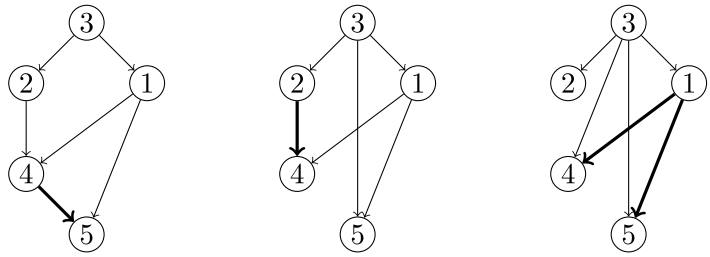
Figure 2.6 illustrates the transitions indicated by the above proof in order to pass between two digraphs in $\mathcal{W}_{5,6}$.

It is immediate to see that M^a can also generate uniformly at random acyclic digraphs on a given number of vertices and a given number of arcs: simply swap two arcs if the resulting digraph remains acyclic. The proof of Theorem 2.3.8 also shows its irreducibility.

Concluding remarks

One direction for further research on combinatorial enumeration of sets is, on the one hand, to enumerate classes of 'minimal' extensional acyclic digraphs, such as *slim* e.a. digraphs (as introduced in Section 1.3). On the other hand, a more challenging direction is to enumerate hyper-extensional digraphs.

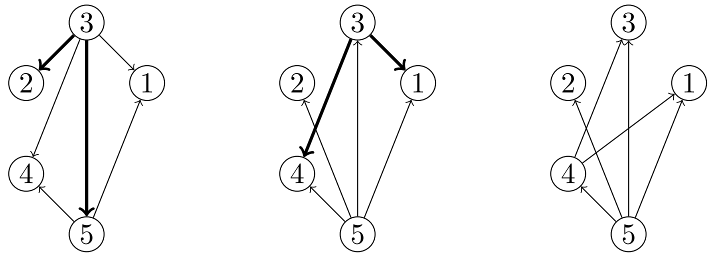
Much like Ackermann's bijection which it generalizes, the bijection we proposed in Section 2.2 is constitutes a 'natural' enumeration (to be precise, a 'natural' order) of hypersets. This embedding into numbers acts as an Occam's razor, by reducing multiplicity to simplicity: in the case of HF-sets, one can implement a full battery of set-handling methods by resorting to natural numbers as their internal representation [24]; likewise, one can implement $\overline{\mathsf{HF}}$-sets on top of rational numbers on the ground of the encoding technique proposed above. A key point in the construction of our bijection has been the notion of *rank*, but so far we cannot exclude that other rank notions might lead to encodings more satisfactory from a logico-mathematical perspective, and along the same lines.



(a) $(4,5) \leftrightarrow (3,5)$

(b) $(2,4) \leftrightarrow (3,4)$

(c) $2 \prec 4 \prec \mathbf{5} \prec \mathbf{1} \prec 3$
$(1,5) \leftrightarrow (5,1)$
$(1,4) \leftrightarrow (5,4)$



(d) $1 \prec 2 \prec 4 \prec \mathbf{5} \prec \mathbf{3}$
$(3,5) \leftrightarrow (5,3)$
$(3,2) \leftrightarrow (5,2)$

(e) $1 \prec 2 \prec \mathbf{4} \prec \mathbf{3} \prec 5$
$(3,4) \leftrightarrow (4,3)$
$(3,1) \leftrightarrow (4,1)$

(f) $1 \prec 2 \prec 3 \prec 4 \prec 5$

Figure 2.6: The sequence of transitions of M^a that transforms $D \in \mathcal{W}_{5,6}$ (Fig. (a)) into $K(D) \in \mathcal{W}_{5,6}$ (Fig. (c)), and then into a digraph $K(H) \in \mathcal{W}_{5,6}$ (Fig. (f))

Another application of Ackermann's bijection is in algorithmics. The linear-time algorithm of Dovier, Piazza and Policriti [47] for computing the maximum bisimulation on an acyclic digraph is deeply rooted in Ackermann's order on well-founded sets. In the case of our encoding of hypersets, we employed instead an algorithmic concept to define an order. It is of interest whether new insight on the open problem of the existence of a linear-time algorithm for the maximum bisimulation problem can be gained. Descriptive complexity issues, a main focus in the study of linear orderings of the universe of hypersets carried out in [79], have not been taken into account here and will be the subject of future work.

Although the Markov chains M, M^c, and M^e are similar to the Markov chains of [82, 83], the proofs of their irreducibility are different and more involved. In the case of M, the fixed element which can be reached by a chain of transitions from every element G of W_n is the same as in [83], namely the totally disconnected digraph. However, the arcs of G must be removed in a particular order, according to the weak extensionality of G. Second, in [82] the fixed element is an arbitrary digraph having a path as underlying graph, which surely cannot be the case for M^c or M^e, as (weak) extensionality would be violated. On the other hand, our proof takes this fixed element to be an acyclic tournament on n vertices (i.e., a digraph isomorphic to the von Neumann numeral of n), ensuring that the proof proposed here can be used to show the irreducibility of (a slightly modified version of) the chain of [82] for the generation of weakly connected acyclic w.e.a. digraphs. Lastly, as noted above, the Markov chain M^a can be easily adapted to generate uniformly at random acyclic digraphs on a given number of labeled vertices and a given number of arcs, a result which we have not found in the literature.

Given this dual usability of the Markov chains considered here, and the fact that the von Neumann numeral of n is a rich structure in which many types of digraphs can be embedded, it would be interesting to characterize in general the class of digraphs whose elements can be generated uniformly at random by these Markov chains.

We regard the generation of hyper-extensional digraphs on n vertices, as the next natural step to take. We conjecture that a similar Markov chain algorithm, having three basic operations, addition of an arc, removal of an arc, move of an arc, can be shown to be irreducible. To be more precise, that any such digraph on n vertices can be transformed by this Markov chain into a digraph isomorphic to the von Neumann numeral of n.

3

Infinite Enumeration of Sets and Well-Quasi-Orders

When enumerating finite discrete objects whose size is no longer bounded by a fixed value, the sequences under consideration become infinite. This radically changes the setting, and raises a multitude of new and interesting questions. Among these, is the existence of *well-quasi-orders* on such objects. A well-quasi-order is a pair $(Q, \preccurlyeq)$ where Q is a set and $\preccurlyeq$ is a transitive and reflexive binary relation on Q and so that for every infinite enumeration $(q_i)_{i=1,2,\ldots}$ of elements of Q, there exist $1 \leqslant i < j$ such that $q_i \preccurlyeq q_j$.

In a graph-theoretic setting, one usually looks for graph immersions that are also well-quasi-orders, so that no 'new' structures can be built up *ad infinitum*. The celebrated theorem of Robertson and Seymour, considered by some to be an indication that graph theory has reached its full bloom as a discipline of mathematics, states that the minor relation between graphs is a well-quasi-order on the class of all graphs.

Not much is known in the case of digraphs. Recently, *strong immersion* between digraphs was proved to be a well-quasi-order on the set of all tournaments, that is, orientations of complete graphs. Complying to our view that sets are digraphs, we focus now on isolating other classes of digraphs where strong immersion becomes a well-quasi-order, with such set-theoretic assistance. The main result of this chapter is to introduce the notion of *slimness*, which together with a bound on the number of sources of a digraph, ensures that strong immersion is a well-quasi-order for slim sets, slim hypersets and, even more generally, for slim digraphs having the property that from every vertex there is a directed path to a sink. This is best possible, in the sense that neither one of these two conditions can be dropped without losing the well-quasi-order property.

3.1 Well-quasi-orders and digraph immersion

A *quasi-order* is a pair $(Q, \preccurlyeq)$ where Q is a set and $\preccurlyeq$ is a transitive and reflexive binary relation on Q. We say that a quasi-order $(Q, \preccurlyeq)$ is a *well*-quasi-order, or *wqo*, if for every infinite sequence $(q_i)_{i=1,2,\ldots}$ of elements of Q, there exist $1 \leqslant i < j$ such that $q_i \preccurlyeq q_j$.

One of the earliest results on well-quasi-orderings of graphs belongs to Kruskal [75], who showed that the class of all finite trees is well-quasi-ordered by the topological minor relation. This study culminated with the celebrated theorem of Robertson and Seymour [128] stating that the minor relation is a well-quasi-order on the class of all finite graphs. Later [129], they showed that this is also the case for weak immersion between graphs, which was a conjecture of Nash-Williams [94]. In the case of *digraphs* not much is known. Immersion between *eulerian* digraphs was studied by Johnson (cf. [9, p. 517], [36]). Recently, Chudnovsky and Seymour [36] proved that *strong immersion* between digraphs is a well-quasi-order on the set of all tournaments, that is, orientations of complete graphs.

In view of this result, we will focus in this chapter on *weak* and *strong immersion* between digraphs, as defined in [36]. A *weak immersion* of a digraph H into G is a map η such that:

- for every $v \in V(H)$, $\eta(v) \in V(G)$;
- for every $u, v \in V(H)$ with $u \neq v$, it holds that $\eta(u) \neq \eta(v)$;
- for each arc $uv \in E(H)$, $\eta(uv)$ is a directed path in G from $\eta(u)$ to $\eta(v)$ (recall that we consider only paths without repeated vertices);
- if $e, f \in E(H)$ are distinct, then $\eta(e)$ and $\eta(f)$ have no arcs in common, although they may share vertices.

The map η is called a *strong immersion* when it also holds that if $v \in V(H)$, $e \in E(H)$, and e is not incident with v in H, then $\eta(v)$ is not a vertex on the directed path $\eta(e)$. We say that a digraph H is *weakly (strongly) immersed* into a digraph G, and write $H \preccurlyeq_i^w G$ ($H \preccurlyeq_i^s G$), if there exists a weak (strong) immersion of H into G.

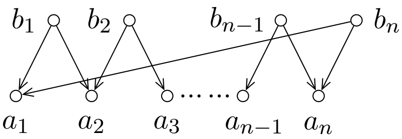


Figure 3.1: Digraphs D_n, $n \geqslant 2$

As already observed in [36], weak immersion is not a wqo on the set of all digraphs. Just consider the acyclic digraphs D_n formed by orienting the edges of a cycle of length $2n$ alternately clockwise and counterclockwise (see Figure 3.1). That being so, the collection $\{D_n \mid n \geqslant 2\}$ has the property that none of its elements can be weakly immersed into another one.

Under the auspices of our thesis that sets are digraphs, and also motivated by the observation that, requiring acyclicity, tournaments are forced to be isomorphic to the

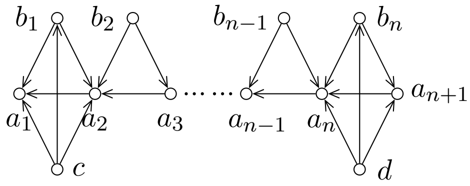


Figure 3.2: Extensional acyclic digraphs H_n, $n \geqslant 3$

membership digraphs of von Neumann's numerals, we can ask whether membership digraphs of hereditarily finite sets are well-quasi-ordered by strong immersion. The answer is *no*, as testified by the sequence of digraphs $(H_n)_{n\geqslant 3}$, depicted in Figure 3.2.

Indeed, given H_n and H_m, with $3 \leqslant n < m$, and supposing that η is a weak immersion of H_n into H_m, observe that $\eta(d) \in \{c, d\}$, since only c or d have three out-neighbors in H_m. Each of the out-neighbors of d in H_n has at least one out-neighbor, which implies that $\eta(d) = d$, as one of the three out-neighbors of c in H_m is a sink. Consequently, $\eta(c) = c$, $\eta(a_1) = a_1$, $\eta(a_2) = a_2$, $\eta(b_1) = b_1$, $\eta(a_n) = a_m$, $\eta(a_{n+1}) = a_{m+1}$, $\eta(b_n) = b_m$. Next, $\eta(b_2) = b_2$, since otherwise the directed paths $\eta(b_2a_2)$ and $\eta(a_3a_2)$ would have to share the arc a_3a_2, against the fact that η is an immersion. This implies that $\eta(a_3) = a_3$. By a similar argument, inductively, $\eta(b_i) = b_i$ and $\eta(a_{i+1}) = a_{i+1}$ hold for all $2 \leqslant i < n-1$. Moreover, $\eta(a_{n-1}) = a_{n-1}$, and for all $1 \leqslant i < n-1$, the image of the arc b_ia_i is the 2-vertex directed path (b_i, a_i), while the image of $a_{i+1}a_i$ is the directed path (a_{i+1}, a_i). At this point, the arc a_na_{n-1} must be mapped to the directed path $(a_m, a_{m-1}, \ldots, a_{n-1})$. This leaves no possibility to map b_{n-1}.

In the next two sections we will show that strong immersion becomes a wqo on membership digraphs of slim hereditarily finite sets of bounded cardinality. A *slim* set is one in which every membership relation is necessary (cf. Section 1.3). This is the best possible result, in the sense that neither slimness nor bounded cardinality can be further dropped without losing the wqo property.

Our proofs are given in a general context, in which slimness is translated as a graph-theoretic property and in which acyclicity and extensionality are no longer assumed. Nevertheless, the set-theoretic basis for our results remain, as from acyclicity and extensionality, we distill the only property that from every vertex there is a directed path to one of the sinks of the digraph (we call such a digraph *channeled*). This result also implies, as a particular case, the fact that strong immersion is a wqo on the class of membership digraphs corresponding to slim hereditarily finite hypersets. The following two sections are thus devoted to proving the following result.

Theorem 3.1.1 *For every $s \geqslant 1$, the collection $\mathcal{D}_s$ of channeled slim finite digraphs with at most s sources is well-quasi-ordered by strong immersion.*

Wqo's proved to be a key ingredient in generalizing and unifying many results concerning the decidability of verification problems (e.g. coverability) on infinite-state transition systems (cf. [1, 53] and the references therein). To be more precise, a transition system is said to be *well-structured* when its transition relation is monotonic w.r.t. a wqo of its states; the classical example is that of Petri nets: the states of the transition system is the set of all configurations of the net, while the wqo is the inclusion between their markings.

In this light, our contribution can also be viewed as laying the set-theoretic groundwork for a class of well-structured transition systems having as states the hereditarily finite (hyper)sets considered here.

3.2 Slim digraphs and their structure

In accordance with Section 1.3, we say that a well-founded set x is *slim* if the digraph obtained by removing any arc from the membership digraph $D(x)$ of x is not extensional. This is equivalent to saying that x is slim if for any vertex y of $D(x)$ and for any out-neighbor of it, z, there exists a vertex y' of $D(x)$ whose set of out-neighbors is precisely $N^+(y) \setminus \{z\}$. In set-theoretic terms, a set x is slim if $\forall y \in \mathsf{TrCl}(x)$ and $\forall z \in y$, it holds that $y \setminus \{z\} \in \mathsf{TrCl}(x)$. Observe that the transitive closure of a slim set x is closed under taking subsets for its elements, in the sense that for any $y \in \mathsf{TrCl}(x)$, $\mathcal{P}(y) \subset \mathsf{TrCl}(x)$.

The following notion of slimness for hypersets avoids the introduction of a bisimilarity check in order to establish if a hyperset is slim.

Definition 3.2.1 *A hyperset x is* slim *if the digraph obtained by removing any arc from $D(x)$ is not extensional.*

We will work in the more general setting of digraphs with no assumptions of (hyper-)extensionality, or of acyclicity. For this, we will employ the following definition, generalizing Definitions 1.3.6 and 3.2.1.

Definition 3.2.2 *A digraph D is* slim *if for any $v \in V(D)$ and for any $u \in N^+(v)$, there exists $v' \in V(D)$ so that $N^+(v') = N^+(v) \setminus \{u\}$.*

Clearly, in any acyclic digraph there is a directed path from every vertex to a sink. As Lemma 1.4.6 shows, when no longer assuming acyclicity, this property is essentially guaranteed by the lack of a non-trivial bisimulation. This shows that we can generalize the notion of rank from well-founded sets to hypersets by taking into account their structural complexity with respect to their sink. Although there is no such standard generalization, we propose below a most natural one, useful for the purposes of our proof.

Definition 3.2.3 *Given a channeled digraph D, the* rank *of a vertex v of D is the length of the longest directed path leading from v to a sink of D.*

Given a channeled digraph D, we denote by $\mathsf{rank}(D)$ the maximum rank of its vertices. For any $0 \leqslant r \leqslant \mathsf{rank}(D)$, we let $D^{=r}$ stand for the set of vertices of D of rank r. Similarly, $D^{\geqslant r}$ stands for the set of vertices of D of rank at least r.

We now argue that dropping slimness or the bound on the number of sources results in a collection of digraphs no longer well-quasi-ordered by strong immersion (in fact, not even by weak immersion). For this means, we can adjust the digraphs H_n of Figure 3.2.

On the one hand, in order to render H_n also slim, it suffices to add seven vertices c_1, c_2, c_3 with out-neighborhoods $\{a_1, b_1\}$, $\{a_2, b_1\}$, $\{b_1\}$, respectively, and vertices d_1, d_2, d_3, d_4 with out-neighborhoods $\{a_n, b_n\}$, $\{a_{n+1}, b_n\}$, $\{b_n\}$, $\{a_{n+1}\}$, respectively. This addition does not disrupt the fact that H_n cannot be weakly immersed in H_m, for any $3 \leqslant n < m$.

On the other hand, in order to make the collection of digraphs H_n of bounded cardinality, it suffices to add to H_n a new source having as out-neighbors all its pre-existing sources.

However, digraphs H_n still have a further peculiarity of unboundedness: the difference among the ranks of their *sources* is unbounded. Even if this is avoided, weak immersion cannot become a wqo. To see this, it suffices to add, for each source of H_n, a set of new vertices forming a directed path which ends in that source. The lengths of these paths can be taken such that the sources of the resulting digraph H'_n all have the same rank. Moreover, if H_n is rendered slim, then H'_n will remain so. The argument given in the previous section still applies for showing the lack of a weak immersion between H'_n and H'_m, for any $3 \leqslant n < m$.

We now prove some lemmas characterizing the structure of slim digraphs.

Lemma 3.2.4 *If D is a slim digraph, then for any $v \in V(D) \setminus O(D)$, there exists $w \in V(D)$ so that $N^+(w) = \{v\}$.*

Proof. If D is slim and v belongs to $V(D) \setminus O(D)$, then take $w \in V(D)$ such that $v \in N^+(w)$ and $|N^+(w)|$ is minimum among the vertices with this property (such a vertex exists since v is not a source of D). If there exists $z \in N^+(w) \setminus \{v\}$ then from the slimness of D we can find $w' \in V(D)$ such that $N^+(w') = N^+(w) \setminus \{z\}$, and hence $v \in N^+(w')$. This contradicts the minimality of w. ■

Lemma 3.2.5 *If D is a slim digraph, then $|\{v \in V(D) \mid |N^+(v)| \geqslant 2\}| \leqslant |O(D)|$.*

Proof. Denoting by A the set $\{v \in V(D) \mid |N^+(v)| \geqslant 2\}$, we construct a map f from A to $\mathcal{P}(V(D))$ so that for any $v \in A$, the following hold:

i) $v \in f(v)$,

ii) $f(v) \cap O(D) \neq \emptyset$,

iii) for any $u \in A$, if $u \neq v$, then $f(u) \cap f(v) = \emptyset$.

The existence of such a map proves the claim. To see that this is indeed the case, for a vertex $v \in A \cap O(D)$ we put $f(v) = \{v\}$. Otherwise, from Lemma 3.2.4 we can obtain a $w_1 \in V(D)$ such that $N^+(w_1) = \{v\}$. If w_1 is a source, then we put $f(v) = \{v, w_1\}$. Otherwise, apply again Lemma 3.2.4, this time to w_1, to obtain $w_2 \in V(D)$ such that $N^+(w_2) = \{w_1\}$. Since D is finite, we can repeat this procedure until obtaining $w_2, \ldots, w_k \in O(D)$, $k \geqslant 2$, such that $N^+(w_k) = \{w_{k-1}\}, \ldots, N^+(w_2) = \{w_1\}, N^+(w_1) = \{v\}$. In this case, we put $f(v) = \{v, w_1, \ldots, w_k\}$. The map f constructed in this way clearly satisfies i) and ii). Finally, since for any $v \in A$ and any $w \in f(v) \setminus \{v\}$, $|N^+(w)| = 1$, iii) is satisfied as well. ■

Corollary 3.2.6 *If D is a slim digraph, then $O(D) \neq \emptyset$.*

Proof. Apply the argument given in the proof of Lemma 3.2.5 to any vertex $v \in V(D)$, and obtain a directed path $(w_k, \ldots, w_1, v)$ of D so that w_k is a source of D. ■

The following more technical lemma is an analogue of Lemma 3.2.5, expressed in terms of in-neighbors.

Lemma 3.2.7 *If D is a slim digraph, then* $|\{v \in V(D) \mid |N^-(v)| \geqslant 2 \wedge \forall w \in N^-(v), N^+(w) = \{v\}\}| \leqslant |O(D)|$.

Proof. The proof will proceed along the lines of the proof of Lemma 3.2.5. Denoting by A the set $\{v \in V(D) \mid |N^-(v)| \geqslant 2 \wedge \forall w \in N^-(v), N^+(w) = \{v\}\}$, we construct a map f from A to $\mathcal{P}(V(D))$ so that for any $v \in A$, the vertices of $f(v)$ form a directed path in D from a source of D to an in-neighbor of v. Moreover, for any $v \in A$ the following properties will hold:

i) $f(v) \cap O(D) \neq \emptyset$,

ii) for any $u \in A$, if $u \neq v$, then $f(u) \cap f(v) = \emptyset$,

iii) for any $w \in f(v)$, $|N^+(w)| = 1$.

As before, the existence of such a map proves the claim. Construct f iteratively, at each step complying with the above three properties. For a $v \in A$, we let w_1 be a vertex of $N^-(v)$ obtained as follows. If $v \in A$ does not belong to the image through f of some vertex $u \in A$, then take w_1 to be an arbitrary vertex of $N^-(v)$. Otherwise, this vertex $u \in A$ so that $v \in f(u)$ is unique, since the map f constructed so far satisfies ii). Moreover, precisely one in-neighbor of v also belongs to $f(u)$; hence there is at least one in-neighbor of v belonging to no image through f of a vertex of A. In this case, we take w_1 to be this vertex.

If w_1 is a source, then we put $f(v) = \{w_1\}$. Otherwise, we obtain a vertex w_2 such that $N^+(w_2) = \{w_1\}$, by considering three cases. If $w_1 \notin A$, then we take w_2 to be an arbitray vertex obtained by Lemma 3.2.4. Otherwise, if $w_1 \in A$, but $f(w_1)$ has not yet been set, take w_2 to be an arbitray vertex of $N^-(w_1)$. Finally, if $w_1 \in A$ and $f(w_1)$ has already been set, observe, as before, that precisely one of its in-neighbors belongs to $f(w_1)$. There is hence an available in-neighbor of w_1, which we take to be w_2.

Since D is finite, we can repeat this procedure until obtaining $w_2, \ldots, w_k \in O(D)$, $k \geqslant 2$, such that $N^+(w_k) = \{w_{k-1}\}, \ldots, N^+(w_2) = \{w_1\}, N^+(w_1) = \{w\}$. In this case, we put $f(v) = \{w_1, \ldots, w_k\}$.

We conclude the proof by observing that for any $v \in A$ and any $w \in f(v)$, $|N^+(w)| = 1$, and hence the new map f constructed in this way satisfies i), ii) and iii). ■

Lemma 3.2.8 *If D is a slim digraph, then for any $v \in V(D)$ it holds that* $|N^+(v)| \leqslant |O(D)| + 1$.

Proof. Suppose, for a contradiction, that $v \in V(D)$ is such that $|N^+(v)| \geqslant |O(D)| + 2$. Since D is slim, then the set $A = \{w \in V(D) \mid \text{there exists } u \in N^+(v) \text{ so that } N^+(w) = N^+(v) \setminus \{u\}\}$ has cardinality at least $|N^+(v)|$. Observe also that for any $w \in A$, $|N^+(w)| \geqslant 2$ holds, since $|O(D)| \geqslant 1$ (Corollary 3.2.6). The set A just considered contradicts Lemma 3.2.5. ■

The following two lemmas characterize the structure of slim sets in terms of the rank of its vertices.

Lemma 3.2.9 *Let D be a channeled digraph. For any $v, w \in V(D)$ such that $N^+(w) = \{v\}$, it holds that* $\mathsf{rank}(w) = \mathsf{rank}(v) + 1$.

Proof. Let P be the longest directed path from v to a sink, of length $\mathsf{rank}(v)$. If $N^+(w) = \{v\}$, then also w followed by the vertices of P is a directed path from w to a sink. Moreover, it is of maximum length, since P is of maximum length. ■

Lemma 3.2.10 *If D is a channeled slim digraph, then the following hold:*

i) $|D^{=r} \setminus O(D)| \leqslant |D^{=r+1}|$, *for any* $0 \leqslant r < \mathsf{rank}(D)$;

ii) $|D^{=r}| \leqslant |D^{\geqslant r} \cap O(D)|$, *for any* $0 \leqslant r \leqslant \mathsf{rank}(D)$.

Proof. To see that i) holds, observe that for any $v \in D^{=r} \setminus O(D)$, by Lemma 3.2.4, there exists $w \in V(D)$ such that $N^+(w) = \{v\}$. From Lemma 3.2.9, w is of rank $r+1$, which proves i).

For ii), note that any vertex of maximum rank of D must be a source, as just argued, by Lemmas 3.2.4 and 3.2.9. This implies that ii) holds for $r = \mathsf{rank}(D)$. Let now $r < \mathsf{rank}(D)$ be the greatest rank for which $|D^{=r}| > |D^{\geqslant r} \cap O(D)|$ and hence that $|D^{=r+1}| \leqslant |D^{\geqslant r+1} \cap O(D)|$. Point i) entails that

$$|D^{=r} \setminus O(D)| \leqslant |D^{=r+1}| \leqslant |O(D) \cap D^{\geqslant r+1}|.$$

This brings the desired contradiction, since

$$|D^{=r}| = |D^{=r} \setminus O(D)| + |D^{=r} \cap O(D)| \leqslant$$
$$|D^{\geqslant r+1} \cap O(D)| + |D^{=r} \cap O(D)| = |D^{\geqslant r} \cap O(D)|.$$

■

3.3 Encoding slim digraphs

Let us begin by considering the collection $\mathcal{D}_s$:

$$\mathcal{D}_s = \{D \mid D \text{ is a channeled slim finite digraph} \wedge |O(D)| \leqslant s\}.$$

The vertices of a channeled digraph can be viewed as disposed on layers, according to their ranks. Point ii) of Lemma 3.2.10 implies that at any rank of $D \in \mathcal{D}_s$ there are at most s vertices. Point i) implies that the number of vertices at any given rank r of D is non-increasing (for decreasing r), save for at most s times, when at most s sources appear in D.

Additionally, we can classify the vertices of D according to whether they are sinks, they have exactly one out-neighbor, or more than one. Lemma 3.2.5 implies that this latter, more complicated case can happen only a bounded number of times in D, so we can include these vertices in the encoding we will give for the digraphs of $\mathcal{D}_s$.

Indeed, let us say that a rank r is a *juncture rank* of a digraph $D \in \mathcal{D}_s$ if one of the following holds:

- there exists $v \in D^{=r}$ with $N^+(v) \geqslant 2$;
- there exists $v \in D^{=r-1}$ with $N^-(v) \geqslant 2$ and for any $w \in N^-(v)$, $N^+(w) = \{v\}$;
- $O(D) \cap D^{=r} \neq \emptyset$;

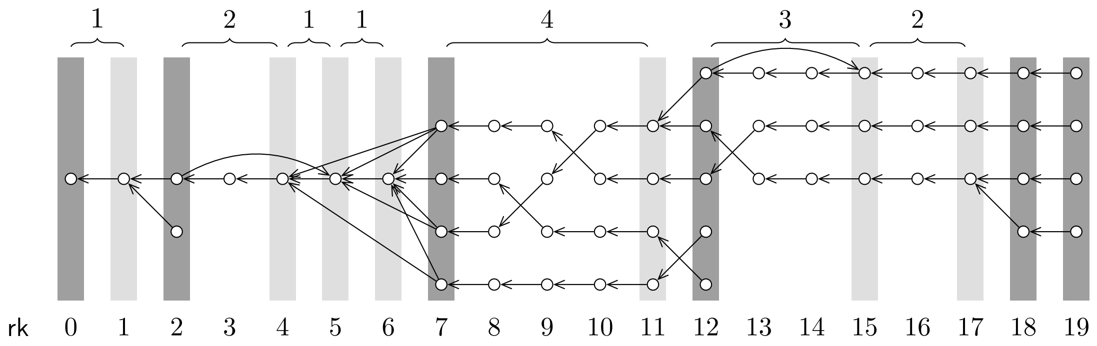


Figure 3.3: A channeled slim digraph D of rank 19 and $|O(D)| = 7$. The juncture ranks of D are marked with dark gray, and its trace ranks are in light gray. Its encoding $\hat{D}$ has rank 13, while $\delta(D) = 1\,2\,1\,1\,4\,3\,2$.

- $r = 0$.

The second bullet above capures those situations in which a vertex v has more than one in-neighbor, all of which have only v as out-neighbor (hence, in particular, they are not captured by the first bullet).

As we are about to see, encoding the vertices at juncture ranks, together with their out-neighbors, is the main ingredient for showing the wqo property. For this, let us also say that a rank r' is a *trace rank* of D if r' is not a juncture rank of D and there exists a juncture rank r of D and $v \in D^{=r}$ such that $N^+(v) \cap D^{=r'} \neq \emptyset$.

Given a digraph $D \in \mathcal{D}_s$, we will employ two encodings for it, $\hat{D}$ and $\delta(D)$. The first is a channeled slim digraph which captures the involved configurations among juncture and trace ranks, while the second is a sequence of positive integers capturing the lengths of the directed paths issuing from vertices at trace ranks.

Entering into details, obtain $\hat{D}$ by applying to D the following graph-theoretic operations. If r is a trace rank of D, then let r' be the greatest rank of D such that $r' < r$ and such that r' is a juncture or a trace rank. Any vertex of rank t of D, $r' < t \leqslant r$, has precisely one out-neighbor, which is a vertex of rank $t-1$ (by Lemma 3.2.9). Moreover, any vertex of rank t of D, $r' \leqslant t < r$, has precisely one in-neighbor, which is a vertex of rank $t+1$. In $\hat{D}$, contract all directed paths $P = (v_1, \ldots, v_k)$ leading from v_1, a vertex of rank r, to v_k, a vertex of rank r', by removing vertices $v_2, \ldots, v_{k-1}$ from D and adding the arc $v_1 \to v_k$. Moreover, if rank r was the ith trace rank of D (starting from the lowest rank), then set the ith character of $\delta(D)$ to be the length of P, that is, k. An example is given in Figure 3.3.

Observe that the vertices at juncture/trace ranks in D correspond precisely to the vertices at juncture/trace ranks in $\hat{D}$.

The proof that strong immersion is a wqo on $\mathcal{D}_s$ proceeds as follows. First, we argue that for any digraph $D \in \mathcal{D}_s$, the number of vertices of $\hat{D}$ is bounded. Therefore, given an infinite sequence $(D_i)_{i=1,2,\ldots}$ of such digraphs, we can extract an infinite subsequence of digraphs having the same encoding, say $\hat{D}$.

This implies that digraphs $(D_{i_j})_{j=1,2,\ldots}$ have essentially the same configurations among juncture and trace ranks. Moreover, the directed paths contracted for obtaining the encoding $\hat{D}$ start at the same trace ranks in any D_{i_j}. This is true since, as observed before, the vertices at trace ranks in any D_{i_j} correspond precisely to the vertices at trace ranks in $\hat{D}$.

Finally, since these directed paths can have arbitrary lengths, we have to use a further basic fact from the theory of wqo's (see e.g. [66]). If $(Q, \preccurlyeq)$ is a wqo, then so is the set of fixed-length sequences over Q, componentwise ordered by $\preccurlyeq$. That is, the pair $(Q^\ell, \preccurlyeq^\ell)$ is a wqo, where for any $(x_1, \ldots, x_\ell), (y_1, \ldots, y_\ell) \in Q^\ell$ we have $(x_1, \ldots, x_\ell) \preccurlyeq^\ell (y_1, \ldots, y_\ell) \Leftrightarrow_{\text{def}} x_i \preccurlyeq y_i$, for all $1 \leqslant i \leqslant \ell$. Since $(\mathbb{N}, \leqslant)$ is a wqo, this implies that we can find D_{i_j} and D_{i_k} in the aforementioned infinite sequence such that taking $\ell = |\delta(D_{i_1})|$ we have $\delta(D_{i_j}) \leqslant^\ell \delta(D_{i_k})$.

Lemma 3.3.1 *For any $s \geqslant 1$ and $D \in \mathcal{D}_s$, $|V(\hat{D})| \leqslant s(s+1)(3s+1) + s(3s+1)$ holds.*

Proof. Observe that the vertices of $\hat{D}$ are precisely the vertices of D at a juncture or at a trace rank of D. By Lemma 3.2.5, the number of vertices of D with two or more out-neighbors is at most s. By Lemma 3.2.7, the same holds for those vertices v of D with at least two in-neighbors, all of which have precisely v as out-neighbor. Taking into account the sources of D and its juncture rank 0, D has at most $3s + 1$ juncture ranks. By Lemma 3.2.10, at every rank of D there are at most s vertices. Therefore, by Lemma 3.2.8, the total number of out-neighbors of the vertices at juncture ranks of D is at most $s(s + 1)(3s + 1)$. To sum up, $|V(\hat{D})| \leqslant s(s+1)(3s+1) + s(3s+1)$. ∎

We can now assemble all results into a proof of our main theorem.

Theorem 3.3.2 *For every $s \geqslant 1$, the collection $\mathcal{D}_s$ of channeled slim finite digraphs with at most s sources is well-quasi-ordered by strong immersion.*

Proof. Let $(D_i)_{i=1,2,\ldots}$ be an infinite sequence of digraphs belonging to $\mathcal{D}_s$. By Lemma 3.3.1, as just argued, there exists an infinite subsequence of it, $(D_{i_j})_{j=1,2,\ldots}$, such that $\hat{D}_{i_1} = \hat{D}_{i_2} = \cdots$. This implies that their δ-encodings all have the same length, say ℓ. Therefore, there exist $1 \leqslant j < k$ such that for D_{i_j} and D_{i_k} we have $\delta(D_{i_j}) \leqslant^\ell \delta(D_{i_k})$.

Construct the strong immersion η of D_{i_j} into D_{i_k} by mapping those vertices of D_{i_j} also present in $\hat{D}_{i_j}$ to those vertices of D_{i_k} also present in $\hat{D}_{i_k}$. Moreover, map the arcs of D_{i_j} between two vertices at a juncture rank, or between a vertex at a juncture rank and a vertex at a trace rank, to the arcs between the corresponding vertices of D_{i_k}.

At this point, the only vertices and arcs of D_{i_j} whose images through η is not yet set belong to directed paths issuing from vertices at trace ranks of D_{i_j}. Finally, given that $\hat{D}_{i_j} = \hat{D}_{i_k}$ and $\delta(D_{i_j}) \leqslant^\ell \delta(D_{i_k})$, we can map these directed paths of D_{i_j} to the corresponding directed paths, of greater lengths, of D_{i_k}. ∎

Note that the proof of Theorem 3.3.2 shows a stronger fact: the map η is not only a strong immersion of D_{i_j} into D_{i_k}, but one such that for any $v \in V(D_{i_j})$, $|N^+(v)| = |N^+(\eta(v))|$ and $|N^-(v)| = |N^-(\eta(v))|$ hold.

Due to Lemma 1.4.5 and to the observations made at the beginning of Section 3.2, the above theorem has the following two corollaries.

Corollary 3.3.3 *The set $\{D(x) \,|\, x$ is a slim hereditarily finite well-founded set $\wedge |x| \leqslant s\}$ is well-quasi-ordered by strong immersion.*

Corollary 3.3.4 *The set $\{D(x)\,|\,x$ is a slim hereditarily finite hyperset $\wedge |x| \leqslant s\}$ is well-quasi-ordered by strong immersion.*

Proof. By Lemma 1.4.6, the membership digraph D of a hereditarily finite hyperset is *almost* channeled, in the sense that at most one vertex $v \in V(D)$ exists from which there is no directed path to the sink. Moreover, $N^+(v) = \{v\}$. We can hence include this vertex v in the encoding $\hat{D}$ of D (for example, by assigning to v the juncture rank 0). Analogously to Lemma 3.3.1, this new encoding still has a number of vertices bounded by a function depending only on s, and hence the proof given above still holds for obtaining the wqo property. ■

Concluding remarks

This result is one of the first steps in studying containment relations between digraphs and well-quasi-orders for various classes of sets. On the one hand, even though we have shown that neither one of the two conditions introduced here can be dropped, various other classes of sets can be considered. For example, a starting point could be dependent or irredundant extensional acyclic digraphs. Somewhat more challenging, the question can be asked for hypersets in which every arc is necessary for having *hyper*-extensionality. On the other hand, other types of digraph immersions can be considered, starting, for example, from an adequate generalization to digraphs of the notion of minor for undirected graphs [128].

On the graph-theoretic side, it would be interesting to investigate if the wqo property holds also when dropping the requirement that the digraphs are channeled (generalizing in a way Corollary 3.3.4). We believe that this is indeed the case, since Lemma 3.2.5 guarantees that the strongly connected components whose vertices have no directed paths towards sinks are nearly directed cycles, hence easy to describe/encode.

Another graph-theoretic question is to what degree the class of digraphs considered in this paper relates to the result of [36] on tournaments: note that tournaments have at most one source.

4

Set Graphs – The Structure Underlying Sets

While hereditarily finite sets capture, without redundancy, the whole *semantic* richness that a well-founded membership can knit, what can be said about the *topological* diversity of their underlying (undirected) graphs? To tackle this question, we put forward the notion of a *set graph*, that is, a graph admitting an extensional acyclic orientation.

Even though hereditarily finite sets can be easily characterized and can be recognized in linear time, the problem of recognizing set graphs is NP-complete. This is also the case for the analogous problem of finding a hyper-extensional orientation of a graph. This complexity result shows that set graphs form a rich class of graphs, but that it is unlikely that a good characterization of them exists. Instead, one can look for a largest hereditary (i.e., closed under taking induced subgraphs) class of graphs such that every connected member of it is a set graph. It turns out that this class is obtained by forbidding the *claw*, $K_{1,3}$. Surprisingly, the *local* condition of being claw-free ensures the *global* property of admitting an extensional acyclic orientation.

We begin by giving necessary or sufficient conditions for being a set graph, by providing some graph-theoretic operations which preserve set graphs and by giving a precise characterization of unicyclic set graphs. As expected, counting the extensional acyclic orientations of a graph is also difficult, in the sense that this problem belongs to the complexity class #P-complete. However, the recognition problem becomes solvable in linear time when the input graphs are restricted to have bounded tree-width. Finally, we study the interplay between 'claw-free' conditions and set graphs. In particular, we identify the maximal hereditary class where being claw-free is equivalent to being a set graph.

4.1 Basic properties

4.1.1 Necessary or sufficient conditions

We say that a graph is a *set graph* is it admits an extensional and acyclic orientation. Whenever in a digraph D we have $N^+(x) = N^+(y)$ for distinct vertices x and y, we say that x and y *collide*. Note that this is not the case if D is acyclic and there is a directed path from x to y.

Every set graph must be connected, since in every acyclic orientation of a disconnected graph G there are at least two sinks (which therefore collide). This necessary condition can be further strengthened as follows:

Proposition 4.1.1 *If G is a set graph, then for every $X \subseteq V(G)$, $G - X$ has at most $2^{|X|}$ connected components.*

Proof. Arguing by contradiction, let D be an e.a.o. of G and suppose there is an $X \subseteq V(G)$ such that $G - X$ has $Y_1, \ldots, Y_t$ as connected components, where $t > 2^{|X|}$. Since D is acyclic, there exist $y_i \in Y_i$, such that y_i is a sink in $D[Y_i]$, for all $1 \leqslant i \leqslant t$. Since D is extensional, we have that all $N^+(y_i) \subseteq X$ are different, for all $1 \leqslant i \leqslant t$, which contradicts the fact that $t > 2^{|X|}$. ■

Proposition 4.1.1 shows that the claw is not a set graph: removing the vertex of degree 3 leaves a graph with three connected components.

A slightly more general result holds:

Proposition 4.1.2 *Let G be a set graph and let $X \subseteq V(G)$. Denote by $Y_1, \ldots, Y_t$ the vertex sets of the connected components of the graph $G - X$. Then,*

$$t \leqslant \max \left| \bigcup_{i=1}^{t} \mathcal{P}\left(N(v_i) \cap X\right) \right| ,$$

where the maximum is taken over all t-tuples $(v_1, \ldots, v_t) \in Y_1 \times \cdots \times Y_t$.

Proof. Let D be an e.a.o. of G and let $X \subseteq V(G)$ with $Y_1, \ldots, Y_t$ as above. Since D is acyclic, there exist $y_i \in Y_i$, such that y_i is a sink in $D[Y_i]$, for all $1 \leqslant i \leqslant t$. Since D is extensional, we have that all $N^+(y_i) \subseteq X$ are different, for all $1 \leqslant i \leqslant t$. Since $N^+(y_i) \in \mathcal{P}\left(N(y_i) \cap X\right)$, the set $\cup_{i=1}^{t} \mathcal{P}\left(N(y_i) \cap X\right)$ has at least t elements. In particular, the maximum value of $\left|\cup_{i=1}^{t} \mathcal{P}\left(N(v_i) \cap X\right)\right|$, taken over all t-tuples $(v_1, \ldots, v_t) \in Y_1 \times \cdots \times Y_t$, is at least t. ■

Example 4.1.3 The graph depicted in Figure 4.1 is not a set graph.

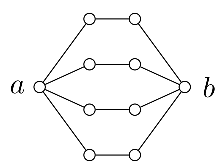


Figure 4.1: A graph violating the condition in Proposition 4.1.2.

Indeed, taking $X = \{a, b\}$, we see that for every $v \in V(G) \setminus X$ we have $N(v) \cap X \in \{\{a\}, \{b\}\}$. Hence $\cup_{i=1}^{4} \mathcal{P}(N(v_i) \cap X) \subseteq \{\emptyset, \{a\}, \{b\}\}$ for every 4-tuple (v_1, v_2, v_3, v_4) as in Proposition 4.1.2, implying $\max \left|\cup_{i=1}^{4} \mathcal{P}(N(v_i) \cap X)\right| = 3 < t = 4$. ■

Proposition 4.1.4 *If G is a set graph, then for every $X \subseteq V(G)$, the set $Y = \{y \in V(G) \mid N(y) = X\}$ has cardinality at most $|X| + 1$.*

Proof. Let X, Y be subsets of $V(G)$ as in the claim. Observe first that in any acyclic orientation of G, for any $y_1, y_2 \in Y$, we have $N^+(y_1) \subseteq N^+(y_2)$ or $N^+(y_2) \subseteq N^+(y_1)$. If this were not the case, then we could find $x_1 \in N^+(y_1) \setminus N^+(y_2)$ and $x_2 \in N^+(y_2) \setminus N^+(y_1)$. Then, we would have $x_1 \to y_2$ and $x_2 \to y_1$, which produces the cycle $y_1 \to x_1 \to y_2 \to x_2 \to y_1$, a contradiction.

If G admits an e.a.o. D, from the above observation and the fact that $N^+(y) \subseteq X$, for any $y \in Y$, we get $|Y| \leqslant |X| + 1$. ■

Proposition 4.1.4 shows that the complete bipartite graph $K_{2,4}$ is not a set graph. More generally:

Corollary 4.1.5 *A complete bipartite graph $K_{m,n}$ is a set graph if and only if $|m - n| \leqslant 1$.*

Proof. Necessity of the condition follows from Proposition 4.1.4. Sufficiency follows from Theorem 4.1.6 below. ■

Notwithstanding, the difference between the cardinalities of the parts of the bipartition of a bipartite set graph can be exponentially large. Just consider the bipartite graph G having as two parts a finite set $X = \{x_1, \ldots, x_n\}$, $n \geqslant 1$, and the power-set of X, $\mathcal{P}(X)$, and whose e.a. orientation is given by the arc relation $\{S \to x : S \in \mathcal{P}(X) \wedge x \in X \wedge x \in S\} \cup \{x_{i+1} \to \{x_i\} : 1 \leqslant i < n\} \cup \{x_1 \to \emptyset\}$.

As already mentioned at the beginning of this section, a directed path between two vertices in an acyclic digraph prevents them from colliding. In particular:

Theorem 4.1.6 *If G has a Hamiltonian path, then G is a set graph.*

Proof. Let $|V(G)| = n$ and denote by $(x_1, x_2, \ldots, x_n)$ a Hamiltonian path of G. To obtain an e.a.o. D of G, orient all edges $x_i x_j$ of G as $x_j \to x_i$, where $1 \leqslant i < j \leqslant n$. Clearly, D is acyclic; since between any x_i and x_j with $i < j$ we have a path in D from x_j to x_i, D is also extensional. ■

Note that the *net*—see Figure 1.1—is a set graph, but does not contain a Hamiltonian path. As a matter of fact, every *corona of a clique*, $K_n \circ K_1$ (made up of a clique $\{\alpha_1, \ldots, \alpha_n\}$, $n \geqslant 3$, together with vertices $\beta_1, \ldots, \beta_n$, where each β_i is adjacent only to α_i, $1 \leqslant i \leqslant n$) is a set graph without a Hamiltonian path, whose unique e.a.o. is obtained along the same lines as in Figure 4.2. (The fact that coronas of cliques are set graphs also follows from Proposition 4.1.8 of Section 4.1.2.)

The next proposition generalizes Corollary 4.1.5 by isolating a larger class of graphs where having a Hamiltonian path is equivalent to being a set graph.

Proposition 4.1.7 *If G is a complete multipartite graph, then G has a Hamiltonian path if and only if G is a set graph.*

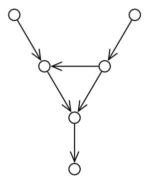


Figure 4.2: The unique e.a. orientation of a net, up to isomorphism.

Proof. We show by induction on $|V(G)|$ that if D is an e.a.o. of G having a vertex s as sink, then there exists a Hamiltonian path in G ending in s.

When $|V(G)| = 1$, the claim is clear. Let thus G be a complete multipartite graph, $|V(G)| \geqslant 2$, and so that D is an e.a.o. of G having s as sink. We claim that $D - s$ is an e.a.o. of $G - s$. Assume, for a contradiction, that in $D - s$ there is a collision between x and y. This implies that x and y belong to the same part of the multipartition of G, different from the one to which s belongs. Additionally, we have that in D the symmetric difference between $N^+(x)$ and $N^+(y)$ is precisely s, so say that $s \in N^+(x) \setminus N^+(y)$. Since G is multipartite, there must be an edge between s and y, which is thus oriented as $s \to y$. This contradicts the fact that s is the sink of D. Moreover, the sink of $D-s$ is a in-neighbor of s in D.

Therefore, we can apply the inductive hypothesis to the complete multipartite graph $G-s$, having the e.a.o. $D-s$, to obtain a Hamiltonian path in $G-s$ ending in a neighbor of s in G. This can be extended by the addition of s for obtaining a Hamiltonian path for G. ∎

Observe that complete multipartite graphs can also be characterized in terms of forbidden induced subgraphs: they are precisely the connected members of the class of (P_4, paw)-free graphs (see, e.g., [84]); the *paw* is depicted in Figure 1.1. However, this is not the largest hereditary class of graphs where having a Hamiltonian path is equivalent to being a set graph, since both P_4 and the paw are set graphs with a Hamiltonian path. It remains an open problem to find a precise characterization of such a largest hereditary class of graphs.

4.1.2 Operations preserving set graphs

In this section, we show that set graphs are closed under certain graph transformations.

Proposition 4.1.8 *Let G be a set graph with $V(G) = \{v_1, \ldots, v_n\}$, and let G' be the graph obtained from G by first adding to it a dominating vertex, and then connecting each vertex of the resulting graph to a new vertex. Formally:*

- $V(G') = V(G) \cup \{v_0\} \cup \{w_i : 0 \leqslant i \leqslant n\}$, *and*
- $E(G') = E(G) \cup \{v_0 v_i : 1 \leqslant i \leqslant n\} \cup \{v_i w_i : 0 \leqslant i \leqslant n\}$.

Then, G' is a set graph.

Proof. Suppose first that G is a set graph and let D be an e.a.o. of G. An e.a.o. D' of G' can be obtained as follows: $D'[V(G)] = D$, vertex v_0 becomes a sink in $D'[V(G) \cup \{v_0\}]$, and the edges $v_i w_i$ are oriented towards v_i for $1 \leqslant i \leqslant n$ and towards w_0 for $i = 0$. Clearly, D' is acyclic, and it is not hard to verify that it is also extensional. ∎

If $G_i = (V_i, E_i), i \in \{1,2\}$, are graphs with $V_1 \cap V_2 = \emptyset$ and x is a vertex of G_1, the *substitution* $H = G_1(x \to G_2)$ of G_2 for x in G_1 is defined as the graph obtained by deleting x from G_1 and joining each vertex of G_2 to each neighbor of x in G_1.

Proposition 4.1.9 *Set graphs are closed under substitution.*

Proof. Let $H = G_1(x \to G_2)$, where G_1 and G_2 are set graphs and x is a vertex of G_1. Let D_i be an e.a.o. of G_i, for $i = 1, 2$. We obtain an e.a.o. D of H as follows:

- All edges completely within $V(G_1)\backslash\{x\}$ are oriented as in D_1.
- All edges completely within $V(G_2)$ are oriented as in D_2.
- For every $v \in V(G_2)$ and every $w \in N^+_{D_1}(x)$, orient the edge vw as $v \to w$.
- For every $w \in V(G_1)$ such that $x \in N^+_{D_1}(w)$ and every $v \in V(G_2)$, orient the edge vw as $w \to v$.

Clearly, D is acyclic. Suppose that there is a collision in D between vertices u and v (so that $uv, vu \notin E(D)$). Then, we cannot have $\{u,v\} \subseteq V(G_2)$ as this would contradict the extensionality of D_2. Similarly, because of the extensionality of D_1, we cannot have $\{u,v\} \cap V(G_2) = \emptyset$. Therefore, $|\{u,v\} \cap V(G_2)| = 1$, say $u \in V(G_2)$ and $v \notin V(G_2)$. But now, since $uv, vu \notin E(D)$ it follows from the construction that $N^+_D(v) \cap V(D_2) = \emptyset$, thus $N^+_D(v) = N^+_{D_1}(v) \subseteq V(D_1)$, and hence $N^+_D(u) = N^+_{D_1}(x)$. This collision between x and v in D_1 contradicts its extensionality. ■

Corollary 4.1.10 *Set graphs are closed under:*

- *adding a dominating vertex (i.e., if G is a set graph, then the graph obtained from G by introducing a new vertex adjacent to all vertices of G is also a set graph),*
- *adding true twins (i.e., if G is a set graph and $v \in V(G)$, then the graph obtained from G by introducing a new vertex v' adjacent precisely to v and to every neighbor of v is also a set graph).*

Note that the converse of Corollary 4.1.10 does not hold, since the set graph in Figure 4.3 can be obtained from the claw $\{\alpha_1, \alpha_0, \alpha_2, \alpha_4\}$ by the addition of α_3 either as a dominating vertex or as a true twin of α_1.

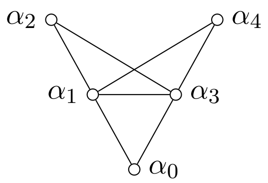


Figure 4.3: A set graph obtained from the claw by one of the two graph operations of Corollary 4.1.10.

We will now introduce an operation that *suppresses* a cut vertex. Let G be a set graph and $x \in V(G)$ is a cut vertex in G. By Proposition 4.1.1, $G - \{x\}$ has precisely two components, say S_1 and S_2. We say that $G \bowtie x$ is the graph (V, E) where

- $V = V(G) \setminus \{x\}$,
- $E = (E(G) \setminus \{xs : s \in N_G(x)\}) \cup \{s_1s_2 : s_1 \in S_1 \cap N_G(x) \wedge s_2 \in S_2 \cap N_G(x)\}$.

Lemma 4.1.11 *If D is extensional and acyclic, then for every $x \in V(D)$, there is at most one component S of $D - \{x\}$ such that $\exists s \in S$, $x \to s$.*

Proof. Arguing by contradiction, if there were two such components S and S' of $D - \{x\}$ so that $s \in S, s' \in S'$, and $x \to s$, $x \to s'$, then denote by t and t' the sinks of $D[S]$ and of $D[S']$, respectively. Since D is extensional and $\{x\}$ is a cut-set, one of these two local sinks, say t, has x as out-neighbor. However, this contradicts the acyclicity of D, since we obtain the cycle t, x, s, followed by the vertices on the path from s to t in $D[S]$. ■

Proposition 4.1.12 *If G is a set graph having a cut vertex x then $G \bowtie x$ is also a set graph.*

Proof. Let G be a set graph having a cut vertex x which gives rise to components S_1 and S_2 of $G - \{x\}$. Denoting by D an e.a.o. of G, observe first that x is not the sink of D, since otherwise there would be a collision between the sinks of $D[S_1]$ and $D[S_2]$, respectively. From Lemma 4.1.11, we thus get that x has out-neighbors in precisely one of S_1 or S_2, say S_1. Moreover, since x is a cut vertex, there are vertices in S_2 having x as out-neighbor.

Obtain the e.a.o. D' of $G \bowtie x$ in the following way:

- all edges completely within S_1 or S_2 are oriented as in D,
- an edge s_1s_2, with $s_1 \in S_1$ and $s_2 \in S_2$ is oriented as $s_1 \to s_2$ if and only if $s_1x \in E(D)$.

To see that D' is acyclic, note that any possible cycle of D' must contain arcs $s_1 \to s_2$ and $s'_2 \to s'_1$, with $s_1, s'_1 \in S_1$, $s_1 \neq s'_1$, and $s_2, s'_2 \in S_2$. Moreover, choose such vertices s_1, s'_1, s_2, s'_2 so that this cycle continues from s'_1 to s_1 using only vertices of S_1. However, this would produce a cycle in D on the vertices s_1, x, s'_1, followed by the vertices on the directed path from s'_1 to s_1 in S_1 belonging to the assumed cycle of D'.

To see that D' is also extensional, argue by contradiction and suppose that there is a collision in D' between s and s' belonging to S_1. Since D is extensional, there must exist a vertex $z \in V(D)$ such that, w.l.o.g., $z \in N^+_D(s) \setminus N^+_D(s')$. If $z \neq x$, then $z \in S_1$ and hence also $z \in N^+_{D'}(s) \setminus N^+_{D'}(s')$. If however $z = x$, then, according to the construction, s receives as out-neighbor an element of S_2, which is not the case for s'. If there is a collision between two vertices of S_2, the argument is identical.

We now have to consider a collision between $s \in S_1$ and $s' \in S_2$, and here there are a few cases to analyze. If $N^+_{D'}(s) \cap S_1 \neq \emptyset$ and $N^+_{D'}(s) \cap S_2 \neq \emptyset$, then $ss' \in E(D')$, against the fact that s and s' collide. Suppose now that $N^+_{D'}(s) \subseteq S_1$, implying that $N^+_{D'}(s') = N^+_D(x)$ and hence that there is a collision in D between x and s. The last case that needs to be considered is when $N^+_{D'}(s) \subseteq S_2$, which implies that s is a sink in $D'[S_1]$. However, this cannot be true, since s would also be a sink in $D[S_1]$, and would hence be in collision in D with the sink of $D[S_2]$. ■

4.1.3 Unicyclic set graphs

Given a connected graph G, let $\mu(G)$ denote its cyclomatic number, $|E(G)| - |V(G)| + 1$. Observe that the problem of finding an e.a.o. of a graph G is easy at extremal values of $\mu(G)$. When G is a tree, by Proposition 4.1.1, it can have no vertex of degree 3 or more, entailing that G is a path. On the other hand, any complete graph is a set graph, since a complete graph has a Hamiltonian path. Note that its orientation is unique, up to isomorphism.[1] We will now give a characterization of unicyclic set graphs (i.e., having cyclomatic number 1)—an example is given in Figure 4.4—which also results in a linear time algorithm for recognizing them.

Given a graph G, a cycle C in G and a sub-tree T of G, we say that T is *pendant* to C if $C \cap V(T) = \{r\}$, where r is a vertex of degree 1 in T; r will be called the *articulation* vertex of T.

Definition 4.1.13 (Jellyfish graph) *A connected unicyclic graph G, having a cycle C, is said to be a* jellyfish graph *if there exist (possibly trivial) paths P, P', P'' in G, pendant to C, having their articulation vertices denoted by r, r', r'', respectively, such that $G = C \cup P \cup P' \cup P''$, $r \neq r''$ and $d(r, r') = d(r', r'') = 1$.*

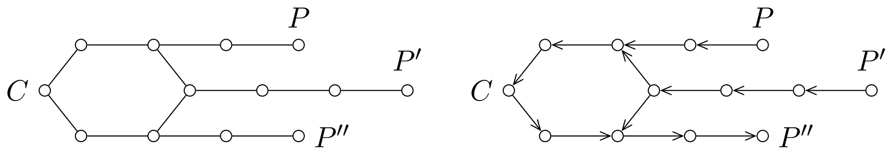


Figure 4.4: A jellyfish graph and its extensional acyclic orientation

Lemma 4.1.14 *If D is extensional and acyclic, then for every $X \subseteq V(D)$, there is at most one component S of $D - X$ such that $\forall x \in X$, $\forall s \in S$, $sx \notin E(D)$.*

Proof. Arguing by contradiction, if there were two such components S and S' of $D - X$, then the sinks of $D[S]$ and $D[S']$ would also be sinks in D, against the extensionality of D. ■

Theorem 4.1.15 *A unicyclic graph is a set graph if and only if it is a jellyfish graph. Moreover, an e.a. orientation of such a graph can be found in linear time.*

Proof. Let D be an e.a.o. whose underlying undirected graph G is connected and unicyclic.

Denoting by C the cycle of G, let us examine the orientation of the arcs between vertices of C. If there were at least two sinks s and s' in $D[C]$, then obtain a contradiction from Lemma 4.1.14, by taking as cut-set $C \setminus \{s, s'\}$. Conversely, if there were two sources in $D[C]$, then this would imply the existence of at least two sinks in $D[C]$. Hence there is exactly one source t and one sink s in $D[C]$. We claim that $s \in N^+(t)$. If not, then consider the two vertices s_1 and s_2 of C such that $s \in N^+(s_1) \cap N^+(s_2)$. Since D is extensional and

[1] It is not difficult to see that this orientation of a complete graph is also the only one which is hyper-extensional.

neither s_1 nor s_2 is a source in $D[C]$, there exists, w.l.o.g, an $s_3 \in N^+(s_1) \setminus (N^+(s_2) \cup C)$. Taking $C \setminus \{s\}$ as cut-set in Lemma 4.1.14, the components of $G - (C \setminus \{s\})$ containing s, and s_3, respectively, produce the desired contradiction.

Consider now $x \in V(G)$ with degree at least three. Since D is extensional, then $x \in C$, implying that all vertices in $V(G) \setminus C$ lie on paths pendant to C. Moreover, from the same Proposition 4.1.1, there can be no two pendant paths having the same articulation vertex.

Let $P = (p_1, \ldots, p_k)$ be a pendant path to C, in G, having p_1 as articulation vertex. If there were two sinks p_i and p_j, $i \neq j$, in $D[P]$, then, by Lemma 4.1.14, we obtain a contradiction by taking $P \setminus \{p_i, p_j\}$ as cut-set. If the sink of $D[P]$ is p_t, where $1 < t < k$, then we obtain again a contradiction, as $N^+(p_{t-1}) = N^+(p_{t+1})$ (since otherwise we would have two sources, and hence also two sinks, in $D[P]$). Therefore, the sink of $D[P]$ is either p_1 or p_k. Note that in both cases P is also a directed path in D.

To simplify notation, let $c_1, \ldots, c_\ell$ be the cyclic order of vertices of C, where $N^+(c_1) \cap C = \emptyset$, $N^+(c_\ell) \cap C = \{c_1, c_{\ell-1}\}$, and $N^+(c_t) \cap C = \{c_{t-1}\}$, for all $1 < t < \ell$. Consider also a pendant path to C, $P = p_1, \ldots, p_k$, where p_1 is its articulation vertex. If p_k is the sink in $D[P]$, then $p_1 = c_1$, since otherwise we can apply Lemma 4.1.14 by taking $C \setminus \{c_1\}$ as cut-set. If p_1 is the sink of $D[P]$, then $p_1 \notin \{c_1, \ldots, c_{\ell-2}\}$. Indeed, if p_1 were some c_t, $1 \leqslant t \leqslant \ell - 2$, then we would have a collision between p_2 and c_{t+1}.

To sum up, we have that either $p_1 = c_1$, in which case p_k is the sink in $D[P]$, or that $p_1 \in \{c_{\ell-1}, c_\ell\}$, in which case p_1 is the sink in $D[P]$. In all these cases, G is a jellyfish graph. On the other hand, each jellyfish graph admits an e.a.o., which can be constructed in linear time, along the same lines of the proof (see also Figure 4.4 as an example). ■

Note that checking whether a connected graph is a jellyfish graph can be done in linear time, by first checking whether G is unicyclic (which can be done even in constant time if the number of edges and vertices is already known), and then finding the cycle C_ℓ of G and ensuring that the pendant trees to C_ℓ, if any, comply with Definition 4.1.13.

4.1.4 Related graph classes and notions

In this section we employ the following notations. Given a graph G and a vertex $v \in V(G)$, the *closed neighborhood* of v is the set $N_G[v] =_{\text{Def}} \{v\} \cup N_G(v)$. Given a digraph D and a vertex $v \in V(D)$, the *closed out-neighborhood* of v is the set $N_D^+[v] =_{\text{Def}} \{v\} \cup N_D^+(v)$, and the *closed in-neighborhood* of v is the set $N_D^-[v] =_{\text{Def}} \{v\} \cup N_D^-(v)$. Because of these concepts, the standard (out-/in-)neighborhood of a vertex is sometimes said to be *open*.

Similarly defined graph classes

We begin by reviewing some graph classes whose definition is quite similar to that of set graphs. We denote by

- $\mathcal{C}_1$: underlying graphs of (not necessarily acyclic) directed graphs with distinct out-neighborhoods;
- $\mathcal{C}_2$: underlying graphs of directed acyclic graphs with distinct closed out-neighborhoods;
- $\mathcal{C}_3$: underlying graphs of (not necessarily acyclic) directed graphs with distinct closed out-neighborhoods;

- $\mathcal{C}_4$: point-determining graphs [140], that is, undirected graphs with distinct neighborhoods;
- $\mathcal{C}_5$: point-distinguishing graphs [140], that is, undirected graphs with distinct closed neighborhoods.

Observe that $\mathcal{C}_2 = \mathcal{C}_3 = \{\text{all undirected graphs}\}$, since every acyclic orientation of a graph G has the property that all closed out-neighborhoods are distinct. Moreover, a graph belongs to $\mathcal{C}_4$ if and ony if its complement belongs to $\mathcal{C}_5$. It is also clear that every set graph is in the class $\mathcal{C}_1$. On the other hand, there are no other inclusion relations among these classes. Denoting by $\mathcal{C}_0$ the class of all set graphs, we collect in Table 4.1 below some examples of graphs $G \in \mathcal{C}_i \setminus \mathcal{C}_j$, for all $i, j \in \{0, 1, 4, 5\}$ such that $i \neq j$ and $(i, j) \neq (0, 1)$.

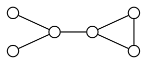
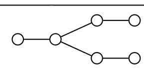
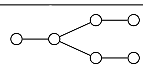
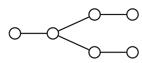
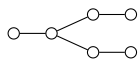
i / j	0	1	4	5
0				
1				
4				
5				

Table 4.1: Separating examples for classes $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_4, \mathcal{C}_5$

Separating codes for digraphs

Related practical applications refer to codes in graphs and digraphs. Let us say that a subset C of vertices of a digraph D is an *open-out-separating code* if the (open) out-neighborhoods of the vertices of D have pairwise distinct intersections with C. It is easy to see that a digraph D admits such a separating code if and only if D is extensional.

To place this in historical context, notice that we are slightly deviating from the nomenclature introduced by [57], where the notion of separating code referred to *closed in*-neighborhoods. We summarize below the various concepts of codes in graphs introduced during the last decade. Given a graph G, a subset $C \subseteq V(G)$ is called:

- *dominating set*, if for all $v \in V(G)$, $N[v] \cap C \neq \emptyset$;
- *separating code*, if for distinct $u, v \in V(G)$ it holds $N[u] \cap C \neq N[v] \cap C$, cf. [57];
- *identifying code*, if C is a dominating set and a separating code, cf. [74];

Moreover, if G is a bipartite graph $G = (A \cup B, E)$, then $C \subseteq B$ is called

- *discriminating code*, if for all $v \in A$, $N(v) \cap C \neq \emptyset$ and for distinct $u, v \in A$ it holds $N(u) \cap C \neq N(v) \cap C$, [28, 29].

In case of digraphs, these notions have been analogously defined in terms of *in*-neighbors. Given a digraph D, a subset $C \subseteq V(D)$ is called:

- *dominating set*, if for all $v \in V(G)$, $N^-[v] \cap C \neq \emptyset$;

- *separating code*, if for distinct $u, v \in V(G)$ it holds $N^-[u] \cap C \neq N^-[v] \cap C$, [57]);
- *identifying code*, if C is a dominating set and a separating code, [74];

As applications of these problems we mention emergency sensor networks in facilities or fault detection in multiprocessor systems (see [30, 127] and the references therein). For the latter, consider a digraph D whose vertices correspond to processors, and whose arcs correspond to unidirectional links between them. Assume that exactly one of the processors is malfunctioning and that it has to be identified. This can only be done by selecting some processors (constituting the open-out-separating code) which are assigned the task of testing their in-neighbors and sending an alarm signal in case one of them is faulty. We require that we can precisely tell which processor is malfunctioning only by looking up which processors produced an alarm.

For a refinement of this problem equivalent to finding an e.a. orientation of a graph, assume that we are given such a network with the additional property that the links between the processors form no directed cycles. Moreover, assume that we are allowed to flip the orientations in any subset of links, as long as the resulting network is acyclic. We want to decide for which networks of processors such a change of the orientations is possible (and find one), so that there is a way to detect any faulty processor under the rules stated above.

Logics capturing PTIME

Assigning an e.a. orientation to a (finite) graph is equivalent to equipping every vertex with a unique identity, given by its Mostowski's collapse. Set graphs are those graphs which provide such a unique identity *internally*, without the need of external labelings (albeit a vertex can have more absolute set identities, corresponding to all e.a. orientations of the set graph).

This relates, among others, to descriptive complexity theory and its quest for logics capturing the complexity class PTIME on finite *unlabeled* graphs (for an introduction, see [60]). If it turned out that there exists no logic capturing PTIME on finite unlabeled graphs, this would show that PTIME $\neq$ NP, since Existential Second Order Logic does capture the complexity class NP on such graphs [51]. On the other hand, a sufficient condition for PTIME to be captured by some logic on finite unlabeled graphs from a class $\mathcal{C}$ is the ability to define a linear order, in that logic, on the vertices of any graph from $\mathcal{C}$ [69, 146]. Since the vertices of an e.a. digraph have a unique identity, a linear order on them is indeed definable in First Order Logic + Least Fixed Point [79]. For this reason, it is interesting to explore deeper connections between (subclasses of) set graphs, together with their e.a. orientations, and logics capturing PTIME. As a motivation, we mention a recent conjecture by Grohe [61] stating that there is a logic capturing PTIME on the class of (connected) chordal claw-free graphs, thus on a subclass of set graphs.

Bayesian networks and essential acyclic digraphs

Finding an orientation of a graph also resembles the problem of learning the structure of a Bayesian network. Such a network, used in machine learning to model real-life phenomena, is an acyclic digraph in which vertices stand for random variables, while arcs encode their conditional interdependencies. When trying to discover the structure of a Bayesian network from input data, one approach is to first discover if there is an interdependence between any two variables (i.e., find the undirected edges), and then to find an acyclic orientation which best explains the data [32].

A more precise connection with Bayesian networks refers to their subclass of *essential* acyclic digraphs (see e.g. [8, 139]). An acyclic digraph D is *essential* if each vertex has a unique identity, in the sense that for every arc $xy \in E(D)$, the set of in-neighbors of x is distinct from the set of in-neighbors of y minus the vertex x. This set-theoretic flavor is further emphasized by numerical evaluations showing that the asymptotic ratio between labeled essential acyclic digraphs and labeled e.a. digraphs is constant (see [124, 139]).

The acyclic orientation game

Let us also mention the somewhat related *acyclic orientation game* in which an unknown acyclic orientation of a given graph has to be discovered by querying edges one by one (see [6, 120] and the references therein). One is usually interested in finding such an orientation with the minimum number of queries, or in characterizing the graphs for which all edges have to be queried. For example, the minimum number of queries for finding an acyclic orientation of the complete graph K_n represents the minimum number of comparisons needed to sort n pairwise distinct elements.

4.2 Complexity issues

4.2.1 Recognizing set graphs is hard

In this section, we prove that the following three problems are NP-complete:

Problem EAO. *Given a graph G, decide whether G is a set graph.*

Problem sEAO. *Given a graph G, decide whether G admits a slim e.a.o.*

Problem HEO. *Given a graph G, decide whether G admits a hyper-extensional orientation.*

Let HP denote the NP-complete problem of determining whether a given graph has a Hamiltonian path [58]. To obtain the above results, we offer a reduction from the following variant of HP:

Problem HP′. *Given a graph G with exactly two leaves, decide whether G has a Hamiltonian path.*

To see that also Problem HP′ is NP-complete, the following reduction from Problem HP suffices. Given a graph G, construct G^+ having $V(G) \cup \{s_1, s_2, t_1, t_2\}$ as vertex set, and $E(G) \cup \{s_1s_2, t_1t_2\} \cup \{s_2v, t_2v \mid v \in V(G)\}$ as edge set. Clearly, G has a Hamiltonian path if and only if G^+ has a Hamiltonian path (having s_1 and t_1 as endpoints). Moreover, the Hamiltonian paths of G are in bijection with the Hamiltonian paths of G^+, an observation which will turn out useful in Section 4.2.2.

Finding a (slim) extensional acyclic orientation

Given a graph $G = (V, E)$, denote by $S(G)$ the *subdivision graph* of G, that is, the bipartite graph obtained by subdividing once every edge of G. Stated formally, $S(G) = (V \cup X, F)$, where

- $X = \{x^e \mid e \in E\}$
- $F = \{ux^{uv} \mid uv \in E\}$

A vertex of X is called an *edge vertex*.

Lemma 4.2.1 *If G is a graph with exactly two leaves that has a Hamiltonian path, then $S(G)$ admits a slim e.a.o.*

Proof. Let $(v_1, v_2, \ldots, v_n)$ be a Hamiltonian path in G. Then $s = v_1$ and $t = v_n$ are the two leaves of G. An edge vertex of X is called *touched* if the above Hamiltonian path of G uses the corresponding edge of G, and *untouched* otherwise. Partition X as $X = T \cup U$ by distinguishing *t*ouched edge vertices from *u*ntouched ones. Choose any total order $\prec$ on the vertices of $S(G)$ with the following properties:

i) every vertex in U is placed after any vertex in $T \cup V$;

ii) $v_i \prec x^{v_i v_{i+1}} \prec v_{i+1}$, for every $i \in \{1, \ldots, n-1\}$.

Notice that such a total order exists. Consider the orientation D of $S(G)$ such that every edge $uv \in E(S(G))$ is oriented in D as $u \to v$ if and only if $u \succ v$. Clearly, this is an acyclic orientation. Furthermore, D is also extensional, since:

- vertex $s = v_1$ is the only vertex with $N^+(s) = \emptyset$;
- every untouched vertex in U, say $x^{uv} \in U$, is the only vertex having $N^+(x^{uv}) = \{u, v\}$;
- every touched vertex in T, say $x^{v_i v_{i+1}} \in T$, is the only vertex having $N^+(x^{v_i v_{i+1}}) = \{v_i\}$;
- every vertex in $V \setminus \{s\}$, say $v_i \in V$ (with $2 \leqslant i \leqslant n$), is the only vertex with $N^+(v_i) = \{x^{v_{i-1} v_i}\}$.

To see that D is also slim, observe first that the out-neighborhood of any vertex $v \in T \cup (V \setminus \{s\})$ is a singleton. Therefore, in the digraph obtained by removing the out-going arc from v, vertex v collides with s. Finally, since both s and t are leaves in G, for every untouched vertex in U, say $x^{v_i v_j} \in U$, we have $i, j \in \{2, \ldots, n-1\}$. The removal of the arc $x^{v_i v_j} v_i$ creates a collision between $x^{v_i v_j}$ and $x^{v_j v_{j+1}}$, and similarly the removal of the arc $x^{v_i v_j} v_j$ creates a collision between $x^{v_i v_j}$ and $x^{v_i v_{i+1}}$. ■

Lemma 4.2.2 *Let G be a graph. If $S(G)$ admits an e.a.o., then G has a Hamiltonian path.*

Proof. Let D be an e.a.o. of $S(G)$ and let its sink be v. We claim that D has a directed path passing through all the vertices of G, which hence produces a Hamiltonian path for G.

Indeed, let P be a longest directed path in D starting in a vertex of G and ending at v. Let $u \in V(G)$ be the endpoint of P other than v. If all vertices of G are on P, we are done. If not, let u' be a vertex of G not on P. Let Q be a longest directed path from u' to v, and let x be the first vertex on Q that belongs to P. Let y and z ($y \neq z$) be the predecessors of x on P and on Q, respectively.

If x is a vertex of G, then y and z are edge vertices (thus different from u and u'). Note that by construction each of y and z have exactly two incident arcs, one in-coming, on P or on Q, and one out-going to x. This implies that $N^+(y) = N^+(z) = \{x\}$, contradicting the extensionality of D.

Otherwise, x is an edge vertex, and x must be the sink of D, since its two incident arcs are in-coming. But y and z are again in collision, since from the maximality of the paths and the acyclicity of D they cannot have other out-neighbors than x. ■

Theorem 4.2.3 *Problems* EAO *and* sEAO *are* NP*-complete, even when the input is restricted to bipartite graphs with exactly two leaves.*

Proof. The problems belong to NP, since acyclicity, extensionality and slimness can be checked in polynomial time; actually, extensionality of an acyclic digraph can be verified in linear time [47]. The hardness follows by reducing from Problem HP′, by Lemmas 4.2.1 and 4.2.2. ■

Remark 4.2.4 Instead of requiring that the digraph obtained by removing any arc from an extensional acyclic digraph creates a collision (as in the definition of slimness), one can consider, in a similar way, extensional acyclic digraphs with the property that *reversing* any arc produces either a cycle or a collision. Notice that the slim e.a. orientation of $S(G)$ given in the proof of Lemma 4.2.1 has this property as well (in fact, reversing any arc produces a collision). In particular, this implies that it is NP-complete to verify whether a given bipartite graph with exactly two leaves admits an e.a.o. such that reversing any arc in it produces either a cycle or a collision. ■

Finding a hyper-extensional orientation

Given digraphs D_1 and D_2 with disjoint vertex sets, and given vertices $v_i \in V(D_i)$, $i = 1, 2$, we denote by $U(D_1, v_1, v_2, D_2)$ the digraph obtained by taking a copy of D_1 and a copy of D_2 and adding the arc $v_1 \to v_2$. Formally, $U(D_1, v_1, v_2, D_2)$ has

- $V(D_1) \cup V(D_2)$ as vertex set,
- $E(D_1) \cup E(D_2) \cup \{v_1 \to v_2\}$ as the arc relation.

We define this operation analogously for graphs.

Our reduction will encode any graph G having two leaves s and t by the graph $U(S(G), s, a_8, G_8)$, where G_8 is the underlying graph of digraph D_8, depicted in Figure 4.5. We start with a preliminary lemma.

Lemma 4.2.5 *Let D_1 and D_2 be two hyper-extensional digraphs. If the sink of D_1 is s and if D_2 has a source t, then the digraph $U(D_1, s, t, D_2)$ is hyper-extensional.*

Proof. Let $D = U(D_1, s, t, D_2)$ and let B be a bisimulation over D. Digraph D is extensional, since D_1 and D_2 are extensional, by Lemma 1.4.5, and t is a source of D_2. To prove that $x = y$ whenever xBy, we argue by contradiction, and consider three cases.

First, by construction, B restricted to $V(D_2)$, that is the relation $B_2 = \{(x, y) \mid xBy \wedge x, y \in V(D_2)\}$, is a bisimulation over D_2. Therefore, xBy cannot hold for distinct $x, y \in V(D_2)$.

Second, suppose that x_0By_0 holds for (distinct) $x_0 \in V(D_1)$ and $y_0 \in V(D_2)$. Take $x_1 \in N^+(x_0)$ so that x_1 is a vertex on the directed path from x_0 to s (or $x_1 = t$, if $x_0 = s$). Since x_0By_0, there exists $y_1 \in N^+(y_0)$, thus $y_1 \neq x_1$, such that x_1By_1. By repeating the above procedure sufficiently many times, we reach a pair (x_i, y_i) (where $i \geqslant 0$) such that $x_i = s$, $y_i \in V(D_2)$ and sBy_i. Since $N^+(s) = \{t\}$, there exists a $y_{i+1} \in N^+(y_i)$ so that tBy_{i+1}. Recall that t is a source of D_2, therefore $t \neq y_{i+1}$. This contradicts the previous case.

Finally, we claim that also the restriction of B to $V(D_1)$, that is the relation $B_1 = \{(x, y) \mid xBy \wedge x, y \in V(D_1)\}$, is a bisimulation over D_1. Observe that neither sBx nor xBs can hold for $x \in V(D_1) \setminus \{s\}$. This is true, since $N^+(s) = \{t\}$, and, by the previous case, there can be no $x_1 \in V(D_1)$ such that tBx_1, or x_1Bt. Hence, also sB_1x or xB_1s cannot hold for $x \in V(D_1) \setminus \{s\}$. If xB_1y, for distinct $x, y \in V(D_1)$ and $s \notin \{x, y\}$, then both conditions *i)* and *ii)* of the bisimulation definition hold, by construction and by the fact that xBy. Therefore, B_1 is a bisimulation over D_1, and by hyper-extensionality of D_1 it follows that $x = y$, a contradiction. ∎

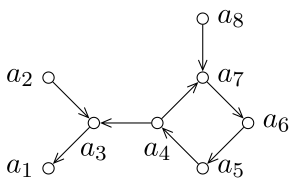


Figure 4.5: Digraph D_8 (denote by G_8 its underlying graph); D_8 is a gadget to force a sink when the orientation can have cycles; one of a_1 or a_2 must be a sink in any *extensional* orientation of G_8.

Lemma 4.2.6 *If G is a graph with two leaves s and t, and if G has a Hamiltonian path, then the graph $U(S(G), s, a_8, G_8)$ admits a hyper-extensional orientation.*

Proof. First, let D be the e.a.o. of $S(G)$ obtained as explained in the proof of Lemma 4.2.1, where s is taken to be its sink. By Lemma 1.4.5, D is also hyper-extensional. Next, observe that also D_8 is hyper-extensional, by applying, for example, the partition refinement algorithm of [107]. Since a_8 is a source of D_8, by Lemma 4.2.5, the digraph $U(D, s, a_8, D_8)$ is hyper-extensional, which proves the claim. ∎

Lemma 4.2.7 *Let G be a graph. If $U(S(G), s, a_8, G_8)$ admits a hyper-extensional orientation, then G has a Hamiltonian path.*

Proof. We reason as in the proof of Lemma 4.2.2. Let D be a hyper-extensional orientation of $U(S(G), s, a_8, G_8)$. Therefore, D

- is extensional,
- has a (unique) sink v that belongs to G_8, and
- from every vertex of D there is a directed path to v.

We claim that D has a directed path passing through all the vertices of G, which thus produces a Hamiltonian path for G.

Indeed, let P be a longest directed path in D starting in a vertex of G and ending at v, the sink of D. Let $u \in V(G)$ be the endpoint of P other than v. If all vertices of G are on P, we are done. If not, let u' be a vertex of G not on P. Let Q be a longest directed path from u' to v, and let x be the first vertex on Q that belongs to P. From construction, we have that x is a vertex of $S(G)$. Let y and z ($y \neq z$) be the predecessors of x on P and on Q, respectively.

If x is a vertex of G, then y and z are edge vertices (thus different from u and u'). Note that by construction, each of y and z have exactly two incident arcs, one in-coming, on P or on Q, and one out-going to x. This implies that $N^+(y) = N^+(z) = \{x\}$, contradicting the extensionality of D.

Otherwise, x is an edge vertex, and x must be the sink of D, since its two incident arcs are in-coming. This contradicts the fact that the sink of D is a vertex of G_8. ■

Theorem 4.2.8 *Problem* HEO *is* NP*-complete, even when the input is restricted to bipartite graphs with exactly three leaves.*

Proof. The problem belongs to NP, since hyper-extensionality can by checked in polynomial time, for example by the algorithm of [107]. The hardness follows by reducing from Problem HP′, by Lemmas 4.2.6 and 4.2.7. ■

The problem of recognizing set graphs appears to be quite delicate, since seemingly negligible variations result in antithetical complexity. Indeed, if one asks for a *weakly* extensional acyclic orientation, then such an orientation always exists and can be found in linear time.

Theorem 4.2.9 *Every graph admits a weakly extensional acyclic orientation. Such an orientation can be found in linear time.*

Proof. Let G be a graph. We may assume that G is connected, since otherwise we can obtain a weakly e.a.o. of G by orienting in a weakly e.a. way the edges of each of its components. Let now T be a depth-first-search tree of G, rooted at an arbitrary vertex r. Note that for every edge $xy \in E(G) \setminus E(T)$, we have that x is an ancestor of y in T, or vice versa. To obtain a weakly e.a.o. D of G, orient all edges xy as $x \to y$ if x is an ancestor of y in D. Clearly, D is acyclic. To see that it is also weakly extensional, suppose that there is a collision between x and y in D. Hence, x is not an ancestor of y, nor y an ancestor of x. Since $N^+(x) \neq \emptyset$, there exists a vertex $x' \in N^+(x)$; but $x' \notin N^+(y)$, as T is a depth-first-search tree. ■

4.2.2 The complexity of counting extensional orientations

We denote by #EAO, #sEAO, #HEO, #HP and #HP′ the corresponding counting variants of the problems considered in Section 4.2.1. For instance, in the #EAO problem the task is to determine the number of all e.a.o.s of a given graph. We consider two orientations $D = (V, E)$ and $D = (V, E')$ of a graph G to be *identical* if the have the same arc set, that is $E = E'$. Similarly, two Hamiltonian paths $P = (v_1, \ldots, v_n)$ and $P' = (v'_1, \ldots, v'_n)$ in a graph G are *identical* if they have the same edge set, that is $\{v_1v_2, \ldots, v_{n-1}v_n\} = \{v'_1v'_2, \ldots, v'_{n-1}v'_n\}$.

Since Problem #HP is #P-complete (see [109, Ch.18], [50]), the simple reduction we gave at the beginning of Section 4.2.1 implies that Problem #HP′ is #P-complete as well.

Theorem 4.2.10 *Problems #EAO and #sEAO are #P-complete, even when the input is restricted to bipartite graphs with exactly two leaves.*

Proof. We first show that if G is a graph with two leaves, s and t, then any e.a.o. D of $S(G)$ is slim. As argued in the proof of Lemma 4.2.2, the vertices of G belong to a directed path P of D. The endpoints of P are s and t, since they are leaves in G. We may assume that s is its last vertex, so that s is the sink of D. Denote by $(t = v_n, v_{n-1}, \ldots, v_1 = s)$ the order in which the vertices of G appear on P. By the construction of $S(G)$ and the fact that P is a directed path, $N^+(x^{v_{i+1}v_i}) = \{v_i\}$, for every $1 \leqslant i \leqslant n-1$. Every vertex of D not on P is an edge vertex $x^{v_iv_j}$, with $i, j \in \{2, \ldots, n-1\}$. If $x^{v_iv_j}$ had only one out-neighbor in D, say v_i, then it would be in collision with $x^{v_{i+1}v_i}$. Therefore, $N^+(x^{v_iv_j}) = \{v_i, v_j\}$. We can conclude that $N^+(v_i) = \{x^{v_iv_{i-1}}\}$, for every $2 \leqslant i \leqslant n$. This shows that D is an orientation obtained as explained in Lemma 4.2.1, hence it is also slim.

We now reduce from #HP′. If G is a graph with two leaves s and t, every Hamiltonian path in G between s and t induces two slim e.a.o.s (having either s or t as sink) for $S(G)$, as argued in Lemma 4.2.1. Moreover, different Hamiltonian paths of G induce different pairs of such slim e.a.o.s for $S(G)$. Conversely, by Lemma 4.2.2 and the above argument, every e.a.o. of $S(G)$ is slim, and it induces a Hamiltonian path of G. This shows that the number of (slim) e.a.o.s of $S(G)$ is exactly twice the number of Hamiltonian paths in G, hence Problems #EAO and #sEAO are #P-complete. ∎

Remark 4.2.11 The above proof implies that it is #P-complete to determine the number of all e.a. orientations in which any arc reversal produces either a cycle or a collision, even when the input is restricted to bipartite graphs with exactly two leaves. ∎

Regarding Problem #HEO, we need the following lemma, describing the possible orientations of the graph G_8.

Lemma 4.2.12 *The digraph D_8 depicted in Figure 4.5, together with D'_8, the digraph obtained from D_8 by reversing the arcs a_2a_3 and a_3a_1, are the only digraphs having G_8 as underlying graph and satisfying properties i), ii) and iii) stated in Lemma 1.4.5.*

Proof. Suppose for a contradiction that there exists an orientation D of G_8 different from D_8 or D'_8 and satisfying the properties *i*), *ii*) and *iii*) stated in Lemma 1.4.5.

Note that exactly one of a_1 or a_2 must be a sink in D, as otherwise they would have the same out-neighborhood. Say $a_3 \to a_1$ and $a_2 \to a_3$. This implies that $a_8 \to a_7$.

Since from every vertex of D there is a directed path to a_1, we have $a_4 \to a_3$. Since $N^+(a_2) \neq N^+(a_4)$, we have that a_4 has at least one other out-neighbor.

Assume first that $a_7 \to a_4$, and hence $a_4 \to a_5$. Since $N^+(a_5) \neq \emptyset$, we have $a_5 \to a_6$. Similarly, $a_6 \to a_7$. Hence, $N^+(a_6) = N^+(a_8)$, contradicting the extensionality of D. Otherwise, since from a_7 there must be a directed path to a_1, we have $a_7 \to a_6 \to a_5 \to a_4$, which contradicts the fact that D is not D_8, nor D'_8. ∎

Theorem 4.2.13 *Problem #*HEO *is #*P*-complete, even when the input is restricted to bipartite graphs with exactly three leaves.*

Proof. We reduce again from #HP′. If G is a graph with two leaves s and t, every Hamiltonian path in G between s and t induces a pair of hyper-extensional orientations of $U(S(G), s, a_8, G_8)$. Indeed, the edges between vertices of $S(G)$ can be oriented as in Lemma 4.2.1 (taking s as a 'local' sink for $S(G)$), whereas the edges between the vertices of G_8 can be oriented as in D_8 or as in D'_8. Moreover, different Hamiltonian paths of G induce different pairs of such hyper-extensional orientations of $U(S(G), s, a_8, G_8)$.

Conversely, if D is a hyper-extensional orientation of $U(S(G), s, a_8, G_8)$, Lemma 4.2.7 and the argument employed in the proof of Theorem 4.2.10 show that $D[V(S(G))]$ must be oriented as indicated by the proof of Lemma 4.2.1. Moreover, Lemma 4.2.12 shows that $D[V(G_8)]$ is either D_8 or D'_8. This allows us to conclude that the number of hyper-extensional orientations of $U(S(G), s, a_8, G_8)$ is exactly twice the number of Hamiltonian paths in G. Hence also Problem #HEO is #P-complete. ∎

4.2.3 Tractability on graphs of bounded tree-width

We now show that set graph recognition is solvable in linear time if the input graph is of bounded treewidth. This follows from Courcelle's Theorem (see [39, 40]): Any property of graphs, or, more generally, relational structures, which is expressible by a Monadic Second Order (MSO) sentence, can be decided in linear time if the treewidth of the input structures is bounded by a fixed constant. (We assume familiarity with these concepts; otherwise, we refer to the excellent monograph [55].) Together with Problem EAO, we also show the tractability of its two generalizations:

Problem EAOsources**.** *Given a graph G and $S \subseteq V(G)$, decide whether G admits an e.a.o. in which all vertices of S are sources.*

Problem EAOarcs**.** *Given a graph G and a partial orientation F of G, decide whether G admits an e.a.o. extending F.*

We observe in passing that these two problems have been shown to be NP-complete also by a direct reduction from the propositional satisfiability problem [86].

Theorem 4.2.14 *For every k, Problems* EAO*,* EAOsources *and* EAOarcs *are all solvable in linear time if the input graph G is of treewidth at most k.*

Proof. By Courcelle's Theorem, it suffices to show that each of these problems can be expressed by a Monadic Second Order sentence. Consider the vocabulary τ consisting of two unary relation symbols V and E, and a binary relation symbol I. We represent a graph

$G = (V', E')$ by a τ-structure $\mathcal{A} = (A, V, E, I)$ with the universe $A := V' \cup E'$, and the following interpretations of the relations: $V := V'$, $E := E'$ and $I := \{(v, e) \mid v \in V', e \in E'$ and $v \in e\}$ is the vertex-edge incidence relation. For our purpose, we will not need the definition of the treewidth and tree decompositions; it will be enough to recall that a σ-structure $\mathcal{B}$ has the same tree decompositions as its Gaifman graph $\mathcal{G}(\mathcal{B}) := (V, E)$ where $V := B$, the universe of the structure, and $E := \{\{a, b\} \mid a, b \in B,\ a \neq b,$ there exists an $R \in \sigma$ and a tuple $(b_1, \ldots, b_r) \in R$ where $r := \text{arity}(R)$, such that $a, b \in \{b_1, \ldots, b_r\}\}$. The Gaifman graph of the structure $\mathcal{A} = (A, V, E, I)$ as above is isomorphic to the graph obtained from G by subdividing each edge precisely once. Since the treewidth of a graph is invariant with respect to edge subdivisions, it follows that if G is of treewidth k, then so is the structure representing it [55].

For unary relations X and R, we will write $x \in X$ to mean $X(x)$, and interpret $X \subseteq V$ and $R \subseteq E$ in the usual way. Moreover, for a subset $R \subseteq E$, we will write $\{x, y\} \in R$ to mean $(\exists e)(e \in R \wedge I(x, e) \wedge I(y, e))$. With these conventions in mind, we now describe MSO sentences expressing the above problems:

- Encoding an orientation D of the graph is possible using some reference orientation and then quantifying over a subset R of edges whose orientations are reversed. The reference orientation can be obtained using a depth-first-search tree: as explained in [41], one can write MSO formulas $\varphi(X, u)$ and $\theta(X, u, x, y)$ such that if $\varphi(X, u)$ holds then for each edge $\{x, y\}$ of G, we have either $\theta(X, u, x, y)$ or $\theta(X, u, y, x)$ but not both. Hence, θ defines an orientation of each edge; however, this orientation depends on two parameters X (a subset of edges) and u (a vertex) that must satisfy formula $\varphi(X, u)$. Such parameters always exist if G is connected. We thus introduce another parameter R, so that the existence of a directed edge xy in the corresponding orientation can be expressed with the following formula:

$$\varphi_{\text{orient}}(X, u, R, x, y) \quad \equiv \quad (\{x, y\} \in R) \Leftrightarrow \theta(X, u, y, x)\,.$$

- Acyclicity can be stated as follows: for every non-empty subset $Y \subseteq V$, there exists a sink in the subgraph of G induced by Y. Stating that a vertex $y \in Y$ is a sink in the subgraph of G induced by Y is equivalent to stating that for every vertex x in Y adjacent to y in G, vertex y is an out-neighbor of x in D. Hence:

$$\varphi_{\text{acycl}}(X, u, R) \quad \equiv \quad (\forall Y \subseteq V)\Big((\exists z \in Y) \Rightarrow (\exists y \in Y)\Big((\forall x \in Y)\big(\{x, y\} \in E \Rightarrow \varphi_{\text{orient}}(X, u, R, x, y)\big)\Big)\Big)\,.$$

- Extensionality can be stated as: for every two distinct vertices x and y, there exists a vertex z which is an out-neighbor of x but not of y, or there exists a vertex z which is an out-neighbor of y but not of x. Formally:

$$\varphi_{\text{ext}}(X, u, R) \quad \equiv \quad (\forall x \in V)(\forall y \in V \setminus \{x\}) \Big((\exists z \in V)\Big(\varphi_{\text{orient}}(X, u, R, x, z) \wedge \neg\varphi_{\text{orient}}(X, u, R, y, z)\Big) \vee (\exists z \in V)\Big(\varphi_{\text{orient}}(X, u, R, y, z) \wedge \neg\varphi_{\text{orient}}(X, u, R, x, z)\Big)\Big)\,.$$

These formulations imply that it can be determined in linear time whether a given graph G of treewidth at most k is a set graph, since this property can be expressed by the following Monadic Second Order sentence:

$$(\exists X \subseteq E)(\exists u \in V)(\exists R \subseteq E)\Big(\varphi(X,u) \wedge \varphi_{\rm ext}(X,u,R) \wedge \varphi_{\rm acycl}(X,u,R)\Big)\,.$$

To obtain the same conclusion for Problems EAOsources and EAOarcs, we must overcome a (minor) technicality: we must show that whenever the input graph G in Problem EAOsources or EAOarcs is of bounded treewidth, so is the structure representing the complete input (a pair (G,S) in Problem EAOsources, or a pair (G,F) in Problem EAOarcs).

For Problem EAOsources, consider the vocabulary τ_1 consisting of three unary relation symbols V, E and S and a binary relation symbol I. A graph $G=(V',E')$ together with a subset S' of its vertex set can be represented by a τ_1-structure $\mathcal{A}_1=(A,V,E,I,S)$ with the universe $A:=V'\cup E'$, interpretations of the relations V, E and I as above (in $\mathcal{A}$), and $S:=S'$. For Problem EAOarcs, consider the vocabulary τ_2 consisting of two unary relation symbols V and E and two binary relation symbols I and F. A graph $G=(V',E')$ together with a partial orientation F' of its edge set can be represented by a τ_2-structure $\mathcal{A}_2=(A,V,E,I,F)$ with the universe $A:=V'\cup E'$, interpretations of the relations V, E and I as in $\mathcal{A}$, and $F:=F'$.

In both cases, the Gaifman graph of $\mathcal{A}_i$ is isomorphic to the graph obtained from G by subdividing each edge precisely once, which implies that if G is of treewidth k, then so is the corresponding structure. Now we can complete our description of Problem EAOsources and Problem EAOarcs by MSO sentences:

- Stating that a vertex $x\in S$ is a source is equivalent to stating that every vertex adjacent to x in G is an out-neighbor of x in D. Formally:

$$\varphi_{\rm sources}(X,u,R) \equiv (\forall x \in S)(\forall y \in V)\Big(\{x,y\}\in E \Rightarrow \varphi_{\rm orient}(X,u,R,x,y)\Big)\,.$$

- Stating that the orientation D extends a given partial orientation F can be done as follows: for all edges $\{x,y\}\in E$, we require that y is an out-neighbor of x in D if this is the case in F. Formally:

$$\varphi_{\rm extend}(X,u,R) \equiv (\forall x \in V)(\forall y \in V)\bigg(\Big(\{x,y\}\in E \wedge F(x,y)\Big) \Rightarrow \varphi_{\rm orient}(X,u,R,x,y)\bigg)\,.$$

Therefore, the property that G admits an e.a.o. in which all vertices in a given set S are sources can be expressed as:

$$(\exists X \subseteq E)(\exists u \in V)(\exists R \subseteq E)\Big(\varphi(X,u) \wedge \varphi_{\rm ext}(X,u,R) \wedge \varphi_{\rm acycl}(X,u,R) \wedge \varphi_{\rm sources}(X,u,R)\Big)\,,$$

and the property that G admits an e.a.o. extending a given partial orientation F can be expressed as:

$$(\exists X \subseteq E)(\exists u \in V)(\exists R \subseteq E)\Big(\varphi(X,u) \wedge \varphi_{\rm ext}(X,u,R) \wedge \varphi_{\rm acycl}(X,u,R) \wedge \varphi_{\rm extends}(X,u,R)\Big)\,.$$

This completes the proof. ■

4.2.4 The complexity of finding a separating code

In this section we are concerned with separating codes in digraphs, with two minor changes: we will be referring to *(open) out*-neighborhoods, instead of closed in-neighborhoods:

Definition 4.2.15 *Given a digraph D and $C \subseteq V(D)$ we say that C is an* open-out-separating code *if for distinct $u, v \in V(G)$ it holds $N^+(u) \cap C \neq N^+(v) \cap C$.*

It can be easily seen that a digraph D has an open-out-separating code if and only if D is extensional.

The problem of finding the minimum size of a separating code of a given graph was shown to be NP-complete in [31,38]. An analogous result holds for digraphs [30], even when restricted to acyclic instances. In what follows, we will show that finding the minimum size of an open-out-separating code is NP-complete.

Problem ooSC. *Given a digraph D and an integer k, decide whether D has an open-out-separating code C of size at most k.*

The following problem was shown to be NP-complete in [29].

Problem DC. *Given a bipartite graph $G = (A \cup B, E)$ and an integer k, decide whether there exists a discriminating code $C \subseteq A$ of size at most k.*

Theorem 4.2.16 *Problem ooSC is NP-complete.*

Proof. Reduce from Problem DC. Let $G = (A \cup B, E)$ be a bipartite graph (with no edges within A or within B), where $B = \{b_1, \ldots, b_m\}$, $m \geqslant 1$. Construct the acyclic digraph $D = (V, F)$ as follows:

- $V = A \cup B \cup \{c_0, c_1, \ldots, c_m\}$,
- $F = \{ab \,|\, a \in A \wedge b \in B \wedge ab \in E\} \cup \{b_i c_i \,|\, 1 \leqslant i \leqslant m\} \cup \{c_i c_j \,|\, 1 \leqslant i \leqslant m, 0 \leqslant j < i\}$.

We claim that G has a discriminating code of size at most k if and only if D has an open-out-separating code of size at most $k + m + 1$.

For the forward implication, note that if C is a discriminating code for G, then $C \cup \{c_0, \ldots, c_m\}$ is an open-out-separating code for D.

For the reverse implication, let C be an open-out-separating code for D. We show that $c_0, \ldots, c_m \in C$. First, $c_0 \in C$, as otherwise $N(c_1) \cap C = \emptyset = N^+(c_0) = N^+(c_0) \cap C$. Assuming now that $c_0, \ldots, c_i$, $0 \leqslant i \leqslant m-2$, belong to C, note that $c_{i+1} \in C$ as well, as otherwise $N^+(c_{i+2}) \cap C = N^+(c_{i+1}) \cap C$. Therefore, $c_0, \ldots, c_{m-1} \in C$. Additionally, $c_m \in C$, as otherwise $N^+(b_m) \cap C = \emptyset = N^+(c_0) \cap C$. This concludes the proof, since $C \cap B$ is a discriminating code for G.

Note that if for distinct $a_1, a_2 \in A$, $N_G(a_1) \neq N_G(a_2)$ holds (which can be assumed w.l.o.g., since otherwise G has no discriminating code), then D is also extensional. ■

4.3 Claw conditions and set graphs

It is not rare that classes of graphs defined in terms of forbidden induced subgraphs are proposed as substitutes for interesting graph properties. For example, Berge's celebrated Strong Perfect Graph Conjecture [19] equates the notion of *perfectness* of a graph to forbidding two families of induced subgraphs from it.

Such classes of graphs are also hereditary, in the sense that any induced subgraph of a graph in the class also belongs to that class. Since recognizing set graphs is NP-complete, we can ask the following question: What is the largest hereditary class of graphs such that every connected member of it is a set graphs? The next section is devoted to elucidating this question, by showing that the sought for class is the class of *claw-free* graphs, obtained by excluding only the smallest connected graph which is not a set graph, the claw, $K_{1,3}$, depicted in Figure 1.1.

Claw-free graphs emerged in the 1960s, as a generalization of line graphs [14, 15]. As mentioned in [52], they caught the attention of the graph theory community once some basic graph-theoretic properties regarding matchings and Hamiltonicity were discovered. Quite worth of notice, although proved true by Chudnovsky et al [33] after a four decades' effort, Berge's conjecture was shown rather early to hold for claw-free graphs [117]. In a recent series of papers, Seymour and Chudnovsky also gave a structural characterization of claw-free graphs [34]– [35].

As it turns out, claws and extensional acyclic orientations are quite intertwined. On the one hand, there exists a largest hereditary class of graphs where being a set graph is equivalent to being claw-free. On the other hand, the claw-freeness condition can be generalized in two ways. First, by requiring that all claws of a graph be vertex-disjoint together with a further connectivity condition, another subclass of set graphs will be isolated, in Section 4.3.2. Second, in Section 4.3.2 we show that if we forbid $K_{1,r+2}$, $r \geqslant 1$, instead of the claw $K_{1,3}$, a pseudo-extensionality property can be guaranteed. More precisely, connected $K_{1,r+2}$-free graphs admit an acyclic orientation in which every collision involves at most r vertices.

4.3.1 Claw-free graphs

Let us say that a digraph is claw-free if its underlying undirected graph is claw-free. We start with two preliminary lemmas.

Lemma 4.3.1 *Let D be an acyclic digraph and let $x, y \in V(D)$ such that $x \to y$ and $N^+(x) \setminus \{y\} \subseteq N^+(y)$. The digraph $D - \{xy\} + \{yx\}$ is acyclic.*

Proof. Assume that in $D - \{xy\} + \{yx\}$ the arc yx belongs to a cycle $x \to x' \to \cdots \to y \to x$. Then, we get $x' \in N^+_D(x) \setminus \{y\} \subseteq N^+_D(y)$. In particular, $x' \neq y$. Hence also in D we have the cycle $y \to x' \to \cdots \to y$. ■

Lemma 4.3.2 *Let D be an acyclic claw-free digraph and let $x, y \in V(D)$ such that $N^+(x) = N^+(y)$ and let $z \in N^+(x)$ be a source in $D[N^+(x)]$. The digraph $D - \{xz\} + \{zx\}$ is acyclic.*

Proof. Assume that in $D - \{xz\} + \{zx\}$ the arc zx belongs to a cycle $z \to x \to \cdots \to z' \to z$. If $z' \to x$, we have the cycle $z' \to x \to \cdots \to z'$ in D. Similarly, $z' \to y$ does not hold. Therefore, to avoid the claw $\{z, z', x, y\}$, we conclude that $z' \in N^+(x)$. This contradicts the fact that z is a source in $D[N^+(x)]$. ■

Theorem 4.3.3 *Let G be a connected claw-free graph and let $r \in V(G)$. G admits an e.a. orientation whose sink is r if and only if r is not a cut vertex of G. Moreover, an e.a. orientation of such a graph can be found in polynomial time.*

We will give two proofs for this theorem. The first starts with an intermediary acyclic orientation, in which every collision is iteratively fixed, until obtaining an extensional acyclic orientation. The second proof is inductive.

First Proof. Let $r \in V(G)$ such that r is not a cut vertex of G. If $V(G) = \{r\}$, the claim is clear. Otherwise, let T' be a spanning tree of $G - r$, and let T be a spanning tree of G obtained from T' by adding r as a leaf to it. Let also $\ell : V(G) \to \{1, \ldots, |V(G)|\}$ be any injective labeling function such that for every $x, y \in V(G)$, $y \neq x$, such that x lies on the path in T from r to y, we have $\ell(x) < \ell(y)$. For instance, such a labeling can be obtained by performing a breadth-first traversal of T starting at r. Obtain D by orienting each edge xy of G as $y \to x$ if $\ell(x) < \ell(y)$. Clearly, D is acyclic, having r as unique sink.

Observe first that in D no vertex can collide with more than one other vertex. If this were not the case, let $x, y, z \in V(D)$ such that x collides with both y and z. Then $xy, xz, yz \notin E(G)$, so taking an arbitrary $u \in N^+(x) = N^+(y) = N^+(z)$ would produce a claw $\{u, x, y, z\}$ in G. Moreover, there is no collision in D between vertices x and y with $N^+(x) = N^+(y) = \{r\}$. This follows from the fact that every vertex v other than r has an out-neighbor $n(v)$ in D such that $vn(v) \in E(T)$. Hence, if r is the unique out-neighbor of x and y in D then $r = n(x) = n(y)$; a contradiction with the fact that r has degree 1 in T.

Onwards, we will show that we can resolve each collision in D locally, with at most two arc reversals. At each step, we preserve the following properties:

(i) r is the unique sink;

(ii) there is no collision between vertices x and y with $N^+(x) = N^+(y) = \{r\}$.

Suppose that in D we have $N^+(x) = N^+(y)$, for some vertices x and y, $x \neq y$. We claim that there is a $z \in N^+(x)$ that is a source in $D[N^+(x)]$ such that $z \neq r$. To see this, we consider the following three cases:

- If $|N^+(x)| = 1$, then let $N^+(x) = \{z\}$. Then $z \neq r$ by property (ii).
- If $D[N^+(x)]$ has at least two sources, then clearly there is one different from r.
- If $|N^+(x)| \geqslant 2$ but $D[N^+(x)]$ has only one source z, then $z \neq r$ since otherwise any source in $D[N^+(x) \setminus \{z\}]$ would also be a source in $D[N^+(x)]$.

From the acyclicity and claw-freeness of D, we conclude that $N^+(z) \subseteq N^+(x)$.

In the digraph $D' = D - \{yz\} + \{zy\}$, which, by Lemma 4.3.2, is acyclic, we have $N^+_{D'}(x) \neq N^+_{D'}(y)$. Moreover, D' still has r as unique sink: denoting by u any vertex in $N^+_D(z)$, we see that $u \in N^+_{D'}(y)$, hence y is not a sink in D'. Notice that in D', there can be no collision between y and another vertex $\tilde{y}$, since this would produce a claw $\{u, x, y, \tilde{y}\}$ in G. So the only possibility for a collision in D' that was not present in D

Algorithm 1: Finding an e.a.o. of a connected claw-free graph

Input: A connected claw-free graph G with $|V(G)| \geqslant 2$, a vertex $r \in V(G)$ that is not a cut vertex of G

Output: An extensional acyclic orientation of G having r as sink

$T' \leftarrow$ a spanning tree of $G - r$;
$T \leftarrow$ a spanning tree of G obtained from T' by the addition of r as a leaf;
perform a breadth-first traversal of T from r;
let $(v_1, \ldots, v_n)$ denote the order of vertices according to the time they were first visited by the traversal;
$D \leftarrow$ orientation of G where each edge $v_i v_j$ of G is oriented as $v_i \rightarrow v_j$ if $i > j$;
while *there exists a collision in D between two distinct vertices x and y* **do**
    $z \leftarrow$ a source in $D[N^+(x)]$ different from r;
    $D' \leftarrow D - \{yz\} + \{zy\}$;
    if *there exists a collision in D' between z and another vertex z'* **then**
        $D \leftarrow D' - \{z'y\} + \{yz'\}$;
    else
        $D \leftarrow D'$;
    end
end
return D;

is between z and some other vertex, say z'. If this is the case, then $zz' \notin E(G)$. Since $N^+_D(z) \subseteq N^+_D(x) = N^+_D(y)$, we have $N^+_{D'}(z') \setminus \{y\} \subseteq N^+_{D'}(y)$. Therefore, by Lemma 4.3.1, the digraph $D'' = D' - \{z'y\} + \{yz'\}$ is acyclic. In D'' we now have $N^+_{D''}(z) \neq N^+_{D''}(z')$, and r still is its unique sink.

Suppose that in D'' there is a collision that was not present in D'. Then it must involve either y or z'. It cannot involve y, as this would result in a claw (similarly as above). However, since $xz' \notin E(G)$ it also cannot involve z': a collision between z' and some other vertex, say $\tilde{z}$, would result in a claw $\{u, z, z', \tilde{z}\}$ in G. Hence in D'' we have resolved the collision between x and y, without introducing any new one.

As argued above, the new orientation will satisfy property (i). Property (ii) follows from the fact that the arc reversals do not produce any new collisions.

For the reverse implication of the claim, let D be an e.a.o. of G whose sink is r and let C_1 and C_2 be two connected components of $G - s$. Since D is acyclic, let r_1, and r_2 be the sinks of $D[C_1]$, and of $D[C_2]$, respectively. Since D is extensional and r is its sink, we have that $N^+(r_1) = N^+(r_2) = \{r\}$ in D, a contradiction.

The proof also suggests a polynomial-time algorithm for finding an e.a. orientation of a given connected claw-free graph. A pseudocode is given in Algorithm 1. ■

Second Proof. We reason by induction on the number of vertices. Let G be a connected claw-free graph and let $x \in V(G)$ that is not a cut vertex of G. If there exists a vertex $y \in N(x)$ which is not a cut vertex of $G - \{x\}$, then from the inductive hypothesis $G - \{x\}$ admits an orientation D having y as sink. Extending the orientation D by orienting the edges incident to x as out-going towards x produces the desired e.a.o. of G.

Observe that if y is a cut vertex of a connected claw-free graph G, then $G - \{y\}$ has exactly two components. Suppose now that every neighbor of x is a cut vertex for $G - \{x\}$. Consider a minimum connected subgraph of $G - x$ which contains all neighbors of x, which must be a tree having as leaves neighbors of x. Let y be such a leaf, and denote by C_1 and C_2 the components of $G - \{x, y\}$, where x has no neighbors in C_2. Observe that y is not a cut vertex for $G - C_i - x$, $i = 1, 2$. Applying the inductive hypothesis to $G - C_i - \{x\}$, we can obtain the e.a. orientation D_i whose sink is y, $i = 1, 2$. Let s_i be the vertex of C_i having y as unique out-neighbor in D_i. From the choice of y, the fact that G is claw-free, and $s_1 s_2 \notin E(G)$, the edge $s_1 x$ must be present in G, while $s_2 x \notin E(G)$. Then, obtain the e.a.o. D of G by extending D_1 and D_2 and orienting the edges incident to x as out-going towards x. ∎

The relation of set graphs to *hereditary* graph classes can be now completely determined. Let us denote by $\mathcal{S}$ the largest hereditary class of graphs every connected member of which is a set graph. The observation that the claw is not a set graph implies that $\mathcal{S}$ is a subclass of claw-free graphs. Conversely, by Theorem 4.3.3, the class of claw-free graphs is contained in $\mathcal{S}$. Therefore, the largest hereditary class of graphs every connected member of which is a set graph is the class of claw-free graphs. On the other hand, the class of set graphs is not contained in any non-trivial hereditary graph class. This follows from Theorem 4.1.6 and by observing that every graph is an induced subgraph of a graph with a Hamiltonian path.[2]

The largest hereditary class where being a set graph is equivalent to being claw-free

We begin by observing the following two consequences of Lemma 4.1.1. Every set graph G

(1) is connected, and

(2) for every vertex v of G, the graph $G - v$ has at most two connected components.

In what follows, we will refer to condition (2) above as the *cut vertex condition.*

An example of graphs in which these two conditions are also sufficient for the property of being a set graph is the class of *block graphs*, that is, graphs in which every maximal connected subgraph without cut vertices is complete.

Lemma 4.3.4 *Let G be a connected block graph. Then, the following conditions are equivalent:*

(1) G is a set graph.

(2) G satisfies the cut vertex condition.

(3) G is claw-free.

[2]This can be seen by taking a linear order $(v_1, \ldots v_n)$ on the vertices of any graph G and building a graph G^{Ham} by adding, for every consecutive two vertices v_i, v_{i+1}, $1 \leqslant i < n$, a new vertex u_i adjacent to v_i and v_{i+1}; G^{Ham} has G as induced subgraph, and contains the Hamiltonian path $v_1, u_1, v_2, u_2, \ldots, v_{n-1}, u_{n-1}, v_n$.

Proof. The conditions (1) ⇒ (2) and (3) ⇒ (1) follow from Lemma 4.1.1 and Theorem 4.3.3, respectively.

(2) ⇒ (3): Let G be a connected block graph satisfying the cut vertex condition. Suppose for a contradiction that G contains a claw K induced by the vertex set $\{a, b, c, d\}$, where a is the vertex of degree 3 in K. Since G satisfies the cut vertex condition, we may assume w.l.o.g. that b and c belong to the same connected component of $G-a$. This implies that there exists a b-c path avoiding a, which, together with the path (b, a, c) forms a cycle. This implies that b and c are contained in some maximal connected subgraph of G without cut vertices. However, since b and c are non-adjacent, this is a contradiction to the fact every such subgraph of G is complete. ■

We now generalize the result of Lemma 4.3.4, by characterizing the largest hereditary class of graphs in which the claw-freeness is not only sufficient but also a necessary condition for a connected graph to be a set graph. The resulting graph class provides a common generalization of claw-free graphs and block graphs. Interestingly, for graphs in this class, the connectedness together with the cut vertex condition are not only necessary but also sufficient conditions for being a set graph.

An *apple* of order $k \geqslant 4$ is the graph obtained from a cycle of order k by adding to it a new vertex and connecting it to precisely one vertex of the cycle. We say that a graph G is *apple-free* if it does not contain any apple as an induced subgraph.

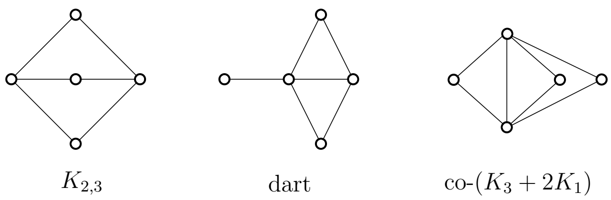


Figure 4.6: The graphs $K_{2,3}$, dart, and co-$(K_3 + 2K_1)$.

It is easy to see that every apple, as well as each of the three graphs depicted in Figure 4.6 is a set graph (for example, by applying Theorem 4.1.6). The following theorem shows that that these are the only minimal set graphs that are not claw-free.

Theorem 4.3.5 *For every graph G, the following conditions are equivalent:*

(1) Every induced subgraph of G that is a set graph is claw-free.

(2) Every induced subgraph of G that satisfies the cut vertex condition is claw-free.

(3) Every induced subgraph of G with a Hamiltonian path is claw-free.

(4) G is (apple, $K_{2,3}$, dart, co-$(K_3 + 2K_1)$)-free.

Proof. The implication (2) ⇒ (1) follows from Lemma 4.1.1.

The implication (1) ⇒ (3) follows from the fact that every graph with a Hamiltonian path is a set graph [86].

The implication (3) ⇒ (4) is straightforward since apples, the $K_{2,3}$, the dart and the co-$(K_3 + 2K_1)$ are graphs with a Hamiltonian path that are not claw-free.

It remains to show (4) $\Rightarrow$ (2). Let G be an (apple, $K_{2,3}$, dart, co-($K_3 + 2K_1$))-free graph and let H be an induced subgraph of G satisfying the cut vertex condition. Suppose for a contradiction that H contains a claw K induced by the vertex set $\{a, b, c, d\}$, where a is the vertex of degree 3 in K. Let $k \geqslant 2$ be the minimum distance in $H - a$ between two leaves of K. Note that k is finite since the graph $H - a$ has at most two connected components. We may assume, without loss of generality, that $P = (b, v_1, \dots, v_{k-1}, c)$ is a path of length k connecting b and c in $H - a$. By the minimality of P, vertex d is not on P. However, d has a neighbor on P since otherwise G would contain either an induced apple (if a does not dominate P) or a dart (otherwise). Let v_j be a neighbor of d on P. Then, by the choice of P, we have that the length of the path $(d, v_j, v_{j-1}, \dots, v_1, b)$ is at least k, and also the length of the path $(d, v_j, v_{j+1}, \dots, v_{k-1}, c)$ is at least k. Consequently $j + 1 \geqslant k$ and $k - j + 1 \geqslant k$, which implies $j = 1$ and $k = 2$. However, now we see that G contains either an induced co-($K_3 + 2K_1$) (if a is adjacent to v_1) or an induced $K_{2,3}$ (otherwise), a contradiction. ■

Corollary 4.3.6 *Let G be a connected (apple, $K_{2,3}$, dart, co-($K_3 + 2K_1$))-free graph. Then, the following conditions are equivalent:*

(1) G is a set graph.

(2) G satisfies the cut vertex condition.

(3) G is claw-free.

The result of Corollary 4.3.6 further elucidates the connection between claw-free graphs and set graphs, and provides another class of graphs in which Problem EAO is solvable in polynomial time. Actually, the complexity of Problem EAO on the class of connected (apple, $K_{2,3}$, dart, co-($K_3 + 2K_1$))-free graphs is in fact linear, due to condition (2): one can verify that the cut vertices of an input graph G satisfy the cut vertex condition by computing, in linear time, the block-tree of G.

4.3.2 Claw disjoint graphs

Observe that set graph recognition remains NP-complete even for graphs none of whose claws share an edge. To see this, first observe that Problem HP remains NP-complete for graphs of maximum degree three. Then, notice that the proof of Lemma 4.2.1 readily yields that if G is a graph with a Hamiltonian path, then $S(G)$ admits a slim e.a.o. (the supplementary condition that G has exactly two leaves was required to ensure slimness as well). Finally, if G has maximum degree at most three, then in the subdivided graph $S(G)$ no two claws share an edge.

Bringing into play a stronger property of claw disjointness, namely the fact that no two claws of a graph share a *vertex* (which we call *claw disjointness*, see below), we now pinpoint a larger subfamily of set graphs, comprising also connected claw-free graphs. We also require a particular, polynomially checkable, connectivity condition, to be introduced as Property Π below. Nevertheless, the status of set graph recognition remains open for claw disjoint graphs.

Definition 4.3.7 *We say that a graph G is* claw disjoint *if for every two distinct claws Y_1 and Y_2 of G, it holds that $V(Y_1) \cap V(Y_2) = \emptyset$.*

Given a graph G, we denote by

- $A(G)$ the set of vertices of G that are the center of some claw of G;
- $B(G)$ the set of vertices of G that are a leaf in some claw of G.

Property Π. *Given a connected claw disjoint graph G and a vertex $s \in V(G)$, we say that $\Pi(G,s)$ holds if the following conditions hold:*

i) s belongs to no claw of G;

ii) no vertex in $B(G) \cup \{s\}$ is a cut vertex of the graph $G - A(G)$;

iii) for every $a \in A(G)$, there exists a connected component of $G - A(G)$ in which a has at least two neighbors.

Notice that if G is claw-free then Property Π amounts to requiring that s not be a cut vertex of G. In Theorem 4.3.10 below, we generalize Theorem 4.3.3 for any connected claw disjoint graph G having a vertex $s \in V(G)$ such that $\Pi(G,s)$ holds. A specification of the algorithm suggested by the theorem is given as Algorithm 2.

We start with two preliminary technical lemmas.

Lemma 4.3.8 *Let G be a connected claw disjoint graph having a vertex s such that $\Pi(G,s)$ holds. Let $a \in A(G)$ be the center of a claw K in G, and let b,c,d be the leaves of K. Then, for every connected component X of $G - a$ and every $y \in X \cap \{b,c,d,s\}$, $\Pi(X,y)$ holds.*

The next lemma is a variant of Lemma 4.3.2.

Lemma 4.3.9 *Let D be an acyclic digraph and let $X \subseteq V(D)$ be a non-empty set such that for every two vertices $x,x' \in X$, $N^+(x) = N^+(x')$ holds. Let also $z \in N^+(x)$ (where $x \in X$) be a vertex of maximum rank among $N^+(x)$. For any $Y \subseteq X$, the digraph $D' = D - \{yz \mid y \in Y\} + \{zy \mid y \in Y\}$ is acyclic.*

Proof. If in D' an arc zy, for some $y \in Y$, belongs to a cycle $z \to y \to z' \to \cdots \to z$, $(z' \neq z)$, then also in D there is a directed path from z' to z, contradicting the maximality of z. ■

Theorem 4.3.10 *Every connected claw disjoint graph G having a vertex s such that $\Pi(G,s)$ holds admits an e.a. orientation having s as sink. Moreover, such an e.a.o. can be found in polynomial time.*

Proof. Suppose for a contradiction that the theorem is false, and let (G,s) be a counterexample minimizing $|V(G)|$. Thus, G is a connected graph and $s \in V(G)$ such that the property $\Pi(G,s)$ holds. Note that G contains a claw since otherwise the claim holds by Theorem 4.3.3.

Let a be the center of some claw of G. We argue first that $G - a$ has at most two connected components. If this were not the case, then let X, Y, and Z be three connected components of $G - a$, so that a has a neighbor in each of them, say x, y, and z, respectively. Since $\Pi(G,s)$ holds, there must be a connected component of $G - a$ in which a has two neighbors. Assume w.l.o.g. that this component is Z so that both z and z' are neighbors of a in Z. However, the claws $\{a, x, y, z\}$ and $\{a, x, y, z'\}$ contradict the fact that G is claw disjoint.

Assume now that the vertices $\{a, b, c, d\}$ induce a claw in G whose center is a, and hence that $G - a$ has at most two components. We will consider several cases and in each of them we will produce an e.a.o. of G.

Suppose first that $G - a$ is connected. Observe that $s \neq a$ and hence by Lemma 4.3.8, $\Pi(G - a, s)$ holds as well. By the minimality of G, we obtain an e.a.o. of $G - a$ having s as sink. To obtain an e.a.o. of G, extend this orientation by orienting any edge between a and vertices of $G - a$ as out-going from a. This leads to no collisions, since any $a' \in V(G) \setminus \{a\}$ colliding with a would have b, c, d among its out-neighbors, and would hence produce the claw $\{a', b, c, d\}$, which intersects $\{a, b, c, d\}$.

Suppose now that $G - a$ has two connected components X and Y. We may assume w.l.o.g. that $b \in X$ and $c, d \in Y$. Observe first that there are no other edges between vertices of X and a (except ba) from the fact that a is a cut vertex and G is claw disjoint. We have to consider the following two cases according to whether s belongs to X or to Y:

Case 1. $s \in X$. By Lemma 4.3.8, $\Pi(X, s)$ and $\Pi(Y, c)$ hold. By the minimality of G we can obtain e.a.o.s for (X, s) and (Y, c). These can be extended to an e.a.o. for G by orienting the edge ab as $a \to b$ and any edge between a vertex of Y and a as going towards a. This orientation is clearly acyclic, and any possible collision that may have arisen is between a and a vertex $a' \in X$. Since b belongs to a claw of G, then $b \neq s$, and thus there exists an $f \in X$ that is an out-neighbor of b in the e.a.o. of (X, s). As there are no other edges between a and vertices of X, this produces the claw $\{b, a, a', f\}$, against the fact that G is claw disjoint.

Case 2. $s \in Y$. Observe that in this case $\Pi(X, b)$ and $\Pi(Y, s)$ hold (again, by Lemma 4.3.8), and hence by the minimality of G we can obtain e.a.o.s for (X, b) and (Y, s). We can extend these to an e.a.o. D for (G, s) by orienting the edge ba as $b \to a$ and any edge between a and vertices of Y as out-going from a. This orientation of G is acyclic, and any collision that may appear is between a and a vertex $a' \in Y$. To regain extensionality, proceed as follows.

Let x be a vertex of maximum rank of $D[N^+(a)]$. Since $|N^+(a)| \geqslant 2$, and s is the unique source of D, then $x \neq s$. By Lemma 4.3.9, we can reverse the arc ax to be $x \to a$, obtaining an acyclic digraph, say D', whose sink is still s. At this point, a and a' no longer collide. Note that in D' there can be no collision between a and some other a'', since this would produce the claw $\{f, a, a', a''\}$, where $f \in \{c, d\} \setminus \{x\}$, against the fact that G is claw disjoint. Moreover, there is no collision in D' involving x since x is the unique vertex of D' having both a and vertices of Y as out-neighbors. ■

A similar approach can be used to characterize set graphs with exactly one induced claw.

Theorem 4.3.11 *If a connected graph G has precisely one induced claw, whose center is a, then G is a set graph if and only if $G - a$ has at most two connected components. If this is the case, an e.a. orientation of G can be found in polynomial time.*

Proof. The forward implication holds by Lemma 4.1.1. For the reverse implication, notice that there are connected graphs G having precisely one claw, but having no vertex s so that $\Pi(G, s)$ holds. However, to prove the claim, it suffices to elucidate the proof of Theorem 4.3.10 for this restricted case. As before, let the claw of G be $\{a, b, c, d\}$.

Algorithm 2: Finding an e.a.o. of claw disjoint graphs satisfying Property Π

Input: A connected claw disjoint graph G and $s \in V(G)$ such that $\Pi(G, s)$ holds;
Output: An e.a. orientation of G having s as sink.

```
if G is claw-free then
    run Algorithm 1 on (G, s);
else
    let {a, b, c, d} induce a claw of G having a as center;
    if G − a is connected then
        recursively apply the algorithm for (G − a, s);
        orient all edges incident to a as out-going from a;
    end
    if G − a has two connected components X and Y then
        suppose c, d ∈ Y;
        if s ∈ X then
            recursively find orientations for (X, s) and (Y, c);
            orient a → b;
            orient all edges between vertices of Y and a as going towards a;
        end
        if s ∈ Y then
            recursively find orientations for (X, b) and (Y, s);
            orient b → a;
            orient all edges between a and vertices of Y as out-going from a;
            if there is a collision between a and some a′ ∈ Y then
                let x be a vertex of maximum rank of D[N⁺(a)];
                reverse the arc ax as x → a.
            end
        end
    end
end
```

If $G - a$ is connected, then let s be a leaf of a spanning tree of $G - a$ (so that s is not a cut vertex of $G - a$). From Theorem 4.3.3 obtain an e.a.o. for $G - a$, and orient every edge incident to a as out-going from a.

If $G - a$ has two connected components X and Y, assume w.l.o.g. that $b \in X$ and $c, d \in Y$. Once again, there are no other edges between vertices of X and a (except ba) from the fact that a is a cut vertex and $\{a, b, c, d\}$ is the unique claw of G. Observe also that b is not a cut vertex for X, as otherwise b, together with a and two neighbors of b belonging to different connected components of X, would induce a claw in G. Moreover, let s be a vertex of Y that is not a cut vertex for Y. From Theorem 4.3.3 we can find e.a.o.s for (X, b) and (Y, s). Proceed now as in Case 2 of the proof of Theorem 4.3.10. ∎

4.3.3 $K_{1,r+2}$-free graphs and r-extensionality

We consider now a generalization of the claw-freeness: the property of being $K_{1,r+2}$-free, for any fixed $r \geqslant 1$. For $r > 1$, extensionality can no longer be guaranteed for connected $K_{1,r+2}$-free graphs, but a suitable generalization of it, which we introduce in the next definition, will follow.

Definition 4.3.12 *Given a digraph D and $A \subseteq V(D)$, we say that A is an r-*collision *of D if $|A| \geqslant r$ and for any $u, v \in A$ we have $N^+(u) = N^+(v)$. We say that D is r-extensional if no $(r+1)$-collision exists.*

We will actually prove a stronger result, by further weakening the condition of being $K_{1,r+2}$-free. Let us define F_r to be the graph obtained from $K_{1,r}$ by subdividing one edge, and let R_r be the graph obtained from $K_{1,r}$ by subdividing one edge and joining its (former) endpoints by a new edge. Our proof will be given for connected $(F_{r+2}, R_{r+2}, K_{1,2r+1})$-free graphs, so that the result for connected $K_{1,r+2}$-free graphs will be an immediate corollary. A specification of the algorithm suggested by Theorem 4.3.13 is given as Algorithm 3.

Theorem 4.3.13 *For every $r \geqslant 1$, every connected $(F_{r+2}, R_{r+2}, K_{1,2r+1})$-free graph admits an r-extensional acyclic orientation. Moreover, an r-extensional acyclic orientation of such a graph can be found in polynomial time.*

Proof. Let G be a connected $(F_{r+2}, R_{r+2}, K_{1,2r+1})$-free graph and let D be an acyclic orientation of it with one sink, obtained as follows. Let T be a spanning tree of G, rooted at s, a vertex of degree 1 in T. Let also $\ell : V(G) \to \{1, \ldots, |V(G)|\}$ be any injective labeling function such that for every $x, y \in V(G)$, $y \neq x$, such that x lies on the path in T from s to y, we have $\ell(x) < \ell(y)$. For instance, such a labeling can be obtained by performing a breadth-first traversal of T from the root. Obtain D by orienting each edge xy of G as $y \to x$ if $\ell(x) < \ell(y)$. Clearly, D is acyclic, with a unique sink s.

Observe first that there can be no $(2r+1)$-collision of non-sink vertices. If this were not the case, let $x_1, \ldots, x_{2r+1} \in V(D)$ such that $N^+(x_1) = \cdots = N^+(x_{2r+1})$. Since $x_i x_j \notin E(G)$, for any $i, j \in \{1, \ldots, 2r+1\}$, taking an arbitrary $u \in N^+(x_1)$, the set $\{u, x_1, \ldots, x_{2r+1}\}$ would induce $K_{1,2r+1}$ in G. Moreover, there is no collision in D containing two vertices x and y of rank 1. This follows from the fact that every vertex v other than s has an out-neighbor $n(v)$ in D such that $vn(v) \in E(T)$. Hence, if s is the unique out-neighbor of x and of y in D then $s = n(x) = n(y)$; a contradiction with the fact that s has degree 1 in T.

Onwards, we will show that we can iteratively 'fix' any $(r+1)$-collision of D locally, with a constant number of arc reversals. At each step, we preserve the property that s is the unique sink of the orientation.

Let thus $A \subseteq V(D)$ be such that for any $x, x' \in A$, $N^+(x) = N^+(x')$ (hence $xx' \notin E(G)$, for any $x, x' \in A$), and such that $r < |A| \leqslant 2r$. Moreover, we consider an $(r+1)$-collision A whose elements have maximum rank among all vertices in $(r+1)$-collisions. (Notice that all elements of a collision have the same rank.) Also, when the rank of the elements of A is 2, consider first the sets A such that the out-neighborhood of their elements is not a singleton.

Fix a vertex $x \in A$. Since $N^+(x) \neq \emptyset$, let z be a vertex of maximum rank among the vertices of $N^+(x)$, and let v be an arbitrary vertex of $N^+(z)$. Observe that the digraph obtained by reversing the arcs between z and any subset of vertices of A is acyclic, by Lemma 4.3.9.

Partition A into $A = B_1 \cup C_1$ so that $|B_1| = r$ and hence $1 \leqslant |C_1| \leqslant r$, and consider the acyclic digraph $D_1 = D - \{y \to z \mid y \in C_1\} + \{z \to y \mid y \in C_1\}$. If in D_1 z belongs to no $(r+1)$-collision, let $D' := D_1$, $B := B_1$, and $C := C_1$. Otherwise, let Z_1 be an $(r+1)$-collision of D_1 containing z. Partition now $A = B_2 \cup C_2$ so that, as before, $|B_2| = r$, $1 \leqslant |C_2| \leqslant r$, but $C_1 \subseteq B_2$. Analogously, consider the acyclic digraph $D_2 = D - \{y \to z \,|\, y \in C_2\} + \{z \to y \,|\, y \in C_2\}$. We claim that the set $Z_2 = \{z' \in V(D) \,|\, N^+_{D_2}(z) = N^+_{D_2}(z')\}$ has cardinality at most r. This follows from the fact that $Z_1 \cap Z_2 = \{z\}$, no two vertices of $Z_1 \cup Z_2$ are adjacent, and $\{v\} \cup Z_1 \cup Z_2$ contains no copy of $K_{1,2r+1}$. In this case, let $D' := D_2$, $B := B_2$, and $C := C_2$.

To show that we have resolved the $(r+1)$-collision A, it remains to show that C is not included in any $(r+1)$-collision of D'. Let thus $\tilde{C} := \{\tilde{y} \in V(D') \,|\, N^+_{D'}(\tilde{y}) = N^+_{D'}(y)\}$, where $y \in C$.

If $|N^+_D(x)| \geqslant 2$ for some (and then all) $x \in A$, then there exists a vertex $u \in \bigcap_{x \in A} N^+_{D'}(x)$. Since no vertex of $\tilde{C}$ is adjacent (in G) to a vertex of B, the cardinality of $\tilde{C}$ is indeed at most r, as otherwise $\{u\} \cup B \cup \tilde{C}$ would contain an induced copy of $K_{1,2r+1}$.

Otherwise, $N^+_D(x) = \{z\}$, for any $x \in A$. Recall now that since at any previous intermediary step no new sinks have been introduced, s is the only sink of D. We claim that in this case any $x \in A$ has rank 2 and hence $N^+_D(z) = \{s\}$. Otherwise, let $w \in N^+(v)$, so that $z \to v \to w$. Since the subgraph of G induced by $B \cup \{z, v, w\}$ together with an arbitrary vertex of C is not F_{r+2}, and D is acyclic, it can only be that $z \to w$. However, this entails that $B \cup \{z, v, w\}$ together with an arbitrary vertex of C is R_{r+2}, a contradiction.

Notice also that there can be no other vertex z' of rank 1 (so that $N^+(z') = \{s\}$, since s is the unique sink of D), as $A \cup \{z, s, z'\}$ would contain a copy of F_{r+2}. Therefore, A consists of all colliding vertices of rank 2 whose out-neighborhood is a singleton, and hence A is the last remaining $(r+1)$-collision of D. Therefore, after fixing the $(r+1)$-collision A our procedure stops and the resulting orientation is acyclic and r-extensional. In D', the set $\tilde{C}$ is actually $C \cup \{s\}$, and hence it has cardinality at most r, as otherwise $\{z\} \cup A \cup \{s\}$ would induce $K_{1,2r+1}$ in G. ∎

Corollary 4.3.14 *For every $r \geqslant 1$, every connected $K_{1,r+2}$-free graph admits an r-extensional acyclic orientation. Moreover, an r-extensional acyclic orientation of such a graph can be found in polynomial time.*

Observe that the proof of Theorem 4.3.13 shows actually a stronger fact than the one stated in the above corollary, namely that for every $r \geqslant 1$, connected $K_{1,r+2}$-free graphs admit an r-extensional acyclic orientation with a *unique* sink. Since any acyclic orientation of a disconnected graph has at least as many sinks as connected components, we have the following:

Corollary 4.3.15 *For every $r \geqslant 1$, a $K_{1,r+2}$-free graph G admits an r-extensional acyclic orientation if and only if G has at most r connected components.*

Algorithm 3: Finding an r-extensional acyclic orientation of connected $(F_{r+2}, R_{r+2}, K_{1,2r+1})$-free graphs

Input: A connected $(F_{r+2}, R_{r+2}, K_{1,2r+1})$-free graph G with $|V(G)| \geqslant 2$.
Output: An r-extensional acyclic orientation of G

let $s \in V(G)$, such that s is not a cut vertex of G;
$T' \leftarrow$ a spanning tree of $G - s$;
$T \leftarrow$ a spanning tree of G obtained from T' by the addition of s as a leaf;
perform a breadth-first traversal of T from s;
let $(v_1, \ldots, v_n)$ denote the order of vertices according to the time they were first visited by the traversal;
$D \leftarrow$ orientation of G where each edge $v_i v_j$ of G is oriented as $v_i \rightarrow v_j$ if $i > j$;
while *there exists an* $(r+1)$*-collision* A *in* D **do**
  take such an A so that its vertices have maximum rank, and, if possible, the out-neighborhood of its vertices is not a singleton;
  let z be a vertex of $N^+(x)$ of maximum rank (for $x \in A$);
  partition A as $A = B_1 \cup C_1$, with $|B_1| = r$;
  let $D' \leftarrow D - \{y \rightarrow z \mid y \in C_1\} + \{z \rightarrow y \mid y \in C_1\}$;
  if z *belongs no* $(r+1)$*-collision in* D' **then**
    $D \leftarrow D'$;
  else
    partition A as $A = B_2 \cup C_2$, where $|B_2| = r$ and $C_1 \subseteq B_2$;
    $D \leftarrow D - \{y \rightarrow z \mid y \in C_2\} + \{z \rightarrow y \mid y \in C_2\}$;
  end
end
return D.

5

Connected Claw-Free Graphs Mirrored into Transitive Sets

Taking as the vertex set of a set graph any of the transitive closures from which it originates, its edge relation need not be defined separately since it can be implicitly read from the membership relation among its vertices: two vertices are adjacent if and only if one is a member of the other. As shown, transitive hereditarily finite sets do *express* connected claw-free graphs.

This point of view leads to shorter proofs of two results concerning connected claw-free graphs. The first is vertex-pancyclicity of the squares of connected claw-free graphs. Our proof method is simple enough not to require the application of a general result equating Hamiltonicity and vertex-pancyclicity in the square of any graph, as the original proof did. The same framework can be employed for proving another well-known result on claw-free graphs, namely that connected claw-free graphs of even order have a perfect matching.

This set-theoretic insight shows that these two properties hold not only for connected claw-free graphs, but also for the more general class of connected graphs admitting an acyclic orientation, with a unique sink, in which only two of the four possible orientations of a claw are forbidden.

The second part of this chapter gives a formalization of these results in the proof-checker Referee. Since Referee deals only with Zermelo-Fraenkel sets, representing a connected claw-free graph by a transitive 'claw-free' set turned out to require the minimal formalism. On the one hand, we avoid explicitly defining graphs, together with an entire armamentarium of graph-theoretic concepts that the original proofs required. On the other hand, we exploit Referee's built-in set manipulating operations to reflect with a minimum degree of encumbrance the two set-theoretic proofs.

5.1 Simpler proofs for two properties of connected claw-free graphs

Given a graph G, we say that a vertex, or an edge, of G is *pancyclic* if it belongs to a cycle of length ℓ, for every $3 \leqslant \ell \leqslant |V(G)|$. If every vertex of G is pancyclic, then G is called *vertex-pancyclic*. The *square* of a graph G, denoted G^2, is the graph with vertex set $V(G)$ in which two vertices are adjacent if their distance in G is one or two. A major result about squares due to Fleischner [54] states that the square of a graph is Hamiltonian if and only if it is vertex-pancyclic. Matthews and Sumner considered connected claw-free graphs, and showed that squares of connected claw-free graphs with at least three vertices are Hamiltonian [81]. Fleischner's result was then used to conclude that they are also vertex-pancyclic.

Inspired by manipulations of hereditarily finite sets, we give below a quite simple way to prove a slightly stronger result than vertex-pancyclicity of squares of connected claw-free graphs without resorting to [54].

Theorem 5.1.1 *If G is a connected claw-free graph with at least three vertices, then G^2 is vertex-pancyclic. Moreover, if $S \subseteq V(G)$ is the set of sources of an acyclic orientation of G with a unique sink, then for every $s \in S$ there exists an edge e_s of G, incident to s, such that:*

i) e_s is pancyclic in G^2, and

ii) there exists a Hamiltonian cycle of G^2 containing all edges e_s.

Proof. We reason by induction on $n = |V(G)|$. The result is immediate for $n = 3$, while for $n = 4$ there are five connected claw-free graphs: G is either P_4, or its square is K_4, and in each of these cases the claim holds. Assume now $n \geqslant 5$ and let G be a connected claw-free graph, and let D be an acyclic orientation of it with a unique sink. Denote by S_G be the set of sources of D. Also, let x be a source of maximum rank r in D, and take $y \in N^+(x)$, of rank $r-1$. Observe that there are no edges between vertices of $N^-(y)$, from the fact that x has maximum rank in D, and the choice of y. Since G is claw-free, it follows that $|N^-(y)| \leqslant 2$. Hence $G - N^-(y)$ has at least three vertices and is connected, since otherwise D would have two sinks. Apply the inductive hypothesis to $H = G - N^-(y)$ and to its orientation $D[V(H)]$, and consider a cycle C_H of H^2 satisfying conditions i) and ii). Denote by S_H the set of sources of $D[V(H)]$, and observe that $y \in S_H$. Let thus $yz \in E(H)$ be the edge of C_H incident to y and pancyclic in H^2 (so that $z \notin S_H$, as $y \in S_H$).

We show that we can extend the cycle C_H to a cycle in G^2, depending on the cardinality of $N^-(y)$, so that every vertex of G becomes pancyclic in G^2. Furthermore, in order to show conditions i) and ii) for G, observe that $N^-(y) \subseteq S_G \subseteq N^-(y) \cup (S_H \setminus \{y\})$. Accordingly, we will show that C_H can also be extended to a cycle C_G for G^2, so that, in particular, every pancyclic edge $e_s \in E(H)$ belonging to C_H, incident to a source s in $S_H \setminus \{y\}$, is present in C_G and is pancyclic in G^2.

If $N^-(y) = \{x\}$, we put $e_x = xy \in E(G)$. Observe first that $xz \in E(G^2)$. Next, all vertices of H are pancyclic in G^2, since they are pancyclic in H^2 and C_H can be extended to a cycle C_G for G^2, of length $|V(G)|$, by replacing the edge yz with the path yxz. To see that e_x is pancyclic in G^2, note first that $xyzx$ is a 3-cycle in G^2; second, yz is pancyclic in H^2, and thus every cycle in H^2 of length ℓ, $3 \leqslant \ell \leqslant |V(G)| - 1$, containing yz can be extended to a cycle of length $\ell + 1$ in G^2 by replacing the edge yz with the path yxz. The above cycle C_G includes every pancyclic edge $e_s \in E(H)$ belonging to C_H, incident to a source in $S_H \setminus \{y\}$, thus every such edge is pancyclic also in G^2. Since C_G includes the pancyclic edge e_x as well, we have that conditions i) and ii) are satisfied for G.

If $N^-(y) = \{x, w\}$, then from the claw-freeness of G at least one of the edges xz or wz belongs to G, say wz. We put $e_x = xy$ and $e_w = wz$. Observe that $xw, xz \in E(G^2)$. Next, all vertices of H are pancyclic in G^2, since they are pancyclic in H^2 and C_H can be extended to cycles C'_G, and C_G for G^2, of length $|V(G)| - 1$, and $|V(G)|$, respectively, by replacing the edge yz with the path yxz, and with $yxwz$, respectively. This argument also shows that every pancyclic edge $e_s \in E(H)$ belonging to C_H, incident to a source in $S_H \setminus \{y\}$ remains pancyclic in G^2.

To see that both $e_x, e_w \in E(G)$ are pancyclic in G^2, observe first that $xyzx, wyzw$ are 3-cycles in G^2; second, yz is pancyclic in H^2, and thus every cycle of H^2 of length ℓ, $3 \leqslant \ell \leqslant |V(G)| - 2$, containing yz can be extended to a cycle of length $\ell + 1$ in G^2 by replacing the edge yz with one of the paths yxz, or ywz. The above cycle C_G, containing both e_x, e_w concludes, on the one hand, the argument that e_x and e_y are pancyclic in G^2. On the other hand, it shows that conditions i) and ii) hold for G as well, since C_G also includes every pancyclic edge $e_s \in E(H)$ belonging to C_H, incident to a source in $S_H \setminus \{y\}$, ■

The framework employed above readily yields a proof for another classic result on claw-free graphs.

Theorem 5.1.2 ([141]) *If G is a connected claw-free graph with $2n$ vertices, then G has a perfect matching.*

Proof. We reason by induction on n. Let G be a connected claw-free graph with $2n$ vertices ($n > 1$) and let D be an acyclic orientation of G having a unique sink. Let r be the maximum rank of its vertices, and let y be a vertex of rank $r - 1$. Observe that there are no edges between vertices of $N^-(y)$, and since G is claw-free, $|N^-(y)| \leqslant 2$.

If $N^-(y) = \{x\}$, then $G - \{x, y\}$ is connected, otherwise D would have two sinks. From the inductive hypothesis, $G - \{x, y\}$ has a perfect matching, which together with the edge xy constitutes a perfect matching for G.

If $N^-(y) = \{x, w\}$, then, similarly, $G - \{x, w\}$ is connected. Let yz be an edge of the perfect matching of $G - \{x, w\}$, obtained from the inductive assumption. Since G is claw-free, assume w.l.o.g. that $wz \in E(G)$. Obtain a perfect matching for G from the perfect matching for $G - \{x, w\}$ by replacing the edge yz with xy and wz. ■

Observe that in the above proofs we have not used the claw-freeness of G to its full extent; we had recourse only to the fact that in the acyclic orientation with a unique sink of G no claw is oriented in the two ways shown in Figure 5.1. Therefore, we can generalize these two results as follows.

Corollary 5.1.3 *If G is a connected graph admitting an acyclic orientation with a unique sink that has none of the two digraphs depicted in Figure 5.1 as induced subdigraph, then*

- *G^2 is vertex-pancyclic;*

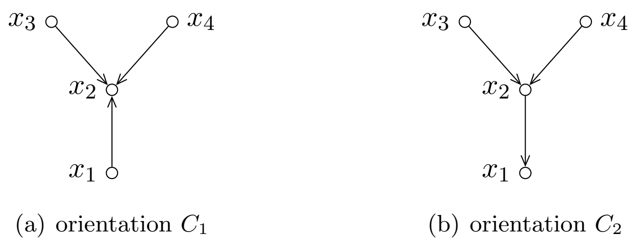


Figure 5.1: Two forbidden orientations of a claw that allow the generalization of Theorems 5.1.1 and 5.1.2.

- *if G has an even number of vertices, then G has a perfect matching.*

Observe also that the graph in Figure 5.2, once suitably oriented, can be handled by the above corollary, whereas the Hamiltonicity of its square and its perfect matchings are not seen either by the traditional results [81, 141, 148], or by subsequent generalizations regarding *quasi claw-free graphs* [4], *almost claw-free graphs* [132], and $S(K_{1,3})$*-free graphs* [65].

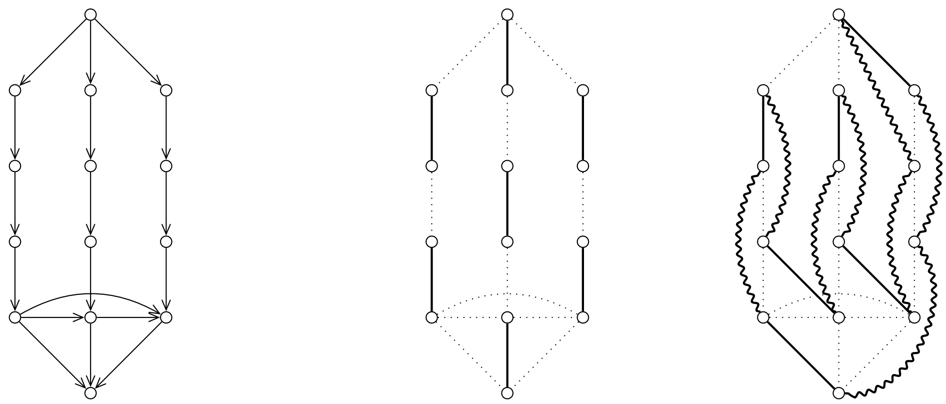
Remark 5.1.4 Stating that the membership digraph of a hereditarily finite well-founded *transitive* set a satisfies the hypothesis of Corollary 5.1.3 can be done by a first-order formula using only the relators $\{=, \in\}$ and such that its prenex form has a purely universal prefix, $\forall\forall\forall\forall$:

$$
\begin{aligned}
&(\forall x_1, x_2, x_3, x_4 \in a)\Big(\\
&\quad (x_1 \neq x_2 \wedge x_1 \neq x_3 \wedge x_1 \neq x_4 \wedge x_2 \neq x_3 \wedge x_2 \neq x_4 \wedge x_3 \neq x_4) \rightarrow \\
&\quad \neg\Big((x_2 \in x_1 \wedge x_2 \in x_3 \wedge x_2 \in x_4) \wedge \qquad \text{orientation } C_1 \\
&\qquad (x_1 \notin x_3 \wedge x_3 \notin x_1 \wedge x_1 \notin x_4 \wedge x_1 \notin x_1 \wedge x_3 \notin x_4 \wedge x_4 \notin x_3)\Big) \wedge \\
&\quad \neg\Big((x_1 \in x_2 \wedge x_2 \in x_3 \wedge x_2 \in x_4) \wedge \qquad \text{orientation } C_2 \\
&\qquad (x_1 \notin x_3 \wedge x_3 \notin x_1 \wedge x_1 \notin x_4 \wedge x_1 \notin x_1 \wedge x_3 \notin x_4 \wedge x_4 \notin x_3)\Big)\Big).
\end{aligned}
$$

■

5.2 Formalizing connected claw-free graphs in a set-based proof-checker

A convenient computerized system for *reasoning* about the entities of our discourse is the proof-checker Referee/ÆtnaNova [96, 133]. This system, in fact, consistently with its foundation which is the Zermelo-Fraenkel theory, ultimately represents every entity in the user's domains of discourse as a set; the framework it provides offers infinite sets also, but these are not relevant for our present purposes.



(a) an e.a.o. D of a graph G (b) a perfect matching of G (c) a Hamiltonian cycle of G^2

Figure 5.2: An extensional acyclic digraph D satisfying the hypothesis of Corollary 5.1.3; its underlying graph has a claw.

We now report on a formalization of the two preceding proofs in Referee; in order to keep the technicalities to a minimum, the first property will be given in its weaker form, that of Hamiltonicity of squares of connected claw-free graphs. The complete Referee proof-scenario is given in Appendix B.

Our endeavor is legitimatized by the following representation theorem, which is an immediate corollary of Theorem 4.3.3 and of Mostowski's collapsing lemma (Lemma 1.3.2).

Theorem 5.2.1 *If $G = (V, E)$ is a connected claw-free graph, then there exists a finite transitive set x_G and a bijection $f : V \to x_G$ so that $uv \in E$ if and only if either $fu \in fv$ or $fv \in fu$.*

Occasionally, in a situation like the one described in this representation theorem, we will refer to G as the graph *underlying* the set x_G, and denote it as $G(x_G)$.

5.2.1 The Referee system in general

The proof-checker Referee, or just 'Ref' for brevity, processes *proof scenarios* to establish whether or not they are formally correct. A scenario, typically written by a working mathematician or computer scientist, consists of definitions, theorem statements, proofs of the theorems, and 'theories' (see below); as shown in Figure 5.3, one can intermix comments with these syntactical entities.

The deductive system underlying Ref is a variant of the Zermelo-Fraenkel set theory: this is evident from the syntax of the language, which borrows from the set-theoretic tradition many constructs, e.g. abstraction terms such as the set-former $\{\mathsf{u} : \mathsf{v} \in \mathsf{X}, \mathsf{u} \in \mathsf{v}\}$ used as *definiens* for the union-set global operation $\bigcup \mathsf{X}$; set theory also reflects into the semantics of the inference rules: for example, the inclusion $\{\mathsf{u} : \mathsf{v} \in \mathsf{x}_0, \mathsf{u} \in \mathsf{v}\} \subseteq \{\mathsf{u} : \mathsf{v} \in \mathsf{x}_0 \cup \{\mathsf{y}_0\}, \mathsf{u} \in \mathsf{v}\}$ can be proved in a single step as an application of the inference rule named Set_monot. Collectively, the inference rules embody almost every feature of the Zermelo-Fraenkel axioms: the only axiom of set theory which Ref maintains as an explicit assumption is, in fact, the one stating that there exist infinite sets.

DEF unionset: [Family of all members of members of a set]

$$\bigcup X =_{\text{Def}} \{u : v \in X, u \in v\}$$

THM 2*e*: [Union of adjunction] $\bigcup(X \cup \{Y\}) = Y \cup \bigcup X$. PROOF:

Suppose_not$(x_0, y_0) \Rightarrow$ *Stat0* : $\bigcup(x_0 \cup \{y_0\}) \neq y_0 \cup \bigcup x_0$

$\langle a \rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $a \in \bigcup(x_0 \cup \{y_0\}) \not\leftrightarrow a \in y_0 \cup \bigcup x_0$

Arguing by contradiction, let x_0, y_0 be a counterexample, so that in either one of $\bigcup(x_0 \cup \{y_0\})$ and $y_0 \cup \bigcup x_0$ there is an a not belonging to the other set. Taking the definition of $\bigcup$ into account, by monotonicity we must exclude the possibility that $a \in \bigcup x_0 \backslash \bigcup(x_0 \cup \{y_0\})$; through variable-substitution, we must also discard the possibility that $a \in \bigcup(x_0 \cup \{y_0\}) \backslash \bigcup x_0 \backslash y_0$.

Set_monot $\Rightarrow$ $\{u : v \in x_0, u \in v\} \subseteq \{u : v \in x_0 \cup \{y_0\}, u \in v\}$

Suppose $\Rightarrow$ *Stat1* : $a \in \{u : v \in x_0 \cup \{y_0\}, u \in v\}$ & $a \notin \{u : v \in x_0, u \in v\}$ & $a \notin y_0$

$\langle v_0, u_0, v_0, u_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Use_def$(\bigcup) \Rightarrow$ *Stat2* : $a \notin \{u : v \in x_0 \cup \{y_0\}, u \in v\}$ & $a \in y_0$

The only possibility left, namely that $a \in y_0 \backslash \bigcup(x_0 \cup \{y_0\})$, is also manifestly absurd. This contradiction leads us to the desired conclusion.

$\langle y_0, a \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM 31*h*: [Less-one lemma for union set]

$\bigcup M = T \backslash \{C\}$ & $S = T \cup X \cup \{V\}$ & $Y = V \vee (C = Y$ & $Y \in S) \rightarrow$
$\langle \exists d \mid \bigcup(M \cup \{X \cup \{Y\}\}) = S \backslash \{d\} \rangle$. PROOF:

Suppose_not$(m, t, c, s, x, v, y) \Rightarrow$ *Stat0* : $\neg\langle \exists d \mid \bigcup(m \cup \{x \cup \{y\}\}) = s \backslash \{d\} \rangle$ & $\bigcup m = t \backslash \{c\}$ & $s = t \cup x \cup \{v\}$ & $y = v \vee (c = y$ & $y \in s)$

For, supposing the contrary, $\bigcup(m \cup \{x \cup \{y\}\})$ would differ from each of $s \backslash \{s\}$, $s \backslash \{c\}$, and $s \backslash \{v\}$, the first of which equals s. Thanks to THM 2*e*, we can rewrite $\bigcup(m \cup \{x \cup \{y\}\})$ as $x \cup \{y\} \cup \bigcup m$; but then the decision algorithm for a fragment of set theory known as 'multi-level syllogistic with singleton' yields an immediate contradiction.

$\langle s \rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $\bigcup(m \cup \{x \cup \{y\}\}) \neq s$

$\langle c \rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $\bigcup(m \cup \{x \cup \{y\}\}) \neq s \backslash \{c\}$

$\langle v \rangle \hookrightarrow$ *Stat0* $\Rightarrow$ $\bigcup(m \cup \{x \cup \{y\}\}) \neq s \backslash \{v\}$

$\langle m, x \cup \{y\} \rangle \hookrightarrow$ *T2e* $\Rightarrow$ AUTO

EQUAL $\Rightarrow$ *Stat1* : $x \cup \{y\} \cup \bigcup m \neq s \backslash \{c\}$ & $x \cup \{y\} \cup \bigcup m \neq s \backslash \{v\}$ & $x \cup \{y\} \cup \bigcup m \neq s$

(*Stat0*, *Stat1*)Discharge $\Rightarrow$ QED

Figure 5.3: Tiny scenario for Ref.

Definitions often introduce abbreviating notation such as the union-set operation in the example just made, sometimes they bring into play sophisticated recursive notions such at the one of rank, to be seen in passing in Section 5.2.2.

Proofs are formed by two-component lines: the second component of each line is the claim being inferred, the first component hints at the inference rule being used to derive it. E.g., the hint Use_def$(\bigcup)$ suggests that one is expanding previous occurrences of the symbol $\bigcup$ inside the proof by the appropriate definition. Most often, the claim of a proof

$\textsc{Theory}$ $\mathsf{finiteInduction}(\mathsf{s}_0, \mathsf{P}(\mathsf{S}))$
$\quad \mathsf{Finite}(\mathsf{s}_0)\ \&\ \mathsf{P}(\mathsf{s}_0)$
$\Rightarrow (\mathsf{fin}_\Theta)$
$\quad \langle \forall \mathsf{S} \mid \mathsf{S} \subseteq \mathsf{fin}_\Theta \rightarrow \mathsf{Finite}(\mathsf{S})\ \&\ \big(\mathsf{P}(\mathsf{S}) \leftrightarrow \mathsf{S} = \mathsf{fin}_\Theta\big)\rangle$
$\textsc{End}$ $\mathsf{finiteInduction}$

Figure 5.4: A finite induction mechanism.

line is not sharply determined by the lines and the hint that precede it in the proof. Thus, for example, it is entirely a matter of taste whether to derive $\bigcup(\mathsf{m} \cup \{\mathsf{x} \cup \{\mathsf{y}\}\}) \neq \mathsf{s}$ or $\bigcup(\mathsf{m} \cup \{\mathsf{x} \cup \{\mathsf{y}\}\}) \neq \mathsf{s}\backslash\{\mathsf{s}\}$ as the second step in the second proof of Figure 5.3.

Proof encapsulation in Ref

> *Beyond this, definitions serve to 'instantiate', that is, to introduce the objects whose special properties are crucial to an intended argument. Like the selection of crucial lines, points, and circles from the infinity of geometric elements that might be considered in a Euclidean argument, definitions of this kind often carry a proof's most vital ideas.* (J. T. Schwartz, [133, p. 9])

The proof-checker Ref has a construct named THEORY, aimed at proof reuse, akin to a mechanism for parameterized specifications of the Clear specification language [22]. Besides providing theorems of which it holds the proofs, a THEORY has the ability to instantiate 'objects whose special properties are crucial to an intended argument'. Like procedures of a programming language, Ref's THEORYs have input formal parameters, in exchange of whose actualization they supply useful information. Actual input parameters must satisfy a conjunction of statements, called the *assumptions* of the THEORY. A THEORY usually encapsulates the definitions of entities related to the input parameters and it supplies, along with some consequences of the assumptions, theorems talking about these internally defined entities, which the THEORY returns as output parameters.[1] After having been derived by the user once and for all inside the THEORY, the consequences of the assumptions, as well as the claims involving the output parameters, are available to be exploited repeatedly.

A simple yet significant example is the THEORY finiteInduction displayed in Figure 5.4, which receives a finite set s_0 along with a property P such that $\mathsf{P}(\mathsf{s}_0)$ holds; in exchange, it will return a 'minimal witness' of P, i.e., a finite set fin_Θ satisfying $\mathsf{P}(\mathsf{fin}_\Theta)$ none of whose strict subsets t satisfies $\mathsf{P}(\mathsf{t})$.

5.2.2 The Referee system in action

For clarity, we revisit here the proof of Theorem 5.1.1, simplified to show only the property recast formally as a Ref proof-scenario, namely that squares of connected claw-free graphs are Hamiltonian.

Theorem 5.2.2 *If G is a connected claw-free graph with at least three vertices, and $S \subseteq V(G)$ is the set of sources of an acyclic orientation of G with exactly one sink, then G^2*

[1] As a visible countersign, the formal output parameters of a THEORY must carry the Greek letter Θ as a subscript.

has a Hamiltonian cycle C such that for every $s \in S$, at least one edge of C incident to s belongs to $E(G)$.

Proof. Arguing as in the preceding proof, unless G has 3 or 4 vertices, we select a 'pivotal' pair x, y, so that $|N^-(y)| \leqslant 2$; moreover, $G - N^-(y)$ has at least 3 vertices and is connected, since otherwise D would have two sinks. In applying the inductive hypothesis to $H = G - N^-(y)$, take the orientation induced by D, so that y is a source, and consider a Hamiltonian cycle C of H containing an edge $yw \in E(G)$.

If $N^-(y) = \{x\}$, notice that $xw \in E(G^2)$. Obtain a Hamiltonian cycle for G by replacing the edge yw in C by the path yxw (so that $xy \in E(G)$). If $N^-(y) = \{x, z\}$, due to the claw-freeness of G at least one of the edges xw or zw, say xw, belongs to G. Moreover, $zx \in E(G^2)$. Obtain a Hamiltonian cycle for G by replacing the edge yw in C with the path $yzxw$ (so that $yz, xw \in E(G)$). ■

Our formal specification of the above stated theorem and of Theorem 5.1.2 will refer to claw-free and transitive sets, instead of to claw-free graphs. Thus, as explained at the beginning of this section, the orientation of edges can be left as implicit; moreover, the unique-sink assumption will readily ensue from extensionality.

Down-to-earth notions for our experiment

In the first place we must define the notions of finiteness and transitivity of a set, for the former of which we can rely on [142]. Both notions presuppose the power-set operation, which we also specify here—its companion union-set operation has been introduced in Section 5.2.1.

DEF $\mathcal{P}$: [Family of all subsets of a given set]	$\mathcal{P}\mathsf{S}$	$=_{\text{Def}}$	$\{\mathsf{x} : \mathsf{x} \subseteq \mathsf{S}\}$
DEF Fin: [Finiteness]	Finite(F)	$\leftrightarrow_{\text{Def}}$	$\langle \forall \mathsf{g} \in \mathcal{P}(\mathcal{P}\mathsf{F}) \setminus \{\emptyset\}, \exists \mathsf{m} \mid \mathsf{g} \cap \mathcal{P}\mathsf{m} = \{\mathsf{m}\} \rangle$
DEF transitivity: [Transitive set]	Trans(T)	$\leftrightarrow_{\text{Def}}$	$\{\mathsf{y} \in \mathsf{T} \mid \mathsf{y} \not\subseteq \mathsf{T}\} = \emptyset$

Pre-existing ancillary properties about these constructs were available for reuse or readaptation in a shared common Ref scenario, cf. Figure 5.5.

Next come our definitions of claws and claw-free *sets*. In the second of these, the assumption that S is transitive is omitted and left pending to be introduced explicitly in the pertaining theorems.

DEF claw: [Pair characterizing a claw, possibly endowed with more than 3 el'ts]
Claw(Y, F) $\leftrightarrow_{\text{Def}}$ $\mathsf{F} \cap \bigcup \mathsf{F} = \emptyset\ \&$
$\langle \exists \mathsf{x}, \mathsf{z}, \mathsf{w} \mid \mathsf{F} \supseteq \{\mathsf{x}, \mathsf{z}, \mathsf{w}\}\ \&\ \mathsf{x} \neq \mathsf{z}\ \&\ \mathsf{w} \notin \{\mathsf{x}, \mathsf{z}\}\ \&\ \{\mathsf{w}\} \cap \mathsf{Y} \supseteq \{\mathsf{v} \in \mathsf{F} \mid \mathsf{Y} \notin \mathsf{v}\} \rangle$
DEF clawFreeness: [Claw-freeness, for a membership digraph]
ClawFree(S) $\leftrightarrow_{\text{Def}}$ $\langle \forall \mathsf{y} \in \mathsf{S}, \mathsf{e} \subseteq \mathsf{S} \mid \neg \mathsf{Claw}(\mathsf{y}, \mathsf{e}) \rangle$

A claw is thereby defined to be a pair y, F of sets such that:

1. F has at least three elements,
2. no element of F belongs to any other element of F,
3. either y belongs to all elements of F or there is a $w \in y$ such that y belongs to all elements of $F \setminus \{w\}$.

Thm $2a$: [Union of doubletons and singletons] $\mathsf{Z} = \{\mathsf{X}, \mathsf{Y}\} \rightarrow \bigcup \mathsf{Z} = \mathsf{X} \cup \mathsf{Y}$
Thm $2c$: [Additivity and monotonicity of monadic union]
$\bigcup(\mathsf{X} \cup \mathsf{Y}) = \bigcup \mathsf{X} \cup \bigcup \mathsf{Y} \;\&\; (\mathsf{Y} \supseteq \mathsf{X} \rightarrow \bigcup \mathsf{Y} \supseteq \bigcup \mathsf{X})$
Thm $2e$: [Union of adjunction] $\bigcup(\mathsf{X} \cup \{\mathsf{Y}\}) = \mathsf{Y} \cup \bigcup \mathsf{X}$
Thm $3a$: [The unionset of a transitive set is included in it] $\mathsf{Trans}(\mathsf{T}) \leftrightarrow \mathsf{T} \supseteq \bigcup \mathsf{T}$
Thm $3c$: [For a transitive set, elements are also subsets] $\mathsf{Trans}(\mathsf{T}) \;\&\; \mathsf{X} \in \mathsf{T} \rightarrow \mathsf{X} \subseteq \mathsf{T}$
Thm $3d$: [Trapping phenomenon for trivial sets] $\mathsf{Trans}(\mathsf{S}) \;\&\; \mathsf{X}, \mathsf{Z} \in \mathsf{S} \;\&\; \mathsf{X} \notin \mathsf{Z} \;\&\; \mathsf{Z} \notin \mathsf{X} \;\&$
$\mathsf{S} \setminus \{\mathsf{X}, \mathsf{Z}\} \subseteq \{\emptyset, \{\emptyset\}\} \rightarrow \mathsf{S} \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
Thm $4b$: [$\emptyset$ belongs to any nonnull transitive set t, $\{\emptyset\}$ also does if $t \not\subseteq \{\emptyset\}$, and so on]
$\mathsf{Trans}(\mathsf{T}) \;\&\; \mathsf{N} \in \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \;\&\; \mathsf{T} \not\subseteq \mathsf{N} \rightarrow$
$\mathsf{N} \subseteq \mathsf{T} \;\&\; \big(\mathsf{N} \in \mathsf{T} \vee (\mathsf{N} = \{\emptyset, \{\emptyset\}\} \;\&\; \{\{\emptyset\}\} \in \mathsf{T})\big)$
Thm $4c$: [Source removal from a transitive set does not disrupt transitivity]
$\mathsf{Trans}(\mathsf{S}) \;\&\; \mathsf{S} \supseteq \mathsf{T} \;\&\; (\mathsf{S} \setminus \mathsf{T}) \cap \bigcup \mathsf{S} = \emptyset \rightarrow \mathsf{Trans}(\mathsf{T})$
Thm 24: [Monotonicity of finiteness] $\mathsf{Y} \supseteq \mathsf{X} \;\&\; \mathsf{Finite}(\mathsf{Y}) \rightarrow \mathsf{Finite}(\mathsf{X})$
Thm $31d$: [Unionset of $\emptyset$ and $\{\emptyset\}$] $\mathsf{Y} \subseteq \{\emptyset\} \leftrightarrow \bigcup \mathsf{Y} = \emptyset$
Thm $31f$: [Unionset of a set obtained through removal followed by adjunction]
$\bigcup \mathsf{M} \supseteq \mathsf{P} \;\&\; \mathsf{Q} \cup \mathsf{R} = \mathsf{P} \cup \mathsf{S} \rightarrow \bigcup(\mathsf{M} \setminus \{\mathsf{P}\} \cup \{\mathsf{Q}, \mathsf{R}\}) = \bigcup \mathsf{M} \cup \mathsf{S}$
Thm $31h$: [Less-one lemma for unionset]
$\bigcup \mathsf{M} = \mathsf{T} \setminus \{\mathsf{C}\} \;\&\; \mathsf{S} = \mathsf{T} \cup \mathsf{X} \cup \{\mathsf{V}\} \;\&\; \mathsf{Y} = \mathsf{V} \vee (\mathsf{C} = \mathsf{Y} \;\&\; \mathsf{Y} \in \mathsf{S}) \rightarrow$
$\langle \exists \mathsf{d} \mid \bigcup(\mathsf{M} \cup \{\mathsf{X} \cup \{\mathsf{Y}\}\}) = \mathsf{S} \setminus \{\mathsf{d}\} \rangle$
Thm 32: [Finite, nonnull sets, own sources] $\mathsf{Finite}(\mathsf{F}) \;\&\; \mathsf{F} \neq \emptyset \rightarrow \mathsf{F} \setminus \bigcup \mathsf{F} \neq \emptyset$

Figure 5.5: Basic laws about $\bigcup$, Trans and Finite.

Accordingly, a *claw-free set* will be one which does not include a claw. For that, it suffices that it does not contain a claw y, F with $|F| = 3$, like the one shown in Figure 5.6.

On the basis of these definitions, one easily proves the monotonicity of claw-freeness, along with two slightly less obvious properties:

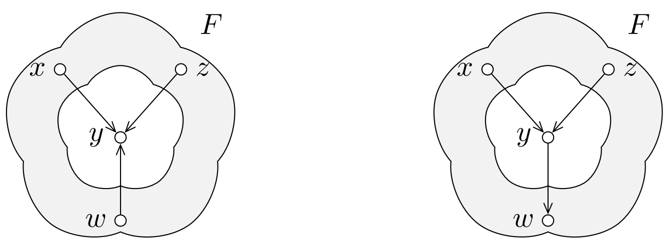


Figure 5.6: The forbidden orientations of a claw in a claw-free set.

THM $\mathsf{clawFreeness_a}$: [Subsets of claw-free sets are claw-free]
$\mathsf{ClawFree(S)} \;\&\; \mathsf{T} \subseteq \mathsf{S} \rightarrow \mathsf{ClawFree(T)}$
THM $\mathsf{clawFreeness_b}$: [In a claw-free set, any potential claw must have a bypass]
$\mathsf{ClawFree(S)} \;\&\; \mathsf{S} \supseteq \{\mathsf{Y},\mathsf{X},\mathsf{Z},\mathsf{W}\} \;\&\; \mathsf{Y} \in \mathsf{X} \cap \mathsf{Z} \;\&\; \mathsf{W} \in \mathsf{Y} \;\&\; \mathsf{X} \notin \mathsf{Z} \cup \{\mathsf{Z}\} \;\&\; \mathsf{Z} \notin \mathsf{X} \rightarrow \mathsf{W} \in \mathsf{X} \cup \mathsf{Z}$
THM $\mathsf{clawFreeness_0}$: [Pivots in a claw-free set own at most two predecessors therein]
$\mathsf{ClawFree(S)} \;\&\; \mathsf{X} \in \mathsf{S} \;\&\; \mathsf{Y} \in \mathsf{X} \cap \mathsf{S} \setminus \bigcup(\mathsf{S} \cap \bigcup \mathsf{S}) \rightarrow \langle \exists \mathsf{z} \in \mathsf{S} \mid \{\mathsf{v} \in \mathsf{S} \mid \mathsf{Y} \in \mathsf{v}\} = \{\mathsf{X},\mathsf{z}\} \;\&\; \mathsf{Y} \in \mathsf{z} \rangle$

To comment on the third of these, we switch back to our view of a set s as being a digraph $D(s)$ with sources $s \setminus \bigcup s$. Relevant for what is to follow, we will focus on the *pivots* of $D(s)$, which we define to be the elements of $(\bigcup s) \setminus \bigcup(s \cap \bigcup s)$; in graph-theoretic terms, $y \in s$ is a pivot of $D(s)$ if y is an out-neighbor of a source of $D(s)$, but is not at the end of any directed path included in s whose length exceeds 1. The salient property of a pivot y is that if x, z are in-neighbors of y, then neither $x \in z$ nor $z \in x$ holds, thanks to the claim

$$\text{THM } 31g.\ \mathsf{Y} \in \mathsf{X} \;\&\; \mathsf{X} \in \mathsf{Z} \;\&\; \mathsf{X},\mathsf{Z} \in \mathsf{S} \rightarrow \mathsf{Y} \in \bigcup(\mathsf{S} \cap \bigcup \mathsf{S})$$

whose contrapositive ensures, when $y \notin \bigcup(s \cap \bigcup s)$, the incomparability of x and z.

When s is transitive, the set of pivots reduces to $(\bigcup s) \setminus \bigcup\bigcup s$; but in order to state THM $\mathsf{clawFreeness_0}$ in its most basic form, we avoid this assumption here. The claim hence is that in a claw-free set the in-neighbors of a pivot $y \in x \in s$ form a set $\{x, z\}$, possibly singleton. This is straightforward: should y have three in-neighbors x, z, w, the pair $y, \{x, z, w\}$ would be a claw.

A crucial auxiliary theory

Two instantiating mechanisms play a key role in our proof-pearl scenario. One relates to the finiteness of the graphs under study here: this assumption conveniently reflects into the induction principle discussed in Section 5.2.1.

The other THEORY more specifically reflects our claw-freeness and transitivity assumptions; it factors out a mathematical insight which is common to the two main proofs on which we are reporting. Essentially, it says that in a transitive claw-free set $\mathsf{s_0} \not\subseteq \{\emptyset\}$ we can always select a pivot $\mathsf{y_\Theta}$ and its in-neighborhood $\{\mathsf{x_\Theta}, \mathsf{z_\Theta}\}$. Along with $\mathsf{y_\Theta}, \mathsf{x_\Theta}, \mathsf{z_\Theta}$, this THEORY returns the set $\mathsf{t_\Theta} = \mathsf{s_0} \setminus \{\mathsf{x_\Theta}, \mathsf{z_\Theta}\} = \{\mathsf{v} \in \mathsf{s_0} \mid \mathsf{y_\Theta} \notin \mathsf{v}\}$, strictly included in $\mathsf{s_0}$; in its turn, $\mathsf{t_\Theta}$ is proved to be claw-free and transitive.

For an intuition of how the quadruple $\mathsf{x_\Theta}, \mathsf{y_\Theta}, \mathsf{z_\Theta}, \mathsf{t_\Theta}$ can be obtained, referring to the classical notion of *rank* recursively definable as

$$\mathsf{rank}(s) \quad =_{\text{Def}} \quad \begin{cases} 0 & \text{if } s = \emptyset \\ \max\{\, \mathsf{rank}(t) + 1 : t \in s \,\} & \text{otherwise,} \end{cases}$$

observe that a transitive set $\mathsf{s_0}$ not included in $\{\emptyset\}$ must have rank $r \geqslant 2$, and hence must have elements $\mathsf{x_\Theta}, \mathsf{y_\Theta}$ such that $\mathsf{y_\Theta} \in \mathsf{x_\Theta}$ and $\mathsf{rank}(\mathsf{y_\Theta}) = r - 2$.

Although a recursive definition such as the one of rank just seen is supported by Ref (as a benefit originating from the assumption that set membership is a well-founded relation), we preferred to avoid it in order to circumvent any possible complication that might ensue from an explicit handling of numbers.

THEORY pivotsForClawFreeness(s_0)
  ClawFree(s_0) & Trans(s_0) & Finite(s_0)
  $s_0 \not\subseteq \{\emptyset\}$
$\Rightarrow$ ($x_\Theta, y_\Theta, z_\Theta, t_\Theta$)
  $\langle \forall x \in s_0,\ y \in x \setminus \bigcup\bigcup s_0 \mid \langle \exists z \in s_0 \mid \{v \in s_0 \mid y \in v\} = \{x, z\}\ \&\ y \in z \rangle\rangle$
  $\{x_\Theta, y_\Theta, z_\Theta\} \subseteq s_0$
  $x_\Theta \notin z_\Theta\ \&\ z_\Theta \notin x_\Theta\ \&\ y_\Theta \in x_\Theta \cap z_\Theta \setminus \bigcup\bigcup s_0$
  $y_\Theta \in t_\Theta \setminus \bigcup t_\Theta\ \&\ t_\Theta = s_0 \setminus \{x_\Theta, z_\Theta\}\ \&\ t_\Theta = \{v \in s_0 \mid y_\Theta \notin v\}$
  ClawFree(t_Θ) & Trans(t_Θ)
END pivotsForClawFreeness

Figure 5.7: A key quadruple associated with a claw-free set.

As a surrogate for the rank notion, we conceal inside this THEORY the definition of the *frontier* of a set s: this consists of those elements s to which a pivot of s belongs:

DEF frontier: [Frontier of a set] $\quad \mathsf{front}(S) =_{\text{Def}} \{x \in S \mid x \cap S \setminus \bigcup(S \cap \bigcup S) \neq \emptyset\}$.

Aided by this notion, we get x_Θ and y_Θ by drawing arbitrarily the former from $\mathsf{front}(s_0)$, the latter from $x_\Theta \setminus \bigcup\bigcup s_0$. This presupposes, of course, a proof that $\mathsf{front}(s_0) \neq \emptyset$, a fact simply ensuing from the more general proposition

THM frontier$_1$. $\mathsf{Finite}(S \cap \bigcup S)\ \&\ S \cap \bigcup S \neq \emptyset \rightarrow \mathsf{front}(S) \neq \emptyset$,

applicable to s_0 thanks to the assumption $s_0 \not\subseteq \{\emptyset\}$ of the THEORY at hand. To conclude the development of this THEORY, one must show that $t_\Theta = \{v \in s_0 \mid y_\Theta \notin v\}$ is transitive, as follows from

THM frontier$_2$. $\mathsf{Trans}(S)\ \&\ X \in \mathsf{front}(S)\ \&\ Y \in X \setminus \bigcup\bigcup S\ \&\ T = \{z \in S \mid Y \notin z\} \rightarrow$
$\mathsf{Trans}(T)\ \&\ T \subseteq S\ \&\ X \notin T\ \&\ Y \in T \setminus \bigcup T$,

in view of THM 4c. from Figure 5.5.

Preparatory lemmas

Since an edge of a graph is represented as membership between two sets, we define a *perfect matching* to be a set of disjoint doubletons $\{x, y\}$ such that $y \in x$ holds.

DEF perfect_matching: [set of disjoint membership pairs]
$\mathsf{perfectMatching}(M) \leftrightarrow_{\text{Def}} \langle \forall p \in M, \exists x \in p, y \in x, \forall q \in M \mid x \in q \vee y \in q \rightarrow \{x, y\} = q \rangle$

The following theorems about perfect matchings admit straightforward proofs.

The last two of these reflect our proof strategy: THM perfectMatching$_3$ claims that we can extend a matching by insertion of a doubleton of new sets, while THM perfectMatching$_4$ states conditions under which we can break a pair $\{y, w\}$ of a matching M into two doubletons $\{y, z\}$ and $\{x, w\}$ (see Figure 5.8).

THM $\mathsf{perfectMatching}_0$: [The null set is a perfect matching] $\mathsf{perfectMatching}(\emptyset)$
THM $\mathsf{perfectMatching}_2$: [All subsets of a perfect matching are perfect matchings]
$\mathsf{perfectMatching}(\mathsf{M})\ \&\ \mathsf{M} \supseteq \mathsf{N} \rightarrow \mathsf{perfectMatching}(\mathsf{N})$
THM $\mathsf{perfectMatching}_3$: [Bottom-up assembly of a finite perfect matching]
$\mathsf{perfectMatching}(\mathsf{M})\ \&\ \mathsf{X} \notin \bigcup \mathsf{M}\ \&\ \mathsf{Y} \notin \bigcup \mathsf{M}\ \&\ \mathsf{Y} \in \mathsf{X} \rightarrow$
$\mathsf{perfectMatching}(\mathsf{M} \cup \{\{\mathsf{X},\mathsf{Y}\}\})$
THM $\mathsf{perfectMatching}_4$: [Deviated perfect matching] $\mathsf{perfectMatching}(\mathsf{M})\ \&$
$\{\mathsf{Y},\mathsf{W}\} \in \mathsf{M}\ \&\ \mathsf{X} \notin \bigcup \mathsf{M}\ \&\ \mathsf{Z} \notin \bigcup \mathsf{M}\ \&\ \mathsf{Y} \in \mathsf{Z}\ \&\ \mathsf{Y} \neq \mathsf{X}\ \&\ \mathsf{X} \neq \mathsf{Z}\ \&\ \mathsf{W} \in \mathsf{X} \rightarrow$
$\mathsf{perfectMatching}(\mathsf{M} \setminus \{\{\mathsf{Y},\mathsf{W}\}\} \cup \{\{\mathsf{Y},\mathsf{Z}\},\{\mathsf{X},\mathsf{W}\}\})$

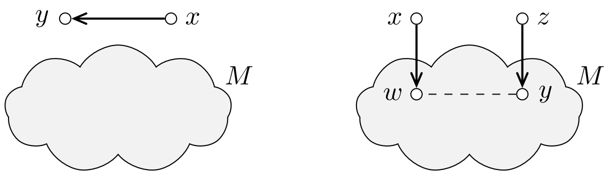


Figure 5.8: Two strategies for extending a perfect matching.

Next come our definitions pertaining to Hamiltonian cycles. These notions must refer to the edges in the square of a claw-free set, which will be formalized as unstructured doubletons. In order to define a Hamiltonian cycle, we can avoid speaking of sequences of vertices of a graph, and refer only to subsets of edges forming a cycle. This is done in two steps: we define $\mathsf{Hank}(H)$ to hold, for an $H \neq \emptyset$, if every element $x \in e \in H$ is a member of another element $q \neq e$ of H. Roughly speaking, this says that every end point of an edge of H has degree at least 2 in H; but notice that for the time being we are not insisting that H is formed by doubletons. Next, we define $\mathsf{Cycle}(C)$ to hold if $\mathsf{Hank}(C)$ holds and C is inclusion-minimal with this property (cf. [55, p. 288]).

DEF cycle_0: [Collection of edges whose endpoints have degree greater than 1]
$\mathsf{Hank}(\mathsf{H}) \quad \leftrightarrow_{\mathrm{Def}} \quad \emptyset \notin \mathsf{H}\ \&\ \langle \forall \mathsf{e} \in \mathsf{H} \mid \mathsf{e} \subseteq \bigcup (\mathsf{H} \setminus \{\mathsf{e}\}) \rangle$
DEF cycle_1: [Cycle (unless null)]
$\mathsf{Cycle}(\mathsf{C}) \quad \leftrightarrow_{\mathrm{Def}} \quad \mathsf{Hank}(\mathsf{C})\ \&\ \langle \forall \mathsf{d} \subseteq \mathsf{C} \mid \mathsf{Hank}(\mathsf{d})\ \&\ \mathsf{d} \neq \emptyset \rightarrow \mathsf{d} = \mathsf{C} \rangle$

Let us briefly digress to show that whenever C is a non-null subset of edges of a graph G and $\mathsf{Cycle}(C)$ holds, the subgraph $G[C]$ of G induced by the edges of C is a cycle in the customary sense. We argue first that $G[C]$ is not a forest and that it must contain a cycle (not necessarily induced). Otherwise, let P be the longest path in $G[C]$ and let its successive vertices be $x_1, \ldots, x_k$. Since $\mathsf{Hank}(C)$ holds, C must have an edge $x_1 x'$ with $x' \neq x_2$, also belonging to G. From the maximality of P we have that $x' \in P$, contradicting the supposed acyclicity of $G[C]$. If $G[C]$ is not a cycle, then we can find a strictly included induced cycle C' by picking a minimal-length closed walk of $G[C]$. Therefore $\mathsf{Hank}(C')$ holds, contradicting the fact that $\mathsf{Cycle}(C)$ holds.

Given an undirected graph (S, E), we say that $H \subseteq E$ is a *Hamiltonian cycle* of it if $\mathsf{Cycle}(H)$ holds, and each vertex v of S is *covered* by an edge e of H, in the sense that $v \in e$. Given a set s, we characterize the set of *square edges* of s by allowing only three of the four possible membership alignments of two sets x, y whose distance in the graph $G(s)$ underlying s is 1 or 2 (see Figure 5.9). These three configurations suffice in a proof

DEF hamiltonian$_1$: [Hamiltonian cycle, in graph without isolated vertices]
$\mathsf{Hamiltonian}(H,S,E) \leftrightarrow_{\text{Def}} \mathsf{Cycle}(H) \,\&\, \bigcup H = S \,\&\, H \subseteq E$
DEF hamiltonian$_2$: [Edges in squared membership]
$\mathsf{sqEdges}(S) =_{\text{Def}} \{\{x,y\} : x \in S, y \in S, z \in S \mid x \in y \vee (x \in z \,\&\, z \in y) \vee (z \in x \cap y \,\&\, x \neq y)\}$
DEF hamiltonian$_3$: [Restraining condition for Hamiltonian cycles]
$\mathsf{SqHamiltonian}(H,S) \leftrightarrow_{\text{Def}} \mathsf{Hamiltonian}\big(H,S,\mathsf{sqEdges}(S)\big) \,\&\, \langle \forall x \in S \setminus \bigcup S, \exists y \in x \mid \{x,y\} \in H \rangle$

of the announced theorem. To complete our setup, we need the notion of SqHamiltonian, which describes a Hamiltonian cycle H of a set s reflecting the claim of Proposition 5.2.2: in the first place, we require H to be Hamiltonian in the square of the underlying graph $G(s)$; secondly, H must cover each source of s by an edge of $G(s)$.

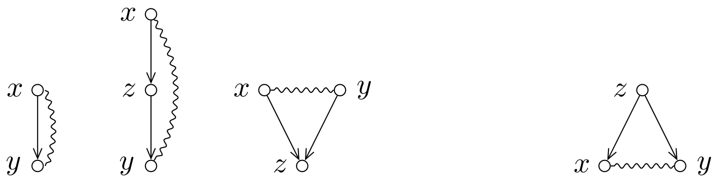
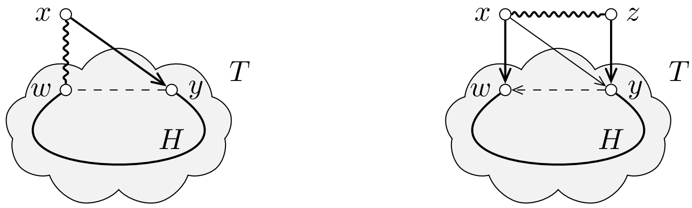


Figure 5.9: Four orientations of a path of length 1 or 2 between two vertices x and y; the last of these is not taken into account by our definition sqEdges.

The following theorems about Hamiltonian cycles admit straightforward proofs.

THM hamiltonian$_1$: [Enriched Hamiltonian cycles]
$S = T \cup \{X\} \,\&\, X \notin T \,\&\, Y \in X \,\&\, \mathsf{SqHamiltonian}(H,T) \,\&\, \{W,Y\} \in H \,\&\, W \in Y \vee (Y \in W \,\&\, K \neq Y \,\&\, \{W,K\} \in H \,\&\, K \in W) \rightarrow \mathsf{SqHamiltonian}(H \setminus \{\{W,Y\}\} \cup \{\{W,X\},\{X,Y\}\}, S)$
THM hamiltonian$_2$: [Doubly enriched Hamiltonian cycles]
$S = T \cup \{X,Z\} \,\&\, \{X,Z\} \cap T = \emptyset \,\&\, X \neq Z \,\&\, Y \in X \cap Z \,\&\, \mathsf{SqHamiltonian}(H,T) \,\&\, \{W,Y\} \in H \,\&\, W \in Y \cap X \rightarrow \mathsf{SqHamiltonian}(H \setminus \{\{W,Y\}\} \cup \{\{W,X\},\{X,Z\},\{Z,Y\}\}, S)$
THM hamiltonian$_3$: [Trivial Hamiltonian cycles]
$S = \{X,Y,Z\} \,\&\, X \in Y \,\&\, Y \in Z \rightarrow \mathsf{SqHamiltonian}(\{\{X,Y\},\{Y,Z\},\{Z,X\}\}, S)$
THM hamiltonian$_4$. [Any nontrivial transitive set whose square is devoid of Hamiltonian cycles must strictly comprise certain sets]
$\mathsf{Trans}(S) \,\&\, S \not\subseteq \{\emptyset,\{\emptyset\}\} \,\&\, \neg\langle \exists h \mid \mathsf{SqHamiltonian}(h,S)\rangle \rightarrow S \neq \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\} \,\&\, S \neq \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\} \,\&\, S \neq \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\} \,\&\, S \supseteq \{\emptyset,\{\emptyset\}\} \,\&\, (\{\{\emptyset\}\} \in S \vee \{\emptyset,\{\emptyset\}\} \in S)$

The last two of these will serve as base case for the proof we are after, namely the case when a transitive set s has 3 or 4 elements. In particular, if $s = \{x,y,z\}$ is a transitive tripleton, then its elements are $x = \emptyset$, $y = \{\emptyset\}$, $z = \{\{\emptyset\}\} \vee z = \{\emptyset,\{\emptyset\}\}$, and x, z form a square edge; therefore $\{x,y\}$, $\{y,z\}$, $\{z,x\}$ form a hank, and then clearly a cycle, because hanks of cardinality 1 or 2 do not exist. When s has 5 elements of more, then, mimicking the proof of Proposition 5.2.2 seen above, we will proceed differently, depending on whether the selected pivot of s belongs to a single element of s, or to two: Thm $\mathsf{hamiltonian}_2$ will serve us when s has two such predecessors, and Thm $\mathsf{hamiltonian}_1$ will settle the other case.



(a) (b)

5.2.3 Specifications of Hamiltonicity proof and of the perfect matching theorem

We will now examine in detail our formal reconstruction of Proposition 5.2.2, as readjusted for membership digraphs and certified correct with Ref.

Assuming the contrary, let s_1 be a finite transitive claw-free set with at least three elements, i.e. $\mathsf{s}_1 \not\subseteq \{\emptyset, \{\emptyset\}\}$, which does not have a Hamiltonian cycle in its square (step **1**). By the finiteInduction THEORY, there would exist an inclusion-minimal finite transitive non-trivial claw-free set s_0 likewise lacking such a cycle (steps **2**, **3**).

The THEORY pivotsForClawFreeness can be applied to s_0 (step **4**): we thereby pick an element x from the frontier of s_0, and an element y of x which is pivotal relative to s_0. This y will have at most two in-neighbors (one of the two being x) in s_0. We denote by z an in-neighbor of y in s_0, such that z differs from x, if possible. Observe, among others, that neither one of x, z can belong to the other.

If the removal of x, z from s_0 leads to a set t included in $\{\emptyset, \{\emptyset\}\}$ (step **5**), then by Thm $3d$ we get $\mathsf{s}_0 \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$. This leads us to a contradiction, in light of Thm $\mathsf{hamiltonian}_4$ (step **7**). Therefore, t is not trivial and the inductive hypothesis applies to it (step **8**): thanks to that hypothesis, we can find a Hamiltonian cycle h_0 for t (step **9**).

Recalling the definitions of Hamiltonian and sqHamiltonian (steps **10**, **11**), it follows from y being a source of $\mathsf{t} = \bigcup \mathsf{h}_0$ that there is an edge $\{\mathsf{y}, \mathsf{w}\}$ in h_0, with $\mathsf{w} \in \mathsf{y}$ (step **12**).

If $\mathsf{x} = \mathsf{z}$, the set $\mathsf{h}_1 = \mathsf{h}_0 \setminus \{\{\mathsf{y}, \mathsf{w}\}\} \cup \{\{\mathsf{x}, \mathsf{y}\}, \{\mathsf{x}, \mathsf{w}\}\}$ is a Hamiltonian cycle for s_0, by Thm $\mathsf{hamiltonian}_1$ (step **14**). This conflicts with the minimality of s_0 (step **15**): in fact $\{\mathsf{x}, \mathsf{w}\}$ is a square edge, since $\mathsf{w} \in \mathsf{y}$ and $\mathsf{y} \in \mathsf{x}$ both hold.

On the other hand, if $\mathsf{x} \neq \mathsf{z}$, claw-freeness implies, via Thm $\mathsf{clawFreeness_b}$, that either $\mathsf{w} \in \mathsf{x}$ or $\mathsf{w} \in \mathsf{z}$ must hold (step **17**). Assume the former (step **18**), and put $\mathsf{h}_2 = \mathsf{h}_0 \setminus \{\{\mathsf{y}, \mathsf{w}\}\} \cup \{\{\mathsf{y}, \mathsf{z}\}, \{\mathsf{z}, \mathsf{x}\}, \{\mathsf{x}, \mathsf{w}\}\}$, where $\{\mathsf{x}, \mathsf{z}\}$ is a square edge and $\{\mathsf{x}, \mathsf{w}\}$ and $\{\mathsf{y}, \mathsf{z}\}$ are genuine edges incident in the sources x, z. By Thm $\mathsf{hamiltonian}_2$, h_2 is a Hamiltonian cycle for s_0 (step **19**), and we are again facing a contradiction (step **20**). The case $\mathsf{w} \in \mathsf{z}$ is entirely symmetric (steps **21**, **22**, **23**), which proves the initial claim.

The result on the existence of a perfect matching is usually referred to graphs whose set of vertices has an even cardinality, as we have done in our Theorem 5.1.2; but here, since numbers pop in only in this place, we omit the evenness constraint: transitive, claw-free

Non-trivial claw-free transitive sets have Hamiltonian squares

THM $\mathsf{clawFreeness_1}$.
$\mathsf{Finite}(S)\ \&\ \mathsf{Trans}(S)\ \&\ \mathsf{ClawFree}(S)\ \&\ S \not\subseteq \{\emptyset,\{\emptyset\}\} \rightarrow \langle \exists h \mid \mathsf{SqHamiltonian}(h,S)\rangle$. PROOF:

1 $\mathsf{Suppose\_not}(s_1) \Rightarrow$ AUTO

2 APPLY $\langle \mathsf{fin}_\Theta : s_0\rangle$ $\mathsf{finiteInduction}\big(s_0 \mapsto s_1,$
$P(S) \mapsto \big(\mathsf{Trans}(S)\ \&\ \mathsf{ClawFree}(S)\ \&\ S \not\subseteq \{\emptyset,\{\emptyset\}\}\ \&\ \neg\langle \exists h \mid \mathsf{SqHamiltonian}(h,S)\rangle\big)\big) \Rightarrow$
$\mathit{Stat1}: \langle \forall s \mid s \subseteq s_0 \rightarrow \mathsf{Finite}(s)\ \&\ \big(\mathsf{Trans}(s)\ \&\ \mathsf{ClawFree}(s)\ \&\ s \not\subseteq \{\emptyset,\{\emptyset\}\}\ \&$
$\neg\langle \exists h \mid \mathsf{SqHamiltonian}(h,s)\rangle \leftrightarrow s = s_0\big)\rangle$

3 $\langle s_0\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathit{Stat2}: \neg\langle \exists h \mid \mathsf{SqHamiltonian}(h,s_0)\rangle\ \&$
$\mathsf{Finite}(s_0)\ \&\ \mathsf{Trans}(s_0)\ \&\ \mathsf{ClawFree}(s_0)\ \&\ s_0 \not\subseteq \{\emptyset,\{\emptyset\}\}$

4 APPLY $\langle x_\Theta : x, y_\Theta : y, z_\Theta : z, t_\Theta : t\rangle$ $\mathsf{pivotsForClawFreeness}(s_0 \mapsto s_0) \Rightarrow$
$\{v \in s_0 \mid y \in v\} = \{x,z\}\ \&\ x,y,z \in s_0\ \&\ y \in x \cap z \setminus \bigcup\bigcup s_0\ \&$
$y \in t \setminus \bigcup t\ \&\ t = s_0 \setminus \{x,z\}\ \&\ t = \{u \in s_0 \mid y \notin u\}\ \&\ s_0 \supseteq t\ \&$
$\mathsf{Trans}(t)\ \&\ \mathsf{ClawFree}(t)\ \&\ x \notin t\ \&\ x \notin z\ \&\ z \notin x$

5 Suppose $\Rightarrow$ $t \subseteq \{\emptyset,\{\emptyset\}\}$

6 $\langle s_0,x,z\rangle \hookrightarrow \mathit{T3d} \Rightarrow$ $s_0 \subseteq \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}$

7 $\langle s_0\rangle \hookrightarrow \mathit{T}\mathsf{hamiltonian_4} \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

8 $\langle t\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathit{Stat9}: \langle \exists h \mid \mathsf{SqHamiltonian}(h,t)\rangle$

9 $\langle h_0\rangle \hookrightarrow \mathit{Stat9} \Rightarrow$ $\mathsf{SqHamiltonian}(h_0,t)$

10 $\mathsf{Use\_def}(\mathsf{Hamiltonian}(h_0,t,\mathsf{sqEdges}(t))) \Rightarrow$ AUTO

11 $\mathsf{Use\_def}(\mathsf{SqHamiltonian}) \Rightarrow$ $\mathit{Stat11}: \langle \forall x \in t \setminus \bigcup t, \exists y \in x \mid \{x,y\} \in h_0\rangle\ \&$
$\mathsf{Cycle}(h_0)\ \&\ \bigcup h_0 = t\ \&\ h_0 \subseteq \mathsf{sqEdges}(t)$

12 $\langle y,w\rangle \hookrightarrow \mathit{Stat11} \Rightarrow$ $w \in y\ \&\ \{w,y\} \in h_0$

13 Suppose $\Rightarrow$ $x = z$

14 $\langle s_0,t,x,y,h_0,w,\emptyset\rangle \hookrightarrow \mathit{T}\mathsf{hamiltonian_1} \Rightarrow$
$\mathsf{SqHamiltonian}(h_0 \setminus \{\{w,y\}\} \cup \{\{w,x\},\{x,y\}\},s_0)$

15 $\langle h_0 \setminus \{\{w,y\}\} \cup \{\{w,x\},\{x,y\}\}\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ false; Discharge $\Rightarrow$ $x \neq z$

16 $\langle s_0,y\rangle \hookrightarrow \mathit{T3c} \Rightarrow$ $w \in s_0$

17 $\langle s_0,y,x,z,w\rangle \hookrightarrow \mathit{T}\mathsf{clawFreeness_b} \Rightarrow$ $w \in x \cup z$

18 Suppose $\Rightarrow$ $w \in x$

19 $\langle s_0,t,x,z,y,h_0,w\rangle \hookrightarrow \mathit{T}\mathsf{hamiltonian_2} \Rightarrow$
$\mathsf{SqHamiltonian}(h_0 \setminus \{\{w,y\}\} \cup \{\{w,x\},\{x,z\},\{z,y\}\},s_0)$

20 $\langle h_0 \setminus \{\{w,y\}\} \cup \{\{w,x\},\{x,z\},\{z,y\}\}\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

21 ELEM $\Rightarrow$ $w \in z$

22 $\langle s_0,t,z,x,y,h_0,w\rangle \hookrightarrow \mathit{T}\mathsf{hamiltonian_2} \Rightarrow$
$\mathsf{SqHamiltonian}(h_0 \setminus \{\{w,y\}\} \cup \{\{w,z\},\{z,x\},\{x,y\}\},s_0)$

23 $\langle h_0 \setminus \{\{w,y\}\} \cup \{\{w,z\},\{z,x\},\{x,y\}\}\rangle \hookrightarrow \mathit{Stat2} \Rightarrow$ false; Discharge $\Rightarrow$ QED

sets admit a 'near-perfect matching' (see [71]), that is to say, a perfect matching which does not cover at most one of its elements. In Ref:

THM clawFreeness$_2$: [Every claw-free transitive set has a near-perfect matching]
$\mathsf{Finite}(\mathsf{S}) \;\&\; \mathsf{Trans}(\mathsf{S}) \;\&\; \mathsf{ClawFree}(\mathsf{S}) \rightarrow \langle \exists \mathsf{m}, \mathsf{y} \mid \mathsf{perfectMatching}(\mathsf{m}) \;\&\; \mathsf{S} \setminus \{\mathsf{y}\} = \bigcup \mathsf{m} \rangle$

As one sees from our previous treatment of Theorem 5.1.2, this theorem's proof bears a close resemblance with the proof about Hamiltonicity just detailed; hence it seems pointless to supply again here many formal details.

5.2.4 An outward look

To take advantage of the set-theoretic foundation of Referee, we exploited set equivalents of the graph-theoretic notions involved in our experiment: edge, source, square, etc. To ease some proofs, we have often resorted to weak counterparts of well-established notions such as cycle, claw-freeness, longest directed path, etc:

- In the above, we could have defined a transitive set to be *claw-free* if none of the four non-isomorphic membership renderings of a claw are induced by any quadruple of its elements; but actually, it sufficed to forbid two out of these four to get the desired proofs. This explains why our results are easier to achieve but under some respects more general. To see the difference, observe that the graph in Figure 5.2, once suitably oriented, can be handled by our theorems, whereas the Hamiltonicity of its square and its perfect matchings are not seen either by the traditional results [81, 141, 148], or by subsequent generalizations regarding *quasi claw-free graphs* [4], *almost claw-free graphs* [132], and $S(K_{1,3})$*-free graphs* [65].

- The graph 'squares' about which our Hamiltonicity proof speaks are actually poorer in edges than the standard ones, since we allow only three out of the four membership alignments, cf. Figure 5.9.

- We have addressed issues regarding graphs, which we see as pre-algorithmic and, as such, application-oriented. Nonetheless, our results are so close to the foundations of mathematics that we found no reason to introduce numbers, and we were able to avoid recursion even in the determination of the pivots, as explained in Section 5.2.2. For the time being, we succeeded even in doing without basic conceptual tools which, as we expect, will enter into play in continuations of this work; for example, the notion of spanning tree.

- The graphs that can be represented by sets form a broad class of graphs, which includes, in partial overlap with connected claw-free graphs, all graphs endowed with a Hamiltonian path. By allowing the presence of 'atoms' in our sets, as will emerge from the next section, we can actually represent all graphs.

To end, let us now place the results presented so far under the more general perspective motivating this work. We display in this section the interfaces of two representation THEORYs (not developed formally with Referee, as of today), and of a THEORY auxiliary to one of these two, explaining why we can work with membership as a convenient surrogate for the edge relationship of general graphs.

One of these, THEORY finGraphRepr, will implement the proof given in Section 4.2.1 that any finite graph $(\mathsf{v}_0, \mathsf{e}_0)$ is 'isomorphic', via a suitable orientation of its edges and an injection f of v_0 onto a set ν, to a digraph $(\nu, \{(x, y) : x \in \nu, y \in x \cap \nu\})$ enjoying weak extensionality. Sinks can, at taste, be seen as pairwise distinct *atoms* (or 'urelements' [70]) entering in the formation of the sets assigned to the internal vertices, or as sets whose internal structure is immaterial.

$$
\begin{array}{l}
\textsc{Theory}\ \mathsf{finGraphRepr}(\mathsf{v}_0, \mathsf{e}_0) \\
\quad \mathsf{Finite}(\mathsf{v}_0)\ \&\ \mathsf{e}_0 \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x},\mathsf{y} \in \mathsf{v}_0 \,|\, \mathsf{x} \neq \mathsf{y}\} \\
\Rightarrow (f_\Theta, \nu_\Theta) \\
\quad 1\text{--}1(f_\Theta)\ \&\ \mathbf{domain}(f_\Theta) = \mathsf{v}_0\ \&\ \mathbf{range}(f_\Theta) = \nu_\Theta \\
\quad \langle \forall \mathsf{x} \in \mathsf{v}_0, \mathsf{y} \in \mathsf{v}_0 \,|\, \{\mathsf{x},\mathsf{y}\} \in \mathsf{e}_0 \leftrightarrow f_\Theta\,\mathsf{x} \in f_\Theta\,\mathsf{y} \vee f_\Theta\,\mathsf{y} \in f_\Theta\,\mathsf{x} \rangle \\
\quad \{\mathsf{x} \in \nu_\Theta \,|\, \mathsf{x} \cap \nu_\Theta \neq \emptyset\} \subseteq \mathcal{P}(\nu_\Theta) \\
\textsc{End}\ \mathsf{finGraphRepr}
\end{array}
$$

Although accessory, the weak extensionality condition (last claim in the THEORY's interface just displayed) is the clue for getting the desired f; in fact, for any weakly extensional digraph, acyclicity always ensures that a variant of Mostowski's collapse is well-defined: in order to get it, one starts by assigning a distinct set Mt to each sink t and then proceeds by putting recursively

$$
Mw \quad = \quad \{\, Mu \,|\, (w, u) \text{ is an arc} \,\}
$$

for all non-sink vertices w; plainly, injectivity of the function $u \mapsto Mu$ can be ensured globally by a suitable choice of the images Mt of the sinks t. The said variant Mostowski's collapse for a well-founded weakly extensional digraph (even an infinite one) can be specified in Ref as a THEORY whose interface reads as follows:

$$
\begin{array}{l}
\textsc{Theory}\ \mathsf{mostowskiCollapse}(\mathsf{v}_0, \mathsf{a}_0) \\
\quad \langle \forall \mathsf{t} \subseteq \mathsf{v}_0, \exists \mathsf{m}, \forall \mathsf{x} \in \mathsf{t} \,|\, \mathsf{m} \in \mathsf{t}\ \&\ (\mathsf{m},\mathsf{x}) \notin \mathsf{a}_0 \rangle \\
\quad \langle \forall \mathsf{w} \in \mathsf{v}_0, \mathsf{w}' \in \mathsf{v}_0, \mathsf{u} \in \mathsf{v}_0 \,| \\
\qquad (\mathsf{w},\mathsf{u}) \in \mathsf{a}_0\ \&\ \{\mathsf{x} \in \mathsf{v}_0 \,|\, (\mathsf{w},\mathsf{x}) \in \mathsf{a}_0\} = \{\mathsf{x} \in \mathsf{v}_0 \,|\, (\mathsf{w}',\mathsf{x}) \in \mathsf{a}_0\} \rightarrow \mathsf{w} = \mathsf{w}' \rangle \\
\Rightarrow (\mathsf{M}_\Theta) \\
\quad 1\text{--}1(\mathsf{M}_\Theta)\ \&\ \mathbf{domain}(\mathsf{M}_\Theta) = \mathsf{v}_0 \\
\quad \langle \forall \mathsf{w} \in \mathsf{v}_0, \mathsf{u} \in \mathsf{v}_0 \,|\, (\mathsf{w},\mathsf{u}) \in \mathsf{a}_0 \rightarrow \mathsf{M}_\Theta \mathsf{w} = \{\mathsf{M}_\Theta \mathsf{u} : \mathsf{u} \in \mathsf{v}_0 \,|\, (\mathsf{w},\mathsf{u}) \in \mathsf{a}_0\} \rangle \\
\textsc{End}\ \mathsf{mostowskiCollapse}
\end{array}
$$

Our second representation THEORY, cfGraphRepr, will specialize finGraphRepr to the case of a *connected, claw-free* (undirected, finite) graph—connectedness and claw-freeness appear, respectively, as the second and the third assumption of this THEORY. For these graphs, we can insist that the orientation be so imposed as to ensure *extensionality* in full: "distinct vertices have different out-neighborhoods".

DEF connectedness: [Connectedness of a graph]
$\mathsf{Connected}(\mathsf{V},\mathsf{E}) \leftrightarrow_{\mathrm{Def}}$
$\langle \forall \mathsf{x} \in \mathsf{V}, \mathsf{y} \in \mathsf{V} \mid \mathsf{x} \neq \mathsf{y} \,\&\, \{\mathsf{x},\mathsf{y}\} \notin \mathsf{E} \rightarrow \langle \exists \mathsf{p} \subseteq \mathsf{E} \mid \mathsf{Cycle}(\mathsf{p} \cup \{\{\mathsf{y},\mathsf{x}\}\})\rangle\rangle$

DEF clawFreeGraph: [Claw-freeness of a graph]
$\mathsf{ClawFreeG}(\mathsf{V},\mathsf{E}) \leftrightarrow_{\mathrm{Def}} \langle \forall \mathsf{w} \in \mathsf{V}, \mathsf{x} \in \mathsf{V}, \mathsf{y} \in \mathsf{V}, \mathsf{z} \in \mathsf{V} \mid \{\mathsf{w},\mathsf{y}\}, \{\mathsf{y},\mathsf{x}\}, \{\mathsf{y},\mathsf{z}\} \in \mathsf{E} \rightarrow$
$\mathsf{x} = \mathsf{z} \vee \mathsf{w} \in \{\mathsf{z},\mathsf{x}\} \vee \{\mathsf{x},\mathsf{z}\} \in \mathsf{E} \vee \{\mathsf{z},\mathsf{w}\} \in \mathsf{E} \vee \{\mathsf{w},\mathsf{x}\} \in \mathsf{E}\rangle$

THEORY $\mathsf{cfGraphRepr}(\mathsf{v}_0, \mathsf{e}_0)$
$\mathsf{Finite}(\mathsf{v}_0) \,\&\, \mathsf{e}_0 \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x},\mathsf{y} \in \mathsf{v}_0 \mid \mathsf{x} \neq \mathsf{y}\}$
$\mathsf{Connected}(\mathsf{v}_0, \mathsf{e}_0)$
$\mathsf{ClawFreeG}(\mathsf{v}_0, \mathsf{e}_0)$
$\Rightarrow (f_\Theta, \nu_\Theta)$
$1\text{–}1(f_\Theta) \,\&\, \mathbf{domain}(f_\Theta) = \mathsf{v}_0 \,\&\, \mathbf{range}(f_\Theta) = \nu_\Theta$
$\langle \forall \mathsf{x} \in \mathsf{v}_0, \mathsf{y} \in \mathsf{v}_0 \mid \{\mathsf{x},\mathsf{y}\} \in \mathsf{e}_0 \leftrightarrow f_\Theta\,\mathsf{x} \in f_\Theta\,\mathsf{y} \vee f_\Theta\,\mathsf{y} \in f_\Theta\,\mathsf{x}\rangle$
$\mathsf{Trans}(\nu_\Theta) \,\&\, \mathsf{ClawFree}(\nu_\Theta)$
END cfGraphRepr

Consequently, the following will hold:

- there is a unique sink, $\emptyset$; moreover,
- the set ν underlying the image digraph is transitive. Also, rather trivially,
- ν is a claw-free set, in an even stronger sense than the definition with which we have been working throughout this paper.

Via the THEORY cfGraphRepr, the above-proved existence results about perfect matchings and Hamiltonian cycles can be transferred from the realm of membership digraphs to the *a priori* more general realm of connected claw-free graphs.

6

Sets Modeling Bernays-Schönfinkel-Ramsey $\forall^*$-formulae

Given a first-order language $\mathcal{L}$, the formulae of $\mathcal{L}$ having $\forall^*$ as their prenex prefix form the so-called $\mathcal{L}$-BSR class. If $\mathcal{L}$ is a language of graphs, the usual primitive relators of the signature of $\mathcal{L}$ are $=$ and E, where E is interpreted as the adjacency relation. In the case of sets, these relators are $=$ and $\in$, where $\in$ is interpreted as the membership relation.

A graph class defined by forbidden induced subgraphs can be characterized by a $\forall^*$-formula: the adjacencies between any tuple of vertices are required not to be the same as the adjacencies between the vertices of one of the forbidden subgraphs. Having a good understanding of the structure of graphs from such a class can guarantee, among others, tractability of otherwise NP-hard problems.

The analogous problem of characterizing sets satisfying a fixed $\forall^*$-formula is certainly rewarding. For example, Section 5.1 extended two properties of claw-free graphs to the case of sets, characterized analogously by a precise $\forall^4$-formula. However, in this chapter we will consider a more general question: If one considers the collection of *all* $\forall^*$-formulae, which are the sets that they can express? Since all hereditarily finite (hyper)sets can be characterized by a $\forall^*$-formula, this question becomes intriguing, and in a truly challenging manner, when asking for the *infinite* (hyper)sets that can be thus characterized.

If in the case of finite graphs the knowledge about their structure provides tractability results, insight about the structure of infinite sets assures a passage from undecidability to decidability. Indeed, after a research of more than 20 years, this understanding proved crucial in the recent proof that it can be decided whether, given a $\forall^*$-formula φ, there exists a tuple of well-founded sets that, substituted for its free variables, render φ true.

The main motivation of this chapter is the analogous decidability problem for hypersets. We therefore focus on $\forall^*$-formulae φ which express infinite non-well-founded sets, in the sense that, on the one hand, there are infinite hypersets which can be substituted for the free variables of φ to render it true; and that, on the other hand, this cannot be the case for hereditarily finite hypersets, or well-founded sets (be they hereditarily finite or not).

6.1 The decidability problem

Many algorithms have been found, over the years (cf. [23, 25]), that can establish whether a set-theoretic formula drawn from a specific fragment of the first-order set-theoretic language—having thus the signature $\{=, \in\}$—can, or cannot, be made true by means of a suitable assignment of set-values to its free variables. In this chapter we will consider one such fragment, called the *Bernays-Schönfinkel-Ramsey class.*

Definition 6.1.1 *The* Bernays-Schönfinkel-Ramsey class *consists of all first-order formulae whose prenex quantificational prefix is purely universal (BSR- or $\forall^*$-formulae, for short).*

The BSR-class has a long history for inspiring deep combinatorial results, starting with the celebrated theorem by Ramsey [126], established in order to study the *spectra* of its formulae. Recent results on the BSR class include [7] and the subsequent work on the expressivity of the BSR class, that the classification starting with that paper stimulated.

The classical decision problem for the logical BSR-class consists of the satisfiability problem for $\forall^*$-formulae whose unquantified matrix is written employing one binary relation symbol[1] and equality (see [20]). The decision problem for the satisfiability of the set theoretic BSR-class over *well-founded sets* asks for an algorithm that, given a BSR-formula $\varphi(x_1, \ldots, x_n)$, decides whether there exist well-founded sets $s_1, \ldots, s_n$ such that $\varphi(s_1, \ldots, s_n)$ is true. This research, started more than 20 years ago [112, 113], recently culminated with a proof that such an algorithm exists [100, 101].

Gaining a comprehensive understanding of what infinite structures are describable within specific syntactic restraints appears to be an essential prerequisite for any decision procedure for a set theoretic context. This line of attack was indeed employed for the BSR-class, where the $\forall\forall$ formulation of infinity of 1988-1990 [112, 113] unearthed a building block of any well-founded and infinite set theoretic interpretation for the free variables of a $\forall\forall$-formula [16]. An analogous, but more involved result holds for the general case as well, where formulations of infinity involving an arbitrary number of universally quantified variables and free variables have to be considered.

This finding proved crucial in the aforementioned decidability result, since it guaranteed that any finite family $\mathcal{F}$ of well-founded sets can be *thinned* into a family $\mathcal{F}'$ which satisfies the same BSR-formulae as $\mathcal{F}$ and whose structure is so devoid of redundancy and so regular that it can be described by means of a finite digraph. Note that although $\mathcal{F}$ is finite (because $\mathcal{F}$ consists of values to be substituted for the free variables of a formula), its members can either be infinite, or involve infinite sets in their structure.

This technique, often brought into play for such decidability results, can be summarized as follows. One starts by singling out those features of a finite family $\mathcal{F}$ that are relevant for the satisfaction of a formula φ in the language of interest under an assignment

$$\mathcal{M} : \mathit{vars}(\varphi) \longrightarrow \mathcal{F}$$

of set values to the free variables of φ. The main subtasks into which the decision analysis task gets subdivided, accordingly, are:

[1]More relational symbols would not make the problem more difficult.

1. distill from $\mathcal{F}$ another set $\mathcal{F}'$, whose structure is as regular as possible, along with an injection $x \mapsto x'$ of $\mathcal{F}$ into $\mathcal{F}'$, so that any φ' involving the same variables as φ gets the same truth value in $\mathcal{M}$ (as above) and in the corresponding assignment $x \mapsto \mathcal{M}(x)'$;

2. taking advantage of the regularities of $\mathcal{F}'$, represent $\mathcal{F}'$ by means of a suitable finite representation $\mathcal{G}$ (typically in the form of a digraph);

3. single out special features that a given $\mathcal{G}$ enjoys when it actually originates from the thinning of a finite family $\mathcal{F}$;

4. indicate how to evaluate, relative to $\mathcal{G}$, any conjunction φ' of constraints that involves at most $\#\mathcal{F}$ variables.

Of the above four items, 1 and 2 must be carried out as existence proofs (of $\mathcal{F}'$ and of $\mathcal{G}$, respectively), and cannot be regarded as algorithmic steps proper; indeed, some of the sets belonging to $\mathcal{F}$ and to $\mathcal{F}'$ may have transfinite cardinality or rank and, as such, cannot be algorithmically manipulated. Concerning, in particular, item 2, note that, due to the fact that there are BSR-formulae satisfied only by infinite sets—as shown in [112, 113] and as it will be seen in Section 6.2—, we cannot assume that the values of the free variables are hereditarily finite; this is why we need to resort to an indirect graph-representation.

On the other hand, if one succeeds, as hinted at at the above points 3 and 4, in implementing rules for evaluating formulae in a representing digraph instead of in the original assignment $\mathcal{M}$, as well as for determining whether a digraph can be induced by a finite family of sets (to wit, the values of the free variables in $\mathcal{M}$), one already owns a semi-decision algorithm. It is necessary to

5. place a computable bound on the overall size of each relevant digraph

to end up with the sought satisfiability decision test. Sometimes, more effective, goal-driven algorithms can be derived from such generate-and-test raw prototypes.

Most often, decidability results of this kind are referred to the standard universe of sets—the von Neumann's cumulative hierarchy—, over which membership behaves as a well-founded relation; but in a few cases (cf. [5, 45, 97, 99]), a known satisfiability decision algorithm could be recast in terms referring to a non-well-founded universe of sets such as Aczel's one. In designing these technically more difficult, out-of-standard, decision algorithms, invention can usually rely upon a certain analogy between the two conceptions of the domain of sets, despite the two being opposite. Staying neutral on whether membership is well-founded or not, does not pay: without a commitment in either direction, one generally loses a result—regarding either Zermelo-Fraenkel or an axiomatic first-order theory germane to it—of deductive completeness relative to a restricted set-theoretic language; while retaining, in its stead, only a minor form of decidability for that language, much less worth of notice [98].

The main motivation of this chapter is the analogous decidability problem over hypersets. We are embarking here on a feasibility study on how much of the solution for the above items 1–5 is transferrable to this non-well-founded context. If in the well-founded case one relies heavily on the ability to do recursion on rank, in the case of hypersets this notion seems to have no natural counter-part, at least for the means of this problem. Therefore, a study of the infinite non-well-founded structures expressible by

BSR-formulae—referring thus to items 1 and 2—is of utmost importance, as it allows one to prepare, test and fine tune his toolbox in tackling this out-of-standard set context.

6.1.1 The computational complexity of deciding satisfiability

Before starting to shed light on two decidable fragments of the BSR-class, we focus on finite set structures, and on their ability to encode other computational problems. We report in this section on a straightforward encoding of the propositional satisfiability problem. This result, emerging at the beginning of the 1990s, is also mentioned in [98] and in the monograph [25, Ch. 6.1.4]. The reduction we present follows [46, 48]; this reduction is simple enough not to make any assumption on the well-foundedness of the membership relation, therefore entailing that deciding whether a BSR-formula can be satisfied by a tuple of hereditarily finite set/hypersets is NP-hard.

For expository purposes, we will let the polyadic construct $a = \{x_1, \ldots, x_n\}$ stand for

$$(\forall y)\Big(y \in a \leftrightarrow \bigvee_{i=1}^{n} y = x_i\Big).$$

In particular, $a = \emptyset$ stands for $(\forall y)(y \notin a)$. Actually, the set theoretic BSR-class is expressive enough to encode also the standard **MLSS** (multilevel syllogistic with singleton) [23] unquantified language of set theory consisting of a denumerable infinity of set variables, the 'null set' constant $\emptyset$, the set operators $\bullet\cap\bullet$, $\bullet\setminus\bullet$, $\bullet\cup\bullet$, $\{\bullet, \ldots, \bullet\}$, and the set predicates $\bullet \in \bullet$, $\bullet = \bullet$, $\bullet \subseteq \bullet$.

Given a propositional formula in conjunctive normal form

$$\varphi \equiv \bigwedge_{i=1}^{n} \bigvee_{j=1}^{k_i} \ell_{i,j},$$

where each literal $\ell_{i,j}$ is drawn from a collection $x_1, \ldots, x_m, \neg x_1, \ldots, \neg x_m$ of propositional variables and their negations, we can consider a collection of set variables such that to each variable x_i there correspond two set variables P_i and N_i. Moreover, for each $i \in \{1, \ldots, n\}$ and each $j \in \{1, \ldots, k_i\}$, we put

$$L_{i,j} = \begin{cases} P_h & \text{if } \ell_{i,j} = x_h, \\ N_h & \text{if } \ell_{i,j} = \neg x_h. \end{cases}$$

The set encoding φ^s of φ is the following BSR-formula:

$$\varphi^s \equiv \Big\{\{\emptyset, \{\emptyset\}\}\Big\} =$$
$$\Big\{\{P_1, N_1\}, \ldots, \{P_n, N_n\},\ \{\emptyset, L_{1,1}, \ldots, L_{1,k_1}\}, \ldots, \{\emptyset, L_{n,1}, \ldots, L_{n,k_n}\}\Big\}.$$

Assume first that φ is satisfiable by a truth assignment to its propositional variables $x_1, \ldots, x_n$. If x_i is true, then put $P_i = \{\emptyset\}$ and $N_i = \emptyset$, otherwise, put $P_i = \emptyset$ and $N_i = \{\emptyset\}$. It is easy to see that this set assignment to the free variables of φ^s renders it true.

Conversely, assume that φ^s is satisfiable. We must have that $\{P_i, N_i\} = \{\emptyset, \{\emptyset\}\}$, for each $i \in \{1, \ldots, n\}$. If $P_i = \{\emptyset\}$, then put x_i true, otherwise, put x_i false. This is a consistent truth assignment. To see that it also satisfies φ, observe that at least one of $L_{i,1}, \ldots, L_{i,k_i}$ is $\{\emptyset\}$, for each $i \in \{1, \ldots, n\}$. Therefore, for each clause of φ there exists a true literal.

6.1.2 A digraph-based satisfiability algorithm for $\exists^*\forall$-sentences and for $\exists\forall^*$-sentences over hypersets

In analogy to the well-founded case, we isolate below two decidable fragments of the BSR-class, over hypersets. Our theorems show that any $\forall$-formula owning at most one universal quantifier and an arbitrary number of free variables, and any $\forall^*$-formula owning an arbitrary number of universal quantifiers, but at most one free variable, if satisfiable by hypersets, are also satisfiable by hereditarily finite hypersets. These proofs of 'finite reflexion' also provide a bound on the size of the hereditarily finite hyperset models; the decidability algorithms they entail, which we leave as implicit, are a generate-and-test method.

We start with the class of $\forall$-formulae, also considered in [98, 104] under various set-theoretic axioms.

Theorem 6.1.2 *For any quantifier-free formula $\varphi(\vec{a}, y)$, where $\vec{a}$ represents a list $a_0, \ldots, a_n$ of distinct variables, the sentence*

$$\exists\,\vec{a}\,\forall\,y\,\varphi(\vec{a}, y) \rightarrow \exists\,\vec{a}\left(\bigwedge_{i=0}^{n} a_i \in \overline{\mathsf{HF}} \wedge \forall\,y\,\varphi(\vec{a}, y)\right)$$

ensues from $\mathsf{ZF}^- - \mathsf{FA} + \mathsf{AFA}$.

Proof. Consider an arbitrary $\forall$-prenex formula $\psi(\vec{a}) = \forall\,y\,\varphi(\vec{a}, y)$ and an array $\vec{\mathbf{a}} = \langle \mathbf{a}_0, \ldots, \mathbf{a}_n \rangle$ of sets such that $\psi(\vec{\mathbf{a}})$ holds. We will see below that if some component $\mathbf{a}_i$ of the satisfying array fails to be hereditarily finite, we can thin $\vec{\mathbf{a}}$ into another array, $\dot{\vec{\mathbf{a}}}$, devoid of this 'drawback'.

Put $\mathcal{A} = \{\mathbf{a}_0, \ldots, \mathbf{a}_n\}$, and let $\mathcal{D}$ be a finite subset of $(\bigcup \mathcal{A}) \setminus \mathcal{A}$ such that $\mathbf{a}_i \neq \mathbf{a}_j$ implies $\mathbf{a}_i \cap (\mathcal{A} \cup \mathcal{D}) \neq \mathbf{a}_j \cap (\mathcal{A} \cup \mathcal{D})$. For example, one could construct $\mathcal{D}$ by selecting an element d_{ij} from each non-void set of the form $(\mathbf{a}_i \setminus \mathbf{a}_j) \setminus \mathcal{A}$, with $i, j \in \{0, \ldots, n\}$.

Next consider the membership digraph $G_{\mathcal{A}\cup\mathcal{D}} = (V, E)$ with vertices $V = \mathcal{A} \cup \mathcal{D}$ and arcs $E = \{\langle u, w \rangle : u \in V \wedge w \in V \wedge w \in u\}$. Associate with each vertex u two sets $\dot{u}, \ddot{u}$ so as to satisfy the following constraints:

$$\dot{u} = \{\,\dot{w} : \langle u, w \rangle \in E\} \cup \ddot{u};$$

$$\ddot{u} = \begin{cases} \emptyset, & \text{if } u \in \mathcal{A}, \\ \{z\}, & \text{if } u \in \mathcal{D}, \text{ where } z \in \mathsf{HF} \text{ and } z \text{ is of cardinality } \#z > \#\mathcal{A} + \#\mathcal{D} + 1; \end{cases}$$

the restriction of the mapping $u \mapsto \ddot{u}$ to $\mathcal{D}$ is *injective*, i.e., it chooses different z's for different u's.

It is plain that such mappings $u \mapsto \dot{u}$ and $u \mapsto \ddot{u}$ can always be obtained: AFA (specifically, its 'existential' part AFA_1) ensures that the former exists. It should also be clear that the mapping $u \mapsto \dot{u}$ is injective, and that its images not only belong to HF, but can even be drawn from a finite repository of hereditarily finite hypersets.

In consequence of these facts, it turns out that a literal $\mathbf{a}_i = \mathbf{a}_j$ (resp., $\mathbf{a}_i \in \mathbf{a}_j$) holds if and only if the corresponding literal $\dot{\mathbf{a}}_i = \dot{\mathbf{a}}_j$ (resp., $\dot{\mathbf{a}}_i \in \dot{\mathbf{a}}_j$) holds. To see that $\psi(\dot{\mathbf{a}}_0, \ldots, \dot{\mathbf{a}}_n)$ is true, we proceed as in [99]. Observe first that each $y \in \dot{\mathbf{a}}_0 \cup \cdots \cup$

$\dot{\mathbf{a}}_n \setminus \{\dot{\mathbf{a}}_0, \ldots, \dot{\mathbf{a}}_n\}$ originates from a set y' such that the membership and equality literals satisfied by $\mathbf{a}_0, \ldots, \mathbf{a}_n, y'$ match precisely the ones satisfied by $\dot{\mathbf{a}}_0, \ldots, \dot{\mathbf{a}}_n, y$; more generally, for each triple I, J, K such that the condition

$$\exists\, y \left(\bigwedge_{i=0}^{n} (\dot{\mathbf{a}}_i \in y \leftrightarrow i \in I) \wedge \bigwedge_{j=0}^{n} (y \in \dot{\mathbf{a}}_j \leftrightarrow j \in J) \wedge \bigwedge_{k=0}^{n} (\dot{\mathbf{a}}_k = y \leftrightarrow k \in K) \right)$$

is met, the analogous condition referring to $\mathbf{a}_0, \ldots, \mathbf{a}_n$ is satisfied as well. This proves our claim. ■

The following theorem shows that a BSR-formula with one free variable cannot force it to designate an infinite set, thus generalizing to Aczel's theory of hypersets an analogous observation made for ordinary sets in [112, p. 276].

Theorem 6.1.3 *For any quantifier-free formula $\varphi(a, \vec{y})$, the sentence*

$$\exists\, a\, \forall\, \vec{y}\, \varphi(a, \vec{y}) \rightarrow \exists\, a \left(a \in \overline{\mathsf{HF}} \wedge \forall\, \vec{y}\, \varphi(a, \vec{y})\right)$$

ensues from $\mathsf{ZF}^- - \mathsf{FA} + \mathsf{AFA}$.

Proof. Consider an arbitrary $\forall^*$-prenex formula $\psi(a) = \forall\, \vec{y}\, \varphi(a, \vec{y})$ and a set $\mathbf{a}$ such that $\psi(\mathbf{a})$ holds. If $\mathbf{a} = \Omega = \{\Omega\}$, then the claim's proof is plain. Otherwise, consider a membership path $\mathcal{P} = \{p_0, p_1, p_2, \ldots, \emptyset\}$ starting from $p_0 = \mathbf{a}$ and ending in $\emptyset$, such that $p_{i+1} \in p_i$, for every i. Assume, moreover, that $\mathcal{P}$ is chosen such that its cardinality is smallest possible. Denote by m be the number of universally quantified variables appearing in ψ; additionally, if $|\mathcal{P}| - 1 \leqslant m$, let r equal $|\mathcal{P}| - 1$, otherwise, take $r = m$.

Denoting by $\mathcal{R}$ the set $\{p_0, p_1, \ldots, p_r\}$, we will now see that the membership digraph

$$D_{\mathcal{R}} \quad = \quad \left(\mathcal{R}, \{\langle u, w\rangle :\ u \in \mathcal{R} \wedge w \in \mathcal{R} \wedge w \in u\}\right)$$

has no distinct bisimilar vertices. Consider, for a contradiction, the largest i, $0 < i \leqslant r$, such that p_i is bisimilar to some p_j, $i > j \geqslant 0$. Plainly, $i < r$, since p_r has no successors in $D_{\mathcal{R}}$, while p_j has p_{j+1} as successor. Therefore, $p_{i+1} \in p_i$, and by the maximality of i we also get that $p_{i+1} \in p_j$. But now the set $\mathcal{P}' = \{p_0, \ldots, p_j, p_{i+1}, \ldots, p_r, \ldots, \emptyset\}$ is a shorter membership path between $\mathbf{a}$ and $\emptyset$, which contradicts the minimality of $\mathcal{P}$.

As one readily sees, the decoration of $D_{\mathcal{R}}$ assigns a hereditarily finite set $\dot{\mathbf{a}}$ satisfying $\psi(\dot{\mathbf{a}})$ to the vertex p_0. This is seen as in Theorem 6.1.2, by checking that all atomic formulae satisfied in the newly defined interpretation correspond to atomic formulae that were already satisfied in the original one. Moreover, note that $\dot{\mathbf{a}}$ can be drawn from a finite collection of hereditarily finite hypersets. ■

An argument similar to the one just given proves that under FA no $\exists\exists\forall^*$-sentence can force only one of the two existential variables to designate an infinite set: to state this more accurately, the sentence

$$(\forall\, a \in \mathsf{HF})\left(\exists\, b \forall\, \vec{y}\, \varphi(a, b, \vec{y}) \rightarrow (\exists\, b \in \mathsf{HF}) \forall\, \vec{y}\, \varphi(a, b, \vec{y})\right)$$

ensues from ZF^-, for any quantifier-free formula $\varphi(a, b, \vec{y})$. Assuming that $\mathbf{a}$ and $\mathbf{b}$ satisfy $\mathbf{b} \notin \mathsf{HF} \wedge \forall\, \vec{y}\ \varphi(\mathbf{a}, \mathbf{b}, \vec{y})$, one can in fact derive $\mathbf{b}' \in \mathsf{HF} \wedge \forall\, \vec{y}\ \varphi(\mathbf{a}, \mathbf{b}', \vec{y})$ for a suitable

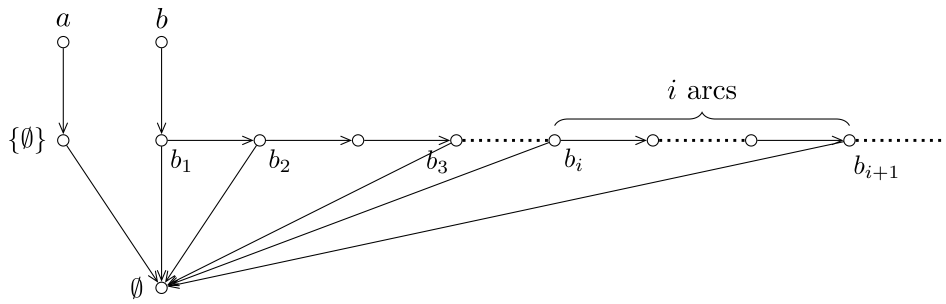


Figure 6.1: The picture of $a = \{\{\emptyset\}\}$ and of the non-well-founded set $b = \{b_1\}$, which contains an infinite descending membership path $\{b_1, b_2, \{b_3\}, b_3, \dots\}$ in its transitive closure; the length of the membership path between b_i and b_{i+1} is i, for any $i \in \omega$. None of the finite induced subdigraphs containing the two points a and b is extensional.

$\mathbf{b}'$. It suffices for that to consider—essentially as done in the proof of Theorem 6.1.3—a membership path starting from $\mathbf{b}$ and ending in a subset x of the transitive closure, $t(\mathbf{a})$, of the other set. The decoration of the digraph consisting of this path extended with $t(\mathbf{a})$ will send $\mathbf{b}$ (now seen as a vertex) to a hereditarily finite set $\mathbf{b}'$ satisfying $\forall \vec{y}\, \varphi(\mathbf{a}, \mathbf{b}', \vec{y})$.

While conjecturing that a result analogous to the one just outlined holds for $\mathsf{ZF}^- - \mathsf{FA} + \mathsf{AFA}$, we see no immediate way of generalizing the above argument to this theory: there are in fact digraphs defeating to such an argument, e.g. the one displayed in Figure 6.1.

6.2 Infinite set models

A customary way to state the existence of an infinite set is

$$\iota(a) \equiv \emptyset \in a \wedge (\forall x \in a)(\exists y \in a)(x \in y).$$

It is plain that any well-founded set $\mathbf{a}$ satisfying $\iota(a)$ must be infinite. However, if we do not assume FA, then $\iota(a)$ becomes satisfiable by the hereditarily finite hyperset $\Omega' = \{\Omega', \emptyset\}$.

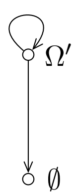


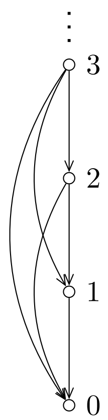
Figure 6.2: The hyperset $\Omega' = \{\Omega', \emptyset\}$.

In order to have a formulation of infinity for whose working FA is immaterial, it suffices to slightly modify $\iota(a)$:

$$\tilde{\iota}(a) \equiv \emptyset \in a \wedge (\forall x \in a)(\exists y \in a)(y = \{x\}).$$

This formulation was actually the one Zermelo gave in 1908 for the Axiom of Infinity:

$$\mathsf{Inf} \equiv (\exists a)\tilde{\iota}(a).$$



$$\omega = \{i : i = \{0, \ldots, i-1\}\}$$

Figure 6.3: The set ω of natural numbers.

The reason why such an infinite set can be so easily summoned up lies in the alternation of universal and existential quantifiers $(\forall x \in a)(\exists y \in a)$, which imposes a 'local' condition at every element of a. The class of $\forall^*\exists^*$-formulae not only has the ability to express infinity, but it actually constitutes an undecidable fragment of the first-order set-theoretic language, even over well-founded sets [111, 115].

Remaining for the moment in the well-founded setting of FA, infinity is expressible by a BSR-formula, with only two universally quantified variables and only two free variables [112, 113]. In order to illustrate this result, together with the machinery that pushes any sets satisfying this formula to be infinite, let us take a step back, and consider one of the simplest infinite set, the set ω of natural numbers, given in Figure 6.3.

In order to devise an infinitely satisfiable formula having ω as model, we can start by observing that ω satisfies[2]

$$\bigcup \omega \subseteq \omega,$$

$$(\forall x \in \omega)(\forall y \in \omega)(x = y \vee x \in y \vee y \in x).$$

The conjunction of these two conditions describes the structure of ω, but it is not sufficient to push ω to be infinite. In fact, this conjunction is satisfied by any natural number defined as above. The missing infinitary ingredient can be provided by the formula $\iota(a)$. If one defines $\iota_\omega(a)$ to be the conjunction of:

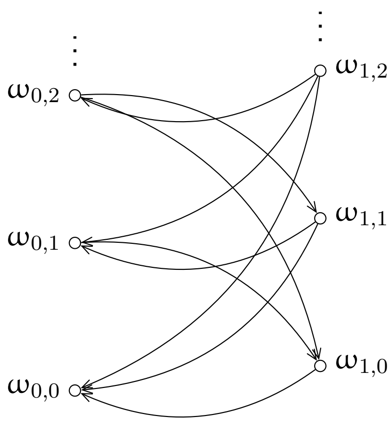
(i) $\iota(a)$

(ii) $\bigcup a \subseteq a$

(iii) $(\forall x \in a)(\forall y \in a)(x = y \vee x \in y \vee y \in x)$,

then we still have that $\iota_\omega(\omega)$ holds, and, as desired, any well-founded set satisfying it must be infinite.

[2] The use of the union set operator $\bigcup$ is merely for readability purposes; indeed, $\bigcup a \subseteq b$ can be rewritten into the BSR-formula $(\forall x \in a)(\forall y \in x)(y \in b)$.



$$\omega_0 = \{\omega_{0,j} : j \in \omega\}, \quad \omega_{0,j} = \{\omega_{1,k} : 0 \leqslant k < j\},$$
$$\omega_1 = \{\omega_{1,j} : j \in \omega\}, \quad \omega_{1,j} = \{\omega_{0,k} : 0 \leqslant k \leqslant j\}, \quad \forall j \in \omega.$$

Figure 6.4: Well-founded sets ω_0 and ω_1 satisfying $\iota\iota$ and $\widetilde{\iota\iota}$.

Trying to replace the $\forall\exists$-subformula $\iota(a)$ with some other purpose-built BSR-formula—which is impossible, in light of the previous section—is a good example of the difficulties arising when trying to capture an infinite (hyper)set by means of a set theoretic formula, and even more so by a BSR-formula: one has to play a subtle game on keeping the formula satisfiable, while avoiding finite satisfiability. What makes this even more interesting is that in order to exclude finite satisfiability we reason only about *finite* sets—since we proceed by contradiction—, turning our proofs into a finitarily combinatorial game.

Since this game cannot succeed in capturing ω, let us try to 'split' the set ω into two parts, to be represented by the two free variables of a BSR-formula. We start again from this intended model that we want to capture, which is made up of the disjoint sets ω_1, ω_0 depicted in Figure 6.4. First, notice that these also satisfy analogs of the two conditions about ω outlined above.

$$\bigcup \omega_0 \subseteq \omega_1 \wedge \bigcup \omega_1 \subseteq \omega_0,$$

$$(\forall x \in \omega_0)(\forall y \in \omega_1)(x \in y \vee y \in x).$$

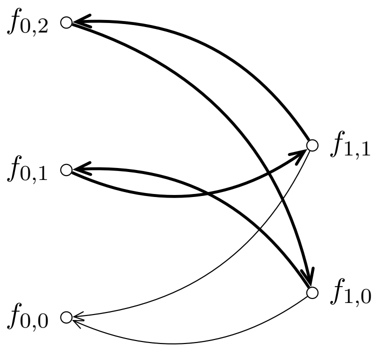
Second, it also holds that $\omega_0 \neq \omega_1$, $\omega_0 \notin \omega_1$, and $\omega_1 \notin \omega_0$. It was observed in [113] that these $\forall\forall$-formulae suffice to obtain a formulation $\iota\iota(a, b)$ of infinity. Indeed, let $\iota\iota(a, b)$ be the conjunction of:

(i) $a \neq b \wedge a \notin b \wedge b \notin a$

(ii) $\bigcup a \subseteq b \wedge \bigcup b \subseteq a$

(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$.

Even though the above condition (i) might seem to bring little information, it turns out to be the right substitute for $\iota(a)$. On the one hand, requiring $a \neq b$ is tantamount to imposing that at least one of the two sets be non-empty, in analogy to the first conjunct of $\iota(a)$. On the other hand, requiring $a \notin b \wedge b \notin a$ proves to be a substitute for the second



$$f_0 = \{f_{0,j} : 0 \leqslant j \leqslant 2\}, \quad f_{0,0} = \emptyset, \quad f_{0,1} = \{f_{1,1}\}, \quad f_{0,2} = \{f_{1,0}\},$$
$$f_1 = \{f_{1,j} : 0 \leqslant j \leqslant 1\}, \quad f_{1,0} = \{f_{0,0}, f_{0,1}\}, \quad f_{1,1} = \{f_{0,0}, f_{0,2}\}.$$

Figure 6.5: Hereditarily finite sets f_0 and f_1 such that $\iota\iota(f_0, f_1)$ holds; they are neither well-founded sets, nor hypersets; the directed cycle in their membership digraph is drawn with thick lines. Vertices $f_{1,0}$ and $f_{1,1}$ get decorated with the same hyperset; this is also the case for vertices $f_{0,1}$ and $f_{0,2}$ [114].

conjunct of $\iota(a)$. Indeed, it can be shown that conditions (ii) and (iii) enact such a rigid structure, that if either one of a or b satisfying $\iota\iota$, say a, turned out to be finite, then it would own an element equal to b, violating thus (ii). We will precisely formalize this argument in the following sections.

Passing now to a setting deprived of FA, are there (hereditarily finite) non-well-founded sets satisfying $\iota\iota(a, b)$? If we simply drop the requirement that the sets be well-founded (that is, if we place ourselves under $\mathsf{ZF}^- - \mathsf{FA}$), then the answer is *yes*. A pair of such sets is depicted in Figure 6.5. However, they are not also hypersets since their membership digraphs are not hyper-extensional. Therefore, the following natural questions arise:

1. Is there a BSR-formula capturing infinity under $\mathsf{ZF}^- - \mathsf{FA}$?
2. If instead of simply dropping FA, we supersede FA and EA by AFA, does $\iota\iota(a, b)$ still express infinity?

Question 1 was answered positively in [114], where the driving engine forcing a and b to be infinite was carefully distilled, by the insertion into $\iota\iota(a, b)$ of the new conjunct

(iv) $(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_2 \in y_2 \in x_1 \in y_1 \rightarrow x_2 \in y_1)$,

clearly derivable from (iii) when FA is assumed. Even if well-foundedness is no longer assumed, the resulting $\forall\forall\forall\forall$-formula is again satisfied by the well-founded sets ω_0 and ω_1, and all sets satisfying it are infinite.

As we are about to see, an even more succinct distillate from FA is

($\star$) $(\forall y_1, y_2 \in b)(y_1 \subseteq y_2 \vee y_2 \subseteq y_1)$,

which in fact follows from $\iota\iota(a, b)$ if the above (iv) is assumed. Although a recasting of this formula into purely universal form seems to require four universal quantifiers, in

1.	$\mathsf{Finite}\,(F) \wedge G \subseteq F$	$\rightarrow$	$\mathsf{Finite}\,(G)$
2.			$\mathsf{Finite}\,(\emptyset)$
3.	$\mathsf{Finite}\,(F)$	$\rightarrow$	$\mathsf{Finite}\,(F \cup \{X\})$
4.	$\mathsf{Finite}\,(F) \wedge \mathsf{Finite}\,(G)$	$\rightarrow$	$\mathsf{Finite}\,(F \cup G)$
5.	$\mathsf{Finite}\,(F)$	$\rightarrow$	$\mathsf{Finite}\,(\{t(x) :\ x \in F\})$
6.	$\mathsf{Finite}\,(A) \wedge A \neq \emptyset$	$\rightarrow$	$(\exists\, m \in A)(\forall\, y \in A \setminus \{m\})(\, m \not\subseteq y\,)$
7.	$\mathsf{Infinite}\,(I)$	$\leftrightarrow$	$\neg\mathsf{Finite}\,(I)$

Figure 6.6: Laws regarding finiteness and infinitude.

Section 6.2.5 we succeed in making the necessary tweaks to $\iota\iota(a, b)$ to produce a $\forall\forall\forall$-formula expressing infinity independent of FA, hence lowering the number of universal quantifiers from four to three.

Regarding Question 2, we will show that, surprisingly, under AFA we can derive $(\star)$ from conditions (i)–(iii) and $a \cap b = \emptyset$. Moreover, even in the non-well-founded setting of AFA, it will turn out that *all* sets satisfying $\iota\iota(a, b) \wedge (a \cap b = \emptyset)$ are *well-founded*. We also give a $\forall\forall\forall$-formula that, under AFA, is satisfied only by infinite hypersets having all peculiarities of non-well-foundedness: membership cycles and infinite descending membership chains with no repeated elements.

6.2.1 Stating infinity

What do we mean by saying that a sentence like Inf *expresses infinity*? In the first place, that we have reasons to believe it to be consistent with $\mathsf{ZF}^- - \mathsf{FA}$. Taking this for granted, there are two readings of the concept in question: one *internal* to $\mathsf{ZF}^- - \mathsf{FA}$, and one *external*, i.e. referring to the models of this theory.

Internal version: This discussion presupposes that complementary predicates $\mathsf{Finite}\,(\cdot)$ and $\mathsf{Infinite}\,(\cdot)$ have been defined in a way reflecting the usual meaning of their names and so that all laws displayed in Figure 6.6 are met in $\mathsf{ZF}^- - \mathsf{FA}$; i.e., one can derive these laws without resorting to Inf or to FA. Concerning the finiteness predicate, a definition drawn, essentially, from [142] and well-suited for our purposes goes as follows:

$$\mathsf{Finite}\,(X) \;\; \leftrightarrow_{\mathrm{Def}} (\forall\, y \in \mathcal{P}(\mathcal{P}(X)) \setminus \{\emptyset\})\,(\exists\, m)\,(y \cap \mathcal{P}(m) = \{m\}).$$

This states that a set X is finite if and only if each nonnull family y of subsets of X owns an element m minimal with respect to $\subseteq$.

Thus, Inf is an internal expression of infinity if $\mathsf{ZF}^- - \mathsf{FA}$ derives $\neg\mathsf{Finite}\,(a)$, hence $\mathsf{Infinite}\,(a)$, from $\tilde{\iota}(a)$. (For Inf, the strategy to achieve this is to show that a certain set a'' of the form $\{t(x) :\ x \in a'\}$, with $a' \subseteq a$, owns no maximal element relative to $\subseteq$; but then $\neg\mathsf{Finite}\,(a'')$ can be derived, thanks to law 6. of Figure 6.6, and so $\neg\mathsf{Finite}\,(a')$ by law 5., and therefore $\neg\mathsf{Finite}\,(a)$ by law 1.).

External version: Before shifting to the semantic level, notice that the scheme $(\neg\mathsf{Finite}\,(a)) \rightarrow a \not\subseteq \{X_1, \ldots, X_n\}$, with any number n of distinct variables X_i, is derivable from the laws 1., 2., and 3.; accordingly, all statements $a \not\subseteq \{X_1, \ldots, X_n\}$ are derivable from $\iota(a)$. It plainly follows, for every model $\mathcal{U} = (\mathcal{U}, \in)$ of $\mathsf{ZF}^- - \mathsf{FA}$, that if $\mathcal{U} \models \tilde{\iota}(\mathbf{a})$ for some $\mathbf{a}$ in $\mathcal{U}$, then such an $\mathbf{a}$ will satisfy infinitely many literals $\mathbf{x} \in \mathbf{a}$ with $\mathbf{x}$ in $\mathcal{U}$.

Likewise, for all of our formulae φ^n, $n \geqslant 2$—to be introduced shortly—it can be seen that the sentence $\exists x_0, \ldots, \exists x_{n-1}\ \varphi^n(x_0, \ldots, x_{n-1})$ expresses infinity in two ways. Our proofs will show that ZF^- (or $\mathsf{ZF}^- - \mathsf{FA}$, or $\mathsf{ZF}^- - \mathsf{FA} + \mathsf{AFA}$) derives from $\varphi^n(x_0, \ldots, x_{n-1})$ that no x_i can own an inclusion maximal element; but then, by laws 6. and 7., we will have

$$\mathsf{ZF}^- \vdash (\forall\, x_0, \ldots, \forall\, x_{n-1})\Big(\varphi^n(x_0, \ldots, x_{n-1}) \rightarrow \bigwedge_{i=0}^{n-1} \mathsf{Infinite}\,(x_i)\Big).$$

As is then plain, for every model $\mathcal{U} = (\mathcal{U}, \in)$ of ZF^-, if $\mathcal{U} \models \exists x_1, \ldots, \exists x_{n-1}\ \iota\iota(\mathbf{x}_0, x_1, \ldots, x_{n-1})$ holds for some $\mathbf{x}_0$, then infinitely many literals $\mathbf{y} \in \mathbf{x}_0$ with $\mathbf{y}$ in $\mathcal{U}$ must be true.

In Appendix A we present a formal proof, certified correct by Referee, that any sets satisfying the predicate Finite just defined also satisfy laws 1., 4. and 6. (blatantly, laws 2. and 3. will also hold). To increase the significance of these proofs, no recourse to FA is made therein.

6.2.2 An apparatus for starting off an infinite well-founded spiral

Having outlined the rules of the game and the main results that will be reached, let us place ourselves in a more general context, that of a BSR-formula involving an arbitrary number $n \geqslant 2$ of free variables. We also momentarily refrain from assuming FA or AFA. Let thus $\iota\iota^n(x_0, \ldots, x_{n-1})$ be the generalization of $\iota\iota(a, b)$, obtained from the conjunction of the following:

(i) $x_0 \neq \emptyset \wedge \bigwedge_{i=0}^{n-1} \big(x_{(i-1) \bmod n} \notin x_i\big)$

(ii) $\bigwedge_{i=0}^{n-1} \big(\bigcup x_i \subseteq x_{(i-1) \bmod n}\big)$

(iii) $(\forall y_0 \in x_0, \ldots, \forall y_{n-1} \in x_{n-1})\big(\bigvee_{i=0}^{n-1} y_i \in y_{(i+1) \bmod n}\big)$

To see that $\iota\iota^n$ is satisfiable by means of well-founded sets, observe that $\iota\iota^n(\omega_0, \ldots, \omega_{n-1})$ holds, where each $\omega_i = \{\omega_{i,j} \mid j \in \omega\}$, and

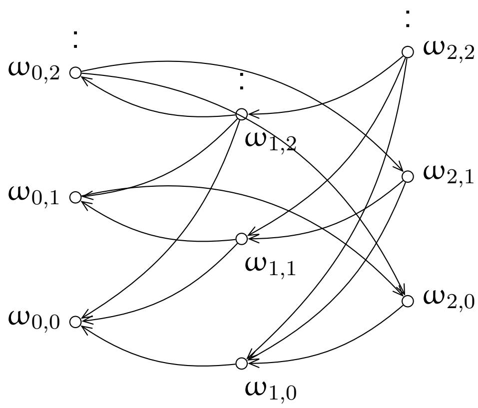
$$\begin{aligned} \omega_{i,j} &= \{\omega_{i-1,k} : k \leqslant j\}, \qquad \text{if } i \in \{1, \ldots, n-1\}, \\ \omega_{0,j} &= \{\omega_{n-1,k} : k < j\}, \end{aligned}$$

so that, in particular, $\omega_{0,0} = \emptyset$. A graphical representation of sets ω_i, for $n = 3$, is depicted in Figure 6.7.

We now prove that, under ZF, $\iota\iota^n$ is satisfied only by infinite sets. Our first lemma shows that $\iota\iota^n(x_0, \ldots, x_{n-1})$ is reminiscent of Zermelo's formulation $\iota(a)$.

Lemma 6.2.1 *Independently of* FA*, for any sets* $\omega_0, \ldots, \omega_{n-1}$ *such that* $\iota\iota^n(\omega_0, \ldots, \omega_{n-1})$ *is true, the following conditions hold, for any* $i \in \{0, \ldots, n-1\}$*:*

- $\omega_i \neq \emptyset$*;*
- $(\forall y \in \omega_i)(\exists z \in \omega_{(i+1) \bmod n})(y \in z)$*.*



$$\begin{aligned}
\omega_0 &= \{\omega_{0,j} : j \in \omega\}, \quad \omega_{0,j} = \{\omega_{2,k} : 0 \leqslant k < j\}, \\
\omega_1 &= \{\omega_{1,j} : j \in \omega\}, \quad \omega_{1,j} = \{\omega_{0,k} : 0 \leqslant k \leqslant j\}, \\
\omega_2 &= \{\omega_{2,j} : j \in \omega\}, \quad \omega_{2,j} = \{\omega_{1,k} : 0 \leqslant k \leqslant j\}, \quad \forall j \in \omega.
\end{aligned}$$

Figure 6.7: Well-founded sets $\omega_0, \omega_1, \omega_2$ satisfying $\iota\iota^3$.

Proof. As a preliminary remark, observe that

$$y \in \omega_i \text{ always implies } \omega_{(i-1) \bmod n} \not\subseteq y \,;$$

for, assuming the contrary, the equality $\omega_{(i-1) \bmod n} = y$ would hold because $y \subseteq \bigcup \omega_i \subseteq \omega_{(i-1) \bmod n}$; but then $\omega_{(i-1) \bmod n} \in \omega_i$ would hold, contradiction.

If some ω_i were empty, given that $\omega_0 \neq \emptyset$, we can consider a k for which $\omega_k = \emptyset$ and $\omega_{(k+1) \bmod n} \neq \emptyset$. We can pick an element $y \in \omega_{(k+1) \bmod n}$ and we know from the above remark that $\omega_k \not\subseteq y$, contradiction.

Let now $y \in \omega_i$; put $y_i = y$. For $k = 0, \ldots, n-2$, by repeatedly taking into account the initial remark, we can pick an element

$$y_{(i-k-1) \bmod n} \in \omega_{(i-k-1) \bmod n} \setminus y_{(i-k) \bmod n} \,.$$

In view of condition (iii) of $\iota\iota^n$, we have that $y \in y_{(i+1) \bmod n} \in \omega_{i+1}$ so that z can be taken to be $y_{(i+1) \bmod n}$. ∎

Our next lemma shows that if one of the free variables of $\iota\iota^n$ is substituted by an infinite set in order to render $\iota\iota^n$ true, then all of them must be substituted by infinite sets.

Lemma 6.2.2 *Independently of* FA, *for any sets* $\omega_0, \ldots, \omega_{n-1}$ *such that* $\iota\iota^n(\omega_0, \ldots, \omega_{n-1})$ *holds, if one* ω_i, $i \in \{0, \ldots, n-1\}$, *is infinite, then all of* $\omega_0, \ldots, \omega_{n-1}$ *are infinite.*

Proof. Assuming that ω_i is infinite, $\omega_{(i-1) \bmod n}$ must be infinite as well, because $\bigcup \omega_i \subseteq \omega_{(i-1) \bmod n}$ ensues from condition (ii) of $\iota\iota^n$. Trivially, in fact, for any set x, the finiteness of $\mathcal{P}(x)$ ensues from x being finite and it holds that $x \subseteq \mathcal{P}(\bigcup x)$; therefore, if $\omega_{(i-1) \bmod n}$ were finite, then $\bigcup \omega_i$ would be finite, $\mathcal{P}(\bigcup \omega_i)$ would be finite, and ω_i would be finite.

This reasoning can be iterated to show that all of $\omega_0, \ldots, \omega_{n-1}$ are infinite. ∎

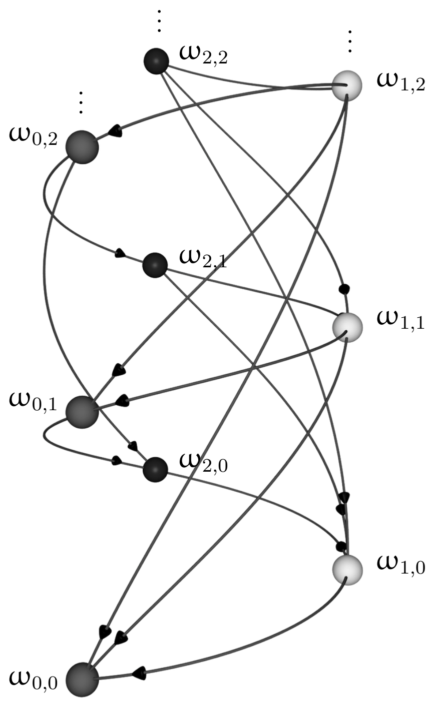


Figure 6.8: A more suggestive three-dimensional representation of the membership digraph of Figure 6.7.

If we guarantee that for at least one of the free variables of $\iota\iota^n$ its elements are pairwise comparable by inclusion, as suggested by condition ($\star$), then $\iota\iota^n$ does become infinitely satisfiable.

Theorem 6.2.3 *Independently of* FA*, for any sets* $\omega_0, \ldots, \omega_{n-1}$ *such that the following two conditions hold*

- $\iota\iota^n(\omega_0, \ldots, \omega_{n-1})$,
- $(\forall y_1, y_2 \in \omega_{n-1})(y_1 \subseteq y_2 \vee y_2 \subseteq y_1)$,

we have that $\omega_0, \ldots, \omega_{n-1}$ *are infinite.*

Proof. Assume for a contradiction that ω_{n-1} is finite. From the second condition of the hypothesis, we can consider y_0 to be the $\subseteq$-maximum element of ω_{n-1}. From Lemma 6.2.1 we have that $\omega_{n-2} \subseteq y_0$. Since $y_0 \subseteq \bigcup \omega_{n-1} \subseteq \omega_{n-2}$, we get $y_0 = \omega_{n-2}$, contradicting condition $\omega_{n-2} \notin \omega_{n-1}$ of $\iota\iota^n$.

Moreover, from Lemma 6.2.2 we get that all of $\omega_0, \ldots, \omega_{n-1}$ are infinite. ■

6.2.3 Infinite well-founded models under ZF, ZF − FA + AFA and ZF − FA

In this section we take a stance on whether we assume FA, or its counterpart AFA. In the former case, even if the second condition of Theorem 6.2.3 is no longer required, infinite satisfiability can still be guaranteed; we will give a refinement of this result is terms of rank. In the latter case, when employing only two free variables, the hypothesis of Theorem 6.2.3 will be met, provided that $a \cap b = \emptyset$ holds. Therefore, basically the same 'minimal' expression of infinitude $\iota\iota$ works immaterial of whether FA, or AFA, is assumed.

Theorem 6.2.4 *Under* ZF*, sets* ω_i*'s for which* $\iota\iota^n(\omega_0, \ldots, \omega_{n-1})$ *is true always share the same rank, which is a limit ordinal. (Consequently each* ω_i *has an infinite cardinality.)*

Proof. Lemma 6.2.1 ensures that all ω_i's are non-empty. If some ω_i had a successor rank, there would exist $y \in \omega_i$, having $\mathsf{rank}(y) + 1 = \mathsf{rank}(\omega_i)$. By iterating n times the application of Lemma 6.2.1, we will construct a membership chain starting in y and ending in a y' belonging to the same ω_k with which we have started. Since we assume that the membership relation is well-founded, $y \neq y'$ holds, and hence $\mathsf{rank}(y) < \mathsf{rank}(y')$ holds as well, a contradiction with the maximality of rank of y.

Therefore, every ω_i has a limit rank; moreover, if $\mathsf{rank}(\omega_i) \neq \mathsf{rank}(\omega_j)$ could hold for some pair i, j, then $\mathsf{rank}(\omega_k) < \mathsf{rank}(\omega_{k+1})$ must hold for some k, and there would be a $z \in \omega_{k+1}$ such that $\mathsf{rank}(z) > \mathsf{rank}(\omega_k)$, which would conflict with the inclusions $z \subseteq \bigcup \omega_{k+1} \subseteq \omega_k$. ■

The previous theorem shows that, under ZF, for $n = 2$, the $\forall\forall$-formula $\iota\iota(a, b)$ obtained by the conjunction of the following conditions is satisfied exclusively by infinite sets.

(i) $a \neq b \wedge a \notin b \wedge b \notin a$

(ii) $\bigcup a \subseteq b \wedge \bigcup b \subseteq a$

(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$.

Since the translation of the condition $a \neq \emptyset$ of $\iota\iota^2(a, b)$ would require the introduction of a new free variable to characterize $\emptyset$, we have modified it into $a \neq b$, which does imply $a \neq \emptyset$, in light of (ii) and of (i). The infinite satisfiability of $\iota\iota(a, b)$ also follows directly from Theorem 6.2.3, since $(\forall y_1, y_2 \in b)(y_1 \subseteq y_2 \vee y_2 \subseteq y_1)$ holds, due to (iii) and to FA.

We show next that, surprisingly, under ZF − FA + AFA, essentially the same formulation of infinity as in the well-founded context is satisfied exclusively by *infinite well-founded* sets. Let $\widetilde{\iota\iota}(a, b)$ be the conjunction of $\iota\iota(a, b)$ with $a \cap b = \emptyset$. In Figure 6.9 we point out a plain consequence of AFA, which we use in our first lemma.

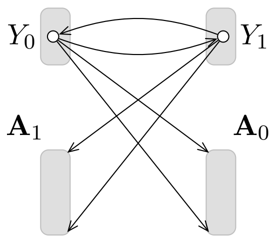
Lemma 6.2.5 *Under* ZF − FA + AFA*, if* ω_0, ω_1 *are sets such that* $\widetilde{\iota\iota}(\omega_0, \omega_1)$ *is true, then there are no infinite descending membership chains in* $\omega_0 \cup \omega_1$.

Proof. First of all, note that for all $x \in \omega_0 \cup \omega_1$, it holds that $x \notin x$, since otherwise we would have $x \in \omega_0 \cap \omega_1$, contradicting the clause $\omega_0 \cap \omega_1 = \emptyset$ of $\widetilde{\iota\iota}(\omega_0, \omega_1)$.

Arguing by contradiction, suppose that the elements of $C = \{c_0, c_1, \ldots\} \subseteq \omega_0 \cup \omega_1$ form an infinite descending membership chain $c_0 \ni c_1 \ni c_2 \ni \cdots$ (so that $c_0 \neq c_1$). Consider now the set

$$\mathbf{X} = \{\, y : \ y \in \omega_0 \cup \omega_1 \quad \wedge \quad \text{there exist } m \in \omega \text{ and } y_1, y_2, \ldots, y_m \in \omega_0 \cup \omega_1 \text{ such that } y \ni y_1 \ni y_2 \ni \cdots \ni y_m \text{ and } y_m \in C \,\}.$$

$$\mathbf{X}_0 = \{Y_0\} = \{\mathbf{X}_1 \cup \mathbf{A}_0\} \qquad \mathbf{X}_1 = \{Y_1\} = \{\mathbf{X}_0 \cup \mathbf{A}_1\}$$



If $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{X}_0$, $\mathbf{X}_1$, are sets such that for $i = 0, 1$, $\mathbf{X}_i \neq \emptyset$, $\mathbf{A}_i \notin \mathbf{X}_i$, and

$$\text{for all } y \in \mathbf{X}_i, \;\; \mathbf{A}_i \subseteq y \wedge y \setminus \mathbf{A}_i \subseteq \mathbf{X}_{i-1} \wedge y \cap \mathbf{X}_i = \emptyset$$

hold, then $\mathbf{X}_0, \mathbf{X}_1$ are those unique sets which solve the system

$$X_0 = \{Y_0\}, \; X_1 = \{Y_1\}, \qquad Y_0 = X_1 \cup \mathbf{A}_0, \; Y_1 = X_0 \cup \mathbf{A}_1$$

of equations.

Figure 6.9: A consequence of AFA.

Then, we can apply the observation in Figure 6.9, by taking $\mathbf{X}_0 = \mathbf{X} \cap \omega_0$, $\mathbf{X}_1 = \mathbf{X} \cap \omega_1$, and $\mathbf{A}_i = (\bigcup \mathbf{X}_i) \setminus \mathbf{X}$ for $i = 0, 1$.

Indeed, the $\mathbf{X}_i$'s are nonnull, as $c_0 \in \mathbf{X}_0$ and $c_1 \in \mathbf{X}_1$ (or vice versa). Since for any $y \in \mathbf{X}$, we have $y \cap \mathbf{X} \neq \emptyset$, we deduce $\mathbf{A}_i \notin \mathbf{X}_i$, for $i = 0, 1$. Let now $y \in \mathbf{X}_i$. To see that $\mathbf{A}_i \subseteq y$, consider a $z \in \mathbf{A}_i$ such that $z \notin y$. From (iii), we get $y \in z$, which implies $z \in \mathbf{X}$, against the choice of z. Requirement $y \setminus \mathbf{A}_i \subseteq \mathbf{X}_{1-i}$ follows from (ii), while $y \cap \mathbf{X}_i = \emptyset$ follows from $\omega_0 \cap \omega_1 = \emptyset$. Moreover, note that $\mathbf{X}_0 \cup \mathbf{A}_1 = \omega_0$ and that $\mathbf{X}_1 \cup \mathbf{A}_0 = \omega_1$.

Therefore, $\mathbf{X}_0 = \{\mathbf{Y}_0\}$, $\mathbf{X}_1 = \{\mathbf{Y}_1\}$, where $\mathbf{Y}_0 = \mathbf{X}_1 \cup \mathbf{A}_0$, $\mathbf{Y}_1 = \mathbf{X}_0 \cup \mathbf{A}_1$. Hence $\mathbf{Y}_0 = \omega_1$ and $\mathbf{Y}_1 = \omega_0$, which, given that $\mathbf{X}_0 \subseteq \omega_0$ and $\mathbf{X}_1 \subseteq \omega_1$, entails $\omega_1 \in \omega_0$ and $\omega_0 \in \omega_1$, which contradicts (i). ■

Lemma 6.2.6 *Under* ZF − FA + AFA, *if* ω_0, ω_1 *are sets such that* $\widetilde{\iota\iota}(\omega_0, \omega_1)$ *is true, then* $(\forall y_1, y_2 \in \omega_1)(y_1 \subseteq y_2 \vee y_2 \subseteq y_1)$ *holds.*

Proof. Arguing by contradiction, assume that $y_1, y_2 \in \omega_1$ and $x_1, x_2 \in \omega_0$ are such that $x_1 \in y_1 \setminus y_2$ and $x_2 \in y_2 \setminus y_1$. By (iii), we get that there is a membership cycle $x_1 \in y_1 \in x_2 \in y_2 \in x_1$ in $\omega_0 \cup \omega_1$, and hence also an infinite descending membership chain (with repeated elements), contradicting Lemma 6.2.5. ■

Theorem 6.2.7 *Under* ZF − FA + AFA, *if* ω_0, ω_1 *are sets such that* $\widetilde{\iota\iota}(\omega_0, \omega_1)$ *is true, then* ω_0 *and* ω_1 *are infinite.*

Proof. The claim follows from Theorem 6.2.3 and Lemma 6.2.6. ■

Passing now to ZF − FA, let $\overline{\iota\iota}(a, b)$ be the conjunction of $\iota\iota(a, b)$ with the following

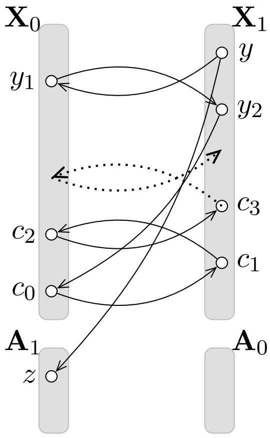


Figure 6.10: A graphical representation of the proof of Lemma 6.2.5.

(iv) $(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_2 \in y_2 \in x_1 \in y_1 \rightarrow x_2 \in y_1)$.

Within the framework proposed in Section 6.2.2, we can easily deduce the following result of [114] about $\overline{\iota\iota}$.

Theorem 6.2.8 *Under* $\mathsf{ZF} - \mathsf{FA}$*, if* ω_0, ω_1 *are sets such that* $\overline{\iota\iota}(\omega_0, \omega_1)$ *holds, then* ω_0 *and* ω_1 *are infinite.*

Proof. By Theorem 6.2.3 it suffices to show that $(\forall y_1, y_2 \in \omega_1)(y_1 \subseteq y_2 \vee y_2 \subseteq y_1)$ holds. Assuming the contrary, we can pick $y_1, y_2 \in \omega$, and $x_1 \in y_1 \setminus y_2$ and $x_2 \in y_2 \setminus y_1$. Condition (iii) of $\overline{\iota\iota}$ implies that $y_2 \in x_1$. Condition (iv) of $\overline{\iota\iota}$ entails $x_2 \in y_1$, which contradicts the choice of x_2. ■

6.2.4 An apparatus for starting off an infinite non-well-founded spiral

Bottomless wonders spring from simple rules, repeated without end.

Benoit Mandelbrot

All the formulations of infinity seen until now are satisfied by (basically the same) well-founded sets. This was just an isolated case, since a plethora of infinite non-well-founded sets can be expressed by BSR-formulae, even in the presence of AFA. We start with a formula involving an arbitrary number $n \geqslant 2$ of free variables, and then analyze the case $n = 2$ in greater detail.

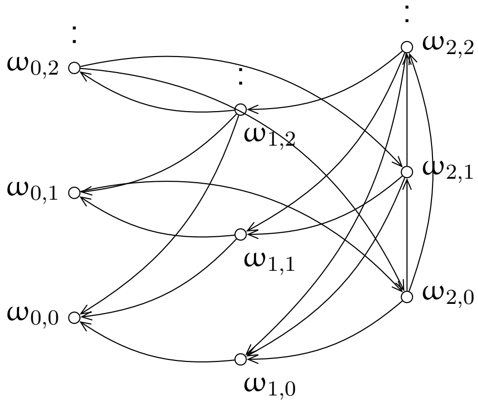
Let $\underline{\iota\iota}^n(x_0, \ldots, x_{n-1})$ be obtained from the conjunction of the following:[3]

(i) $x_0 \neq \emptyset \wedge \bigwedge_{i=0}^{n-1} \big(x_{(i-1) \bmod n} \notin x_i\big)$

(ii′) $\bigwedge_{i=0}^{n-2} \big(\bigcup x_i \subseteq x_{(i-1) \bmod n}\big) \wedge \bigcup x_{n-1} \subseteq x_{n-2} \cup x_{n-1}$

(iii) $(\forall y_0 \in x_0, \ldots, \forall y_{n-1} \in x_{n-1})\big(\bigvee_{i=0}^{n-1} y_i \in y_{(i+1) \bmod n}\big)$

[3]We use the shorthand $\subseteq$ in the context $a \in b \rightarrow b \subseteq a$ with the meaning $(\forall x)(a \in b \rightarrow (x \in b \rightarrow x \in a))$.



$$\begin{array}{ll} \omega_0 = \{\omega_{0,i} : i \in \omega\}, & \omega_{0,j} = \{\omega_{2,k} : 0 \leqslant k < j\}, \\ \omega_1 = \{\omega_{1,i} : i \in \omega\}, & \omega_{1,j} = \{\omega_{0,k} : 0 \leqslant k \leqslant j\}, \\ \omega_2 = \{\omega_{2,i} : i \in \omega\}, & \omega_{2,j} = \{\omega_{1,k} : 0 \leqslant k \leqslant j\} \cup \{\omega_{2,k} : k > j\}, \quad \forall j \in \omega. \end{array}$$

Figure 6.11: Hypersets $\omega_0, \omega_1, \omega_2$ satisfying $\underline{\iota\iota}^3$.

(iv′) $(\forall y_1, y_2 \in x_{n-1})(y_1 \in y_2 \rightarrow y_2 \subseteq y_1)$

To see that $\underline{\iota\iota}^n$ is satisfiable by means of hypersets, observe that $\underline{\iota\iota}^n(\omega_0, \ldots, \omega_{n-1})$ holds, where each $\omega_i = \{\omega_{i,j} \mid j \in \omega\}$, and

$$\begin{array}{rcl} \omega_{0,j} & = & \{\omega_{n-1,k} : k < j\}, \\ \omega_{i,j} & = & \{\omega_{i-1,k} : k \leqslant j\}, \text{ if } i \in \{1, \ldots, n-2\}, \\ \omega_{n-1,j} & = & \{\omega_{n-2,k} : k \leqslant j\} \cup \{\omega_{n-1,k} : k > j\}. \end{array}$$

so that, in particular, $\omega_{0,0} = \emptyset$. We omit a proof that sets ω_i are hypersets, that is, that their membership digraphs are hyper-extensional; we will prove this fact for $n = 2$ in Section 6.2.5. A graphical representation of sets ω_i, for $n = 3$, is depicted in Figure 6.11.

The main difference between $\underline{\iota\iota}^n$ and $\iota\iota^n$ lies in the relaxation of condition $\bigcup x_{n-1} \subseteq x_{n-2}$ into $\bigcup x_{n-1} \subseteq x_{n-2} \cup x_{n-1}$. Once we allow membership relations to hold between elements of x_{n-1}, we can use them to 'propagate' membership relations across x_{n-2} and x_{n-1}. For example, when $n = 2$, condition $(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_2 \in y_2 \in x_1 \in y_1 \rightarrow x_2 \in y_1)$ of $\overline{\iota\iota}(a, b)$ is implied by condition (iv′) if $y_1 \in y_2$ holds (see Figure 6.13).

We start by laying the groundwork for a proof that $\underline{\iota\iota}^n$ is infinitely satisfiable—closely following the proof method employed in Section 6.2.2—and then show two ways in which the conditions governing the membership relations among elements of x_{n-1} can be instantiated. In doing this, we will repeatedly exploit the plain consequence of AFA given in Figure 6.14.

Lemma 6.2.9 *Under* ZF − FA + AFA*, for any sets* $\omega_0, \ldots, \omega_{n-1}$ *such that* $\underline{\iota\iota}^n(\omega_0, \ldots, \omega_{n-1})$ *is true,* $(\forall y \in \omega_{n-1})(\omega_{n-2} \nsubseteq y)$ *holds.*

Proof. Arguing by contradiction, suppose this is not the case, so that the set $\mathbf{X} = \{y \in \omega_{n-1} \mid \omega_{n-2} \subseteq y\}$ is nonnull. Then, by (ii′) and (iv′), we have $(\forall y \in \mathbf{X})(y \setminus \omega_{n-2} \subseteq \mathbf{X})$, and, by (i), that $(\forall y \in \mathbf{X})(y \cap \mathbf{X} \neq \emptyset)$. The observation made in Figure 6.14 implies that $\mathbf{X} = \{\Omega_{\omega_{n-2}}\}$, contradicting the last conjunct of (ii′). ■

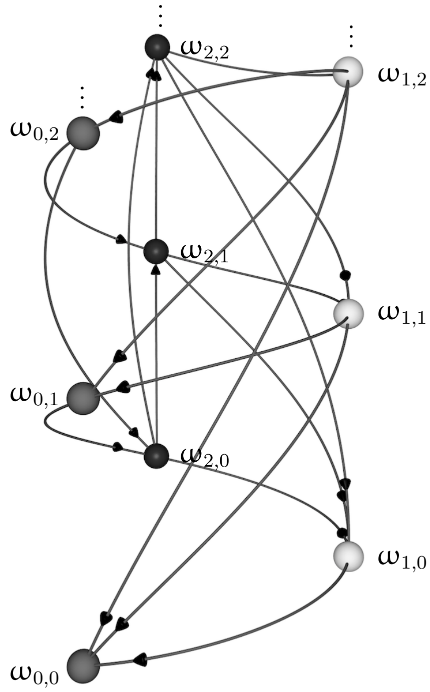


Figure 6.12: A more suggestive three-dimensional representation of the membership digraph of Figure 6.11.

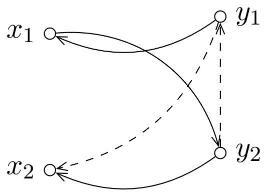


Figure 6.13: Sketch of the derivation of condition (iv) of $\overline{\iota\iota}$.

Lemma 6.2.10 *Under* ZF − FA + AFA*, for any sets* $\omega_0, \ldots, \omega_{n-1}$ *such that* $\underline{\iota\iota}^n(\omega_0, \ldots, \omega_{n-1})$ *is true, the following conditions hold, for any* $i \in \{0, \ldots, n-1\}$*:*

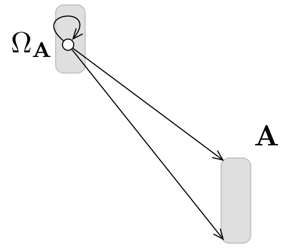
- $\omega_i \neq \emptyset$*;*
- $(\forall y \in \omega_i)(\exists z \in \omega_{(i+1) \bmod n})(y \in z)$.

Proof. To simplify notation, throughout this proof operations on indices are assumed to be performed modulo n. If some ω_i were empty, given that $\omega_0 \neq \emptyset$, we can consider a k for which $\omega_k = \emptyset$ and $\omega_{k+1} \neq \emptyset$; let y be an element of ω_{k+1}. If $k+1 \neq n-1$, then, from (ii′), $y \subseteq \emptyset$ follows, implying $y = \emptyset = \omega_k$, contradicting condition (i). Otherwise, we have that $\omega_{n-2} \subseteq y \in \omega_{n-1}$, which contradicts Lemma 6.2.9.

Suppose now that there is some $k \in \{0, \ldots, n-1\}$ such that $(\exists y_k \in \omega_k)(\forall z \in \omega_{k+1})(y_k \notin z)$. We claim that, for all $i = 1, \ldots, n-1$, we can pick an element y_{k-i} such that

$$y_{k-i} \in \omega_{k-i} \text{ and } y_{k-i} \notin y_{k-i+1}.$$

$\mathbf{X} = \{\Omega_{\mathbf{A}}\} = \{\mathbf{X} \cup \mathbf{A}\}$



If $\mathbf{A}, \mathbf{X}$ are sets such that $\mathbf{X} \neq \emptyset$, $\mathbf{A} \notin \mathbf{X}$, and for all $y \in \mathbf{X}$

$$\mathbf{A} \subseteq y \wedge y \setminus \mathbf{A} \subseteq \mathbf{X} \wedge y \cap \mathbf{X} \neq \emptyset$$

holds, then $\mathbf{X}$ is that unique set—to be designated as $\{\Omega_{\mathbf{A}}\}$ in what follows—which solves the equation $X = \{X \cup \mathbf{A}\}$.

Figure 6.14: A plain consequence of AFA.

If this were not the case, then for some $y_{k-j+1} \in \omega_{k-j+1}$, with $j \in \{1, \ldots, n-1\}$, it would hold that $(\forall y \in \omega_{k-j})(y \in y_{k-j+1})$, or, equivalently, $\omega_{k-j} \subseteq y_{k-j+1}$. The case $k-j+1 = n-1$ cannot hold, by Lemma 6.2.9; hence $k-j+1 \neq n-1$. As $\bigcup \omega_{k-j+1} \subseteq \omega_{k-j}$, we also have $y_{k-j+1} \subseteq \omega_{k-j}$, implying $y_{k-j+1} = \omega_{k-j}$ and $\omega_{k-j} \in \omega_{k-j+1}$. This violates condition (i).

The n-tuple $y_0, \ldots, y_k, y_{k+1}, \ldots, y_{n-1}$ violates condition

$$(\forall y_0 \in \omega_0, \ldots, \forall y_{n-1} \in \omega_{n-1})\big(\bigvee_{i=0}^{n-1} y_i \in y_{i+1}\big),$$

of $\underline{\iota\iota}^n$, as $y_k \notin y_{k+1}$ follows from the initial assumption, in consequence of $y_{k+1} \in \omega_{k+1}$. ■

Theorem 6.2.11 *Under* ZF − FA + AFA*, for any sets* $\omega_0, \ldots, \omega_{n-1}$ *such that the following two conditions hold*

- $\underline{\iota\iota}^n(\omega_0, \ldots, \omega_{n-1})$,
- $(\forall y_1, y_2 \in \omega_{n-1})(y_1 \subseteq y_2 \vee y_2 \subseteq y_1)$,

we have that $\omega_0, \ldots, \omega_{n-1}$ *are infinite.*

Proof. If ω_{n-2} is finite, let $Y \subseteq \omega_{n-1}$ be a finite set such that $(\forall y \in \omega_n)(\exists z \in Y)(y \in z)$. From Lemma 6.2.10 it follows that Y is non-empty, while the second condition of the hypothesis guarantees that we can take $y_0 \in Y$ to be $\subseteq$-maximal in Y. This entails that $\omega_{n-2} \subseteq y_0$, which contradicts Lemma 6.2.9.

Let now k, $0 \leqslant k \leqslant n-1$, be the greatest index such that ω_k is infinite, but $\omega_{(k-1) \bmod n}$ is finite. Since $\bigcup \omega_k \subseteq \omega_{(k-1) \bmod n}$ ensues from condition (ii) of $\underline{\iota\iota}^n$, the claim readily follows, as in the proof of Lemma 6.2.2. ■

Corollary 6.2.12 *Under* ZF − FA + AFA*, for any sets* $\omega_0, \ldots, \omega_{n-1}$ *such that the following two conditions hold*

$\underline{\iota\iota}^n(\omega_0, \ldots, \omega_{n-1})$,

(v) $(\forall y_1, y_2 \in \omega_{n-1})(y_1 = y_2 \vee y_1 \in y_2 \vee y_2 \in y_1)$

we have that $\omega_0, \ldots, \omega_{n-1}$ *are infinite.*

Proof. The above condition (v), together with condition (iv′) of $\underline{\iota\iota}^n$ imply $(\forall y_1, y_2 \in \omega_{n-1})(y_1 \subseteq y_2 \vee y_2 \subseteq y_1)$; the claim then follows, by Theorem 6.2.11. ■

6.2.5 Infinite non-well-founded models under ZF − FA + AFA and ZF − FA

We focus now on capturing non-well-founded infinity with the least number of universally quantified variables or free variables, and propose three such ∀∀∀-formulae.

The only ∀∀ formulation of infinity in a context deprived of FA is $\widetilde{\iota\iota}$ given in the Section 6.2.3, which however is satisfied exclusively by well-founded sets. This raises the question of whether an infinite and 'genuinely' ill-founded set can be captured with only two universal quantifiers; should a negative answer emerge, it would suggest the likelihood that the decision algorithm of [16], devised for ∀∀-formulae about ordinary well-founded sets, can be recast to cope with Aczel's sets.

As just done in Corollary 6.2.12, our first formula requires that a membership relation be present between any two distinct elements of x_{n-1}. Let thus $\underline{\iota\iota}_1(a, b)$ be the conjunction of the following sub-formulae:

(i) $a \neq b \wedge a \notin b \wedge b \notin a$

(ii′) $\bigcup a \subseteq b \wedge \bigcup b \subseteq a \cup b \wedge (\forall y \in b)(y \notin y)$

(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$

(iv′) $(\forall x \in a)(\forall y_1, y_2 \in b)(y_1 \in y_2 \rightarrow y_2 \subseteq y_1)$

(v) $(\forall y_1, y_2 \in b)(y_1 = y_2 \vee y_1 \in y_2 \vee y_2 \in y_1)$

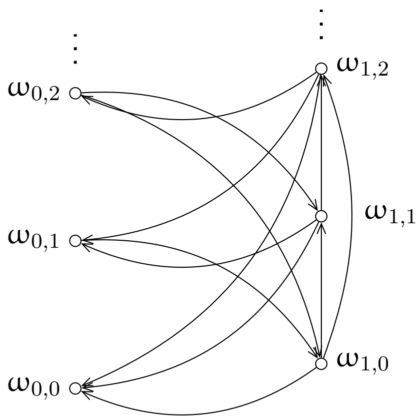
Consider now three elements $x \in a$ and $y_1, y_2 \in b$, such that $y_2 \in x \in y_1$. Condition (v) of $\underline{\iota\iota}_1(a, b)$ imposes a membership arc to connect y_1 with y_2, but tells us nothing about its orientation. As our goal is to find formulae which have no well-founded models, it comes natural to impose $y_1 \in y_2$, in order to obtain the membership cycle $y_2 \in x \in y_1 \in y_2$. Consequently, we introduce the formula $\underline{\iota\iota}_2(a, b)$ be obtained from $\overline{\iota\iota}_1(a, b)$ by replacing condition (v) with:

(v′) $(\forall x \in a)(\forall y_1, y_2 \in b)(y_1 \neq y_2 \wedge y_2 \in x \in y_1 \rightarrow y_1 \in y_2)$

Notice that, analogously to the well-founded case, $\omega_0 \cap \omega_1$ holds by (iii) and the last conjunct of (ii′), for any sets ω_0, ω_1 satisfying $\underline{\iota\iota}_1$ or $\underline{\iota\iota}_2$.

Theorem 6.2.13 *Under* ZF − FA + AFA*,* $\underline{\iota\iota}_1$ *and* $\underline{\iota\iota}_2$ *are satisfiable.*

Figure 6.15: Condition (iv′) of $\underline{\iota\iota}_1(a,b)$ and of $\underline{\iota\iota}_2(a,b)$ *(left)*. Condition (v′) of $\underline{\iota\iota}_2(a,b)$ *(right)*.



$$\omega_0 = \{\omega_{0,j} : j \in \omega\}, \quad \omega_{0,j} = \{\omega_{1,k} : 0 \leqslant k < j\},$$
$$\omega_1 = \{\omega_{1,j} : j \in \omega\}, \quad \omega_{1,j} = \{\omega_{0,k} : 0 \leqslant k \leqslant j\} \cup \{\omega_{1,k} : k > j\}, \quad \forall j \in \omega.$$

Figure 6.16: A model of $\underline{\iota\iota}_1 \wedge \underline{\iota\iota}_2$.

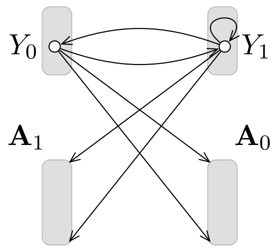
Proof. A shared model ω_0, ω_1 for $\underline{\iota\iota}_1, \underline{\iota\iota}_2$ is shown in Figure 6.16. We claim that $\underline{\iota\iota}_1(\omega_0, \omega_1)$ and $\underline{\iota\iota}_2(\omega_0, \omega_1)$ are true. At the outset, we will prove by induction on n that $\omega_{0,i} \neq \omega_{1,j}$ for all $i, j \in \{0, \ldots, n\}$, and that $\omega_{t,i} \neq \omega_{t,j}$ for all $i, j \in \{0, \ldots, n\}, i \neq j$, and $t \in \{0, 1\}$.

For $n = 0$, we have that $\omega_{0,0} \neq \omega_{1,0}$, as $\omega_{0,0} = \emptyset$, and $\emptyset \in \omega_{1,0}$. Supposing that the claim is true for n, we will show that it is also true for $n+1$. Since $\omega_{1,n} \in \omega_{0,n+1}$, and since $\omega_{1,n} \notin \omega_{0,i}$ holds for any $i \in \{0, \ldots, n\}$, we have that $\omega_{0,n+1} \neq \omega_{0,i}$ for any $i \in \{0, \ldots, n\}$. Moreover, for any $i \in \{0, \ldots, n\}$, $\omega_{0,n+1}$ differs from $\omega_{1,i}$, since $\omega_{1,i} \in \omega_{0,n+1}$, but $\omega_{0,n+1} \notin \omega_{1,i}$. In a similar manner, one can check that $\omega_{1,n+1} \neq \omega_{0,i}$ for any $i \in \{0, \ldots, n+1\}$, and that $\omega_{1,n+1} \neq \omega_{1,i}$ for any $i \in \{0, \ldots, n\}$.

The above statement guarantees that $\omega_0 \cap \omega_1 = \emptyset$ and that $\omega_0 \neq \omega_1$. Conditions (ii′)–(v), (v′) are also satisfied by the way ω_0 and ω_1 were constructed. Suppose now that there exists $i \in \omega$ such that $\omega_{0,i} = \omega_1$. Since $\omega_{1,i} \in \omega_1$ but $\omega_{1,i} \notin \omega_{0,i}$, we obtain a contradiction. Hence $\omega_1 \notin \omega_0$. Similarly, supposing that there exists $i \in \omega$ such that $\omega_{1,i} = \omega_0$, we observe that $\omega_{0,i+1} \in \omega_0$ but $\omega_{0,i+1} \notin \omega_{1,i}$, another contradiction, leading us to the conclusion $\omega_0 \notin \omega_1$. ■

Even though it may seem that $\underline{\iota\iota}_2$ is just a particular case of $\underline{\iota\iota}_1$, in the following proposition we show that, in fact, under ZF − FA + AFA, $\underline{\iota\iota}_2(a,b)$ holds whenever $\underline{\iota\iota}_1(a,b)$ holds. However, the converse is not true, as testified by the two hypersets ω_0', ω_1'—given

$\mathbf{X}_0 = \{Y_0\} = \{\mathbf{X}_1 \cup \mathbf{A}_0\}$ $\qquad$ $\mathbf{X}_1 = \{Y_1\} = \{\mathbf{X}_0 \cup \mathbf{X}_1 \cup \mathbf{A}_1\}$



If $\mathbf{A}_0$, $\mathbf{A}_1$, $\mathbf{X}_0$, $\mathbf{X}_1$, are sets such that the following conditions hold,

- for $i = 0, 1$, $\mathbf{X}_i \neq \emptyset$, $\mathbf{A}_i \notin \mathbf{X}_i$,
- for all $y \in \mathbf{X}_0$, $\mathbf{A}_0 \subseteq y$, $y \setminus \mathbf{A}_0 \subseteq \mathbf{X}_1$, and $y \cap \mathbf{X}_0 = \emptyset$,
- for all $y \in \mathbf{X}_1$, $\mathbf{A}_1 \subseteq y$, $y \setminus \mathbf{A}_1 \subseteq \mathbf{X}_1 \cup \mathbf{X}_0$, and $y \cap \mathbf{X}_1 \neq \emptyset$,

then $\mathbf{X}_0, \mathbf{X}_1$ are those unique sets which solve the system

$$X_0 = \{Y_0\},\ X_1 = \{Y_1\}, \qquad Y_0 = X_1 \cup \mathbf{A}_0,\ Y_1 = X_0 \cup X_1 \cup \mathbf{A}_1$$

of equations.

Figure 6.17: Another consequence of AFA.

in Proposition 6.2.15—, which satisfy $\underline{\iota\iota}_2$, but without satisfying $\underline{\iota\iota}_1$. For this, let us point out in Figure 6.17 another plain consequence of AFA, similar to the one given for $\widetilde{\iota\iota}$.

Lemma 6.2.14 *Under* ZF − FA + AFA*, if* ω_0, ω_1 *are sets such that* $\underline{\iota\iota}_1(\omega_0, \omega_1)$ *is true, then* $\underline{\iota\iota}_2(\omega_0, \omega_1)$ *also holds.*

Proof. Let $c_1, c_3 \in \omega_1$, $c_1 \neq c_3$, and $c_2 \in \omega_0$ such that $c_1 \in c_2 \in c_3$, but $c_3 \notin c_1$. From condition (v) of $\underline{\iota\iota}_1$ we have $c_1 \in c_3$, which implies that $c_2 \in c_1$ holds as well, from (iv′).

Consider now the subset of elements of $\omega_0 \cup \omega_1$ from which there is an alternating membership chain between ω_0 to ω_1 to one of c_1 or c_2, that is,

$$\begin{aligned}\mathbf{X} \;=\; \{\, y :\ & y \in \omega_0 \cup \omega_1 \wedge \text{there exist } m \in \omega \text{ and } y_1, y_2, \ldots, y_m \in \omega_0 \cup \omega_1 \\ & \text{such that } y = y_0 \ni y_1 \ni y_2 \ni \cdots \ni y_m \text{ and } y_m \in \{c_1, c_2\} \text{ and} \\ & y_{2k} \in \omega_0 \text{ and } y_{2k+1} \in \omega_1, \text{ or } y_{2k} \in \omega_1 \text{ and } y_{2k+1} \in \omega_0,\ \forall k,\ 0 \leqslant k \leqslant (m-1)/2 \,\}.\end{aligned}$$

Then, we can apply the observation in Figure 6.17, by taking $\mathbf{X}_0 = \mathbf{X} \cap \omega_0$, $\mathbf{X}_1 = \mathbf{X} \cap \omega_1$, and $\mathbf{A}_i = (\bigcup \mathbf{X}_i) \setminus \mathbf{X}$ for $i = 0, 1$ (thus, $\mathbf{X}_0 \cup \mathbf{A}_1 = \omega_0$ and $\mathbf{X}_1 \cup \mathbf{A}_0 = \omega_1$).

Indeed, the $\mathbf{X}_i$'s are nonnull, as $c_2 \in \mathbf{X}_0$ and $c_1 \in \mathbf{X}_1$. Since for any $y \in \mathbf{X}$, we have $y \cap \mathbf{X} \neq \emptyset$, we deduce $\mathbf{A}_i \notin \mathbf{X}_i$, for $i = 0, 1$. Let now $y \in \mathbf{X}_i$. To see that $\mathbf{A}_i \subseteq y$, consider a $z \in \mathbf{A}_i$ such that $z \notin y$. From (iii), we get $y \in z$, which implies $z \in \mathbf{X}$, against the choice of z.

Moreover, on the one hand, for any $y \in \mathbf{X}_0$, $y \setminus \mathbf{A}_0 \subseteq \mathbf{X}_1$ follows from (ii′), while $y \cap \mathbf{X}_0 = \emptyset$ follows from $\omega_0 \cap \omega_1 = \emptyset$ and (ii′). On the other hand, for any $y \in \mathbf{X}_1$, also $y \setminus \mathbf{A}_1 \subseteq \mathbf{X}_0 \cup \mathbf{X}_1$ follows from (ii′). Supposing now that $y \cap \mathbf{X}_1$ were empty, by (v) we get that for any $z \in \mathbf{X}_1$, $y \in z$. By Lemma 6.2.10 we get that for any $v \in \mathbf{X}_0$, there exists $z \in \omega_1$ so that $v \in z$. This also entails that $z \in \mathbf{X}_1$. Therefore, by (iv′), we have $\mathbf{X}_0 \subseteq y$, which entails $\omega_0 \subseteq y$. This contradicts Lemma 6.2.9.

Therefore, $\mathbf{X}_0 = \{\mathbf{Y}_0\}$, $\mathbf{X}_1 = \{\mathbf{Y}_1\}$, where $\mathbf{Y}_0 = \mathbf{X}_1 \cup \mathbf{A}_0$, $\mathbf{Y}_1 = \mathbf{X}_0 \cup \mathbf{X}_1 \cup \mathbf{A}_1$. Hence $\mathbf{Y}_0 = \omega_1$, which, given that $\mathbf{X}_0 \subseteq \omega_0$, entails $\omega_1 \in \omega_0$, in contradiction with (i). ■

Proposition 6.2.15 *Under* ZF − FA + AFA*, there exist hypersets* ω_0' *and* ω_1' *such that* $\underline{\iota\iota}_2(\omega_0', \omega_1') \wedge \neg\underline{\iota\iota}_1(\omega_0', \omega_1'))$ *holds.*

Proof. To obtain such a model ω_0', ω_1' of $\underline{\iota\iota}_2 \wedge \neg\underline{\iota\iota}_1$, it will suffice to replace, in the model ω_0, ω_1 of $\underline{\iota\iota}_2$, vertex $\omega_{1,0}$ by five new vertices connected to one another as shown in Figure 6.18 (and connecting each of them with the rest of the vertices in the same way $\omega_{1,0}$ was connected with). The proof of Theorem 6.2.13 can be closely followed to show that these sets are indeed hypersets.

It can easily be seen that $\underline{\iota\iota}_2(\omega_0', \omega_1')$ is true. However, $\underline{\iota\iota}_1(\omega_0', \omega_1'')$ does not hold, since $\omega_{1,0}^3, \omega_{1,0}^4 \in \omega_0''$ but $\omega_{1,0}^3 \neq \omega_{1,0}^4$, $\omega_{1,0}^3 \notin \omega_{1,0}^4$, and $\omega_{1,0}^4 \notin \omega_{1,0}^3$, in conflict with condition (v) of $\underline{\iota\iota}_1(\omega_0', \omega_1')$. ■

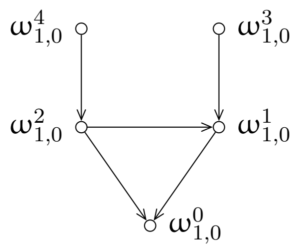


Figure 6.18: Five new vertices replacing $\omega_{1,0}$ in the model of $\underline{\iota\iota}_2 \wedge \neg\underline{\iota\iota}_1$.

Theorem 6.2.16 *Under* ZF − FA + AFA*, if* ω_0, ω_1 *are sets such that either* $\underline{\iota\iota}_1(\omega_0, \omega_1)$ *or* $\underline{\iota\iota}_2(\omega_0, \omega_1)$ *is true, then* ω_0 *and* ω_1 *are infinite.*

Proof. We will show that the second condition of the hypothesis of Theorem 6.2.11 holds in both cases. If $\underline{\iota\iota}_2(\omega_0, \omega_1)$ is true, then assume for a contradiction that $y_1, y_2 \in \omega_1$ and $x_1, x_2 \in \omega_0$ are such that $x_1 \in y_1 \setminus y_2$ and $x_2 \in y_2 \setminus y_1$. By (iii), we get $y_2 \in x_1$, which by (v′) implies $y_1 \in y_2$, and thus $y_2 \subseteq y_1$, by (iv′). If $\underline{\iota\iota}_1(\omega_0, \omega_1)$ is true, the claim readily follows from conditions (iv′) and (v) (and also from the above argument, in light of Lemma 6.2.14). ■

Having established that any sets ω_0, ω_1 satisfying $\underline{\iota\iota}_1$ or $\underline{\iota\iota}_2$ are infinite, we are concerned with characterizing their models. The following proposition, similar to a statement in the proof of [114, Proposition 6], characterizes in more detail the structure of $\underline{\iota\iota}_1$ and $\underline{\iota\iota}_2$.

Proposition 6.2.17 *Assuming that ω_0 and ω_1 satisfy $\underline{\iota\iota}_2$, if X is a finite nonnull subset of $\omega_0 \cup \omega_1$, then there is an element $c_X \in X$ such that one of the following two properties hold:*

- $c_X \in \omega_0$ *and* $X \cap \omega_1 \subseteq c_X$,
- $c_X \in \omega_1$ *and* $X \cap \omega_0 \subseteq c_X$.

Proof. We will prove the claim by induction on the cardinality of X. If X is a singleton the claim is clear, since $\omega_0 \cap \omega_1 = \emptyset$. Otherwise, if $X \cap \omega_0 = \emptyset$, then every element in X can be taken as c_X. Otherwise, pick $a_* \in X \cap \omega_0$ and let $X' = X \setminus \{a_*\}$.

By the induction hypothesis applied to X', there is a $c_{X'} \in X'$ satisfying our claim. If $c_{X'} \in \omega_0$ and $X' \cap \omega_1 \subseteq c_{X'}$, then, since $\omega_0 \cap \omega_1 = \emptyset$, we have $X \cap \omega_1 = X' \cap \omega_1$. Hence $X \cap \omega_1 \subseteq c_{X'}$ and we can take c_X to be $c_{X'}$. On the other hand, if $c_{X'} \in \omega_1$ and $X' \cap \omega_0 \subseteq c_{X'}$, we have two cases: $a_* \in c_{X'}$ and $a_* \notin c_{X'}$.

In the former case, it suffices to take $c_X = c_{X'}$. In the latter, since $(\forall x \in \omega_0)(\forall y \in \omega_1)(x \in y \vee y \in x)$, $c_{X'} \in a_*$. If $\omega_1 \cap X \subseteq a_*$, then it suffices to let $c_X = a_*$. Otherwise there must be a $b_* \in \omega_1 \cap X$ such that $b_* \notin a_*$. Hence, as before, we have $a_* \in b_*$. Since $b_* \neq c_{X'}$, from condition (v′) we have $b_* \in c_{X'}$, and from condition (iv′) we get $X' \cap \omega_0 \subseteq b_*$; so we can take $c_X = b_*$. ■

Actually, this rich information about the structure of any model of $\underline{\iota\iota}_2$ is enough to provide a second proof of the fact that $\underline{\iota\iota}_2$ (and hence, by Lemma 6.2.14, that also $\underline{\iota\iota}_1$) is infinitely satisfiable.

Second Proof of Theorem 6.2.16. Assume that ω_0, ω_1 satisfy $\underline{\iota\iota}_2$ and that $\omega_0 \cap \omega_1$ is finite. From Proposition 6.2.17 applied to $X = \omega_0 \cap \omega_1$, we can find $c_X \in X$ satisfying one of the two claims of that proposition. In the first case, due to (ii′), we have $c_X = \omega_1$, contradicting (i). In the second case, $\omega_0 \subseteq c_X$ contradicts Lemma 6.2.9. ■

To see that that all models of $\underline{\iota\iota}_2$ are non-well-founded, let ω_0 and ω_1 satisfy $\underline{\iota\iota}_1$ or $\underline{\iota\iota}_2$. By Lemma 6.2.10, we can take $y_1, y_2 \in \omega_1$ and $x \in \omega_0$ such that $y_2 \in x \in y_1$. If $\underline{\iota\iota}_2(\omega_0, \omega_1)$ holds, then $y_1 \in y_2$ follows, which produces a membership cycle in both $\mathsf{TrCl}(\omega_0)$ and $\mathsf{TrCl}(\omega_1)$ (transitive closures which, incidentally, due to (ii) and to Lemma 6.2.10, coincide).

Moreover, in both models ω_0, ω_1 proposed for $\underline{\iota\iota}_1$ or for $\underline{\iota\iota}_2$ there is an infinite descending membership chain in ω_1 with no repeated elements. In Proposition 6.2.20 we show that this property holds for any model of either $\underline{\iota\iota}_1$ or $\underline{\iota\iota}_2$, in blatant violation of FA. The following is a preparatory lemma.

Lemma 6.2.18 *If ω_0 and ω_1 satisfy $\underline{\iota\iota}_2$, then for all $y \in \omega_1$, $y \cap \omega_1 \neq \emptyset$.*

Proof. Let $y_1 \in \omega_1$. By Lemma 6.2.10, we can find $y_2 \in \omega_0$ and $y_3 \in \omega_1$ such that $y_1 \in y_2 \in y_3$. This implies, by condition (v′), that $y_3 \in y_1$, and hence $y_1 \cap \omega_1 \neq \emptyset$. ■

In the following proposition we resort to a useful variant of the transitive closure operation: the *relativized transitive closure operation*, depending on a parameter b, which sends every set s into the set $\mathsf{TrCl}_b(s)$ formed by those s' which can reach s through membership without ever leaving b. The following definition readjusts the earlier definition of transitive closure to our current needs:

Definition 6.2.19 *Given sets b and s, we define $\mathsf{TrCl}_b(s)$ to be the set formed by those s' for which there is a finite-length path*

$$\begin{array}{ccccccc} s & = & s_0 & \ni \cdots \ni & s_{n+1} & = & s'\,. \\ & & \Cap & \cdots & \Cap & & \\ & & b & & b & & \end{array}$$

Proposition 6.2.20 *If ω_0 and ω_1 satisfy $\underline{\iota\iota}_1$ or $\underline{\iota\iota}_2$ then ω_1 contains an infinite descending membership chain with no repeated elements.*

Proof. As ω_1 is infinite (from Theorem 6.2.16), Lemma 6.2.18 implies that the set $Z = \{z \in \omega_1 \mid z \in \mathsf{TrCl}_{\omega_1}(\{z\})\}$ is nonnull. Moreover, we can find a $z_0 \in Z$ such that for all $z \in Z \setminus \{z_0\}$, $\mathsf{TrCl}_{\omega_1}(\{z_0\}) \nsubseteq \mathsf{TrCl}_{\omega_1}(\{z\})$. From condition (iv$'$) and the choice of z_0, we have that $(\forall y \in \mathsf{TrCl}_{\omega_1}(\{z_0\}))(y \cap \omega_0 = z_0 \cap \omega_0)$. Additionally, $(\forall y \in \mathsf{TrCl}_{\omega_1}(\{z_0\})(\emptyset \neq y \cap \omega_1 \subseteq \mathsf{TrCl}_{\omega_1}(\{z_0\}))$, and hence $z_0 = \Omega_{z_0 \cap \omega_0}$ entailing $z_0 \in z_0$. This violates the condition $(\forall y \in \omega_1)(y \notin y)$. ■

For the remainder of this section, we refrain again from assuming FA or AFA. The following formula $\overline{o\iota}$—the conjunction of the subsequent four conditions—is a variation of $\underline{\iota\iota}_2$, so that the inclusion appearing in condition (iv$'$) is reversed; accordingly, (ii$'$) has to be changed into (ii$''$). These changes also guarantee that no further condition governing the membership relations among elements of b has to be required. Just like for $\overline{\iota\iota}$, the working of $\overline{o\iota}$ is immaterial of FA, or of AFA; however, this is now obtained with only three universal quantifiers, instead of the four of $\overline{\iota\iota}$. Moreover, $\overline{o\iota}$ is satisfied only by non-well-founded sets, since $(\forall y \in b)(y \in y)$ follows from the last conjunct of (ii$''$) and from (iii).

(i) $a \neq b \wedge a \notin b \wedge b \notin a$

(ii$''$) $\bigcup(a \setminus b) \subseteq b \wedge \bigcup b \subseteq a \wedge b \subseteq a$

(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$

(iv$''$) $(\forall y_1, y_2 \in b)(y_1 \in y_2 \rightarrow y_1 \subseteq y_2)$

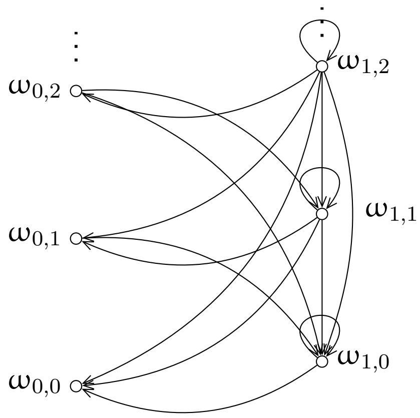
A hyperset model of $\overline{o\iota}$ is depicted in Figure 6.19; the proof of Theorem 6.2.13 can be closely followed to show that these sets are indeed hypersets. Our first lemma is closely analogous to Lemma 6.2.1.

Lemma 6.2.21 *If ω_0, ω_1 are sets such that $\overline{o\iota}(\omega_0, \omega_1)$ is true, then $(\forall x \in \omega_0 \setminus \omega_1)(\exists y \in \omega_1)(x \in y)$.*

Proof. Assume for a contradiction that $(\forall y \in \omega_1)(x_0 \notin y)$ holds for some $x_0 \in \omega_0 \setminus \omega_1$. By (iii), we have that $(\forall y \in \omega_1)(y \in x_0)$. Thus, since $\omega_1 \subseteq x_0 \subseteq \bigcup(\omega_0 \setminus \omega_1) \subseteq \omega_1$ by (ii$'$), we have $x_0 = \omega_1$, contradicting $\omega_1 \notin \omega_0$. ■

Lemma 6.2.22 *If ω_0, ω_1 are sets such that $\overline{o\iota}(\omega_0, \omega_1)$ is true, then $(\forall y_1, y_2 \in \omega_1)(y_1 \subseteq y_2 \vee y_2 \subseteq y_1)$.*

Proof. Immediate from the conjunct $\omega_1 \subseteq \omega_0$ of (ii$''$), (iii) and (iv$''$). ■



$$\omega_0 = \{\omega_{1,j} : i \in \omega\}, \qquad \omega_{0,j} = \{\omega_{1,k} : 0 \leqslant k < j\},$$
$$\omega_1 = \{\omega_{0,j} : i \in \omega\} \cup \omega_0, \quad \omega_{1,j} = \{\omega_{0,k} : 0 \leqslant k \leqslant j\} \cup \{\omega_{1,k} : k \leqslant j\}, \quad \forall j \in \omega.$$

Figure 6.19: Hypersets ω_0 and ω_1 such that $\overline{o\iota}(\omega_0, \omega_1)$ holds.

Lemma 6.2.23 *If ω_0, ω_1 are sets such that $\overline{o\iota}(\omega_0, \omega_1)$ is true, and ω_1 is finite, then $(\exists y \in \omega_1)(\omega_1 \subseteq y)$.*

Proof. Suppose that this were not the case, and consider $y_0 \in \omega_1$ such that $y_0 \cap \omega_1$ is $\subseteq$-maximal. Let $y_1 \in \omega_1 \setminus y_0$. From (ii$''$) and (iii) we get $y_0 \in y_1$, which implies, by (iv$''$), that $y_0 \subseteq y_1$. Likewise we have $y_1 \in y_1$, which contradicts the maximality of y_0. ■

Theorem 6.2.24 *Under* ZF − FA*, if ω_0, ω_1 are sets such that $\overline{o\iota}(\omega_0, \omega_1)$ is true, then ω_0 and ω_1 are infinite.*

Proof. Notice first that $\omega_1 \neq \emptyset$, since otherwise, by (ii$''$), either $\omega_0 = \emptyset$ or $\omega_0 = \{\emptyset\}$ would hold. This contradicts (i). Now, assume for a contradiction that ω_1 is finite. By Lemma 6.2.22, let y_0 stand for the $\subseteq$-maximum element of ω_1. From Lemma 6.2.21, we have that $\omega_0 \setminus \omega_1 \subseteq y_0$. From Lemma 6.2.23, let $y_1 \in \omega_1$ such that $\omega_1 \subseteq y_1$. From the maximality of y_0 we have $\omega_1 \subseteq y_1 \subseteq y_0$, and hence $\omega_0 \subseteq y_0$. Moreover, $y_0 \subseteq \bigcup \omega_1 \subseteq \omega_0$, entailing $y_0 = \omega_0$, contradicting $\omega_0 \notin \omega_1$. Note that the infinitude of ω_1 also implies the infinitude of ω_0, since $\bigcup \omega_1 \subseteq \omega_0$. ■

6.3 Expressiveness of the set theoretic BSR class

6.3.1 A satisfiability-preserving translation from a logical context

We argue now that the set theoretic BSR class is strictly more expressive than the logical one. It is well-known since [126] that one can analyze the *full spectrum* of interpretations modeling an $\mathcal{L}$-BSR-sentence when $\mathcal{L}$ is an uninterpreted language with equality. That is, given an $\mathcal{L}$-BSR sentence Φ, one can determine the set $\mathcal{S}_\Phi$ of all positive integers M such that Φ owns a model whose underlying domain has cardinality M. Ramsey's analysis implies that if $\mathcal{S}_\Phi$ comprises an adequately large number R_Φ, then $\mathcal{S}_\Phi$ will include all integers greater than R_Φ. Our expectation, in view of the richness of a set-solution of a $\forall^*$-formula such as $\widetilde{\iota\iota}(a,b)$, is that $\mathcal{S}_\Phi$ can be fully described, for any given $\mathcal{L}$-BSR Φ, by a single set theoretic BSR-sentence. With this goal in mind, for the time being we simply discuss how to perform a satisfiability-preserving translation of BSR-sentences from an uninterpreted, purely logical context into one referring to sets. As one notices in this exploratory phase, for a straightforward translation of this kind one can best rely on Aczel's hypersets.

Let us make the inessential simplifying assumption that the signature consists of a dyadic relator ϱ and equality, in short that $\mathcal{L} = \mathcal{L}_\varrho$; and refer by $\mathcal{U} = (\mathcal{U}, \in)$ to a model of $\mathsf{ZF}^- - \mathsf{FA} + \mathsf{AFA}$. To convert any $\mathcal{L}_\varrho$-BSR-sentence Φ into a BSR-sentence $\dot{\Phi}$ of the language $\mathcal{L}_\in$ *interpreted* in $\mathcal{U}$, proceed as follows. If

$$\Phi \quad \equiv \quad \exists\, x_1 \cdots \exists\, x_n\, \forall\, y_1 \cdots \forall\, y_m\, \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)\,,$$

then put

$$\dot{\Phi} \quad \equiv \quad \exists\, d\, (\exists\, x_1 \in d) \cdots (\exists\, x_n \in d) \quad (\forall\, y_1 \in d) \cdots (\forall\, y_m \in d) \qquad \varphi_\in^\varrho(x_1, \ldots, x_n, y_1, \ldots, y_m)\,,$$

where $\varphi_\in^\varrho$ results from φ through uniform replacement of ϱ by $\in$.

We will now see that Φ and $\dot{\Phi}$ are equi-satisfiable. On the one hand, in fact, it is plain that for any tuple $\mathbf{d}, \mathbf{s}_1, \ldots, \mathbf{s}_n$ of sets such that the membership relations $\mathbf{s}_1 \in \mathbf{d}, \ldots, \mathbf{s}_n \in \mathbf{d}$ hold in $\mathcal{U}$ and

$$\mathcal{U} \models (\forall\, y_1 \in \mathbf{d}) \cdots (\forall\, y_m \in \mathbf{d})\, \varphi_\in^\varrho(\mathbf{s}_1, \ldots, \mathbf{s}_n, y_1, \ldots, y_m)\,,$$

the interpreting structure $\mathcal{M} = (\mathbf{d}, \varrho^\mathcal{M})$ with

$$\varrho^\mathcal{M} \quad = \quad \{\, \langle u, v\rangle :\; u \in \mathbf{d} \wedge v \in \mathbf{d} \wedge u \in v\,\},$$

enforces

$$\mathcal{M} \models \forall\, y_1 \cdots \forall\, y_m\, \varphi(\, \mathbf{s}_1, \ldots, \mathbf{s}_n,\; y_1, \ldots, y_m\,)\,.$$

(To avoid that $\mathbf{d}$ can be void, in case $n = 0$ we tab Φ with a dummy existential variable before translating it).

On the other hand, consider an interpreting structure $\mathcal{M} = (\mathcal{D}, \varrho^\mathcal{M})$ and an n-tuple $\mathbf{d}_1, \ldots, \mathbf{d}_n$ of elements of $\mathcal{D}$ such that

$$\mathcal{M} \models \forall\, y_1 \cdots \forall\, y_m\, \varphi(\, \mathbf{d}_1, \ldots, \mathbf{d}_n,\; y_1, \ldots, y_m\,)\,.$$

Since the restriction of $\mathcal{M}$ to the domain $\{\mathbf{d}_1, \ldots, \mathbf{d}_n\}$ gives us a model of $\forall\, y_1, \ldots, \forall\, y_m\, \varphi(\, \mathbf{d}_1, \ldots, \mathbf{d}_n,\; y_1, \ldots, y_m\,)$, we will assume that $\mathcal{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_n\}$.

Associate with each u in $\mathcal{D}$ two sets $\dot{u}, \ddot{u}$ subject to the following constraints:

$\dot{u} = \{\, \dot{w} :\; \langle w, u\rangle \in \varrho^\mathcal{M}\} \cup \ddot{u}$;

$\ddot{u} = \{z\}$, where z is of cardinality $\#z > \#\mathcal{D}$, and $z \in \mathsf{HF}$;

the mapping $u \mapsto \ddot{u}$ is injective over $\mathcal{D}$, i.e., we choose different z's for different u's.

Much as in the proof of Theorem 6.1.2, it should be clear that the mapping $u \mapsto \dot{u}$ is injective and that its images are hereditarily finite hypersets. In consequence of this injectivity and of how the $\dot{u}$'s have been constructed, it turns out that

$$\mathcal{U} \models (\forall\, y_1 \in \mathbf{d}) \cdots (\forall\, y_m \in \mathbf{d})\; \varphi^{\varrho}_{\in}(\dot{\mathbf{d}}_1, \ldots, \dot{\mathbf{d}}_n, y_1, \ldots, y_m),$$

where $\mathbf{d} = \{\, \dot{\mathbf{d}}_1, \ldots, \dot{\mathbf{d}}_n \,\}$. We conclude thus our proof of equi-satisfiability:

Lemma 6.3.1 *A BSR-sentence $\exists \vec{x}\, \Psi(\vec{x})$ of $\mathcal{L}_\varrho$ is satisfiable if and only if for a suitable array $\vec{t}$ of hereditarily finite hypersets, $\Psi^{\varrho}_{\in}(\vec{t}\,)$ is true in any model $\mathcal{U}$ of* $\mathsf{ZF}^- - \mathsf{FA} + \mathsf{AFA}$.

Example 6.3.2 Consider the $\mathcal{L}_\varrho$-BSR-sentence

$$\Phi \equiv \exists x_0 \exists x_1 \forall y_1 \forall y_2 \big(\, x_0 \neq x_1 \textstyle\bigwedge_{i=0}^{1} x_i \,\varrho\, x_{1-i} \wedge ((\, x_0 \,\varrho\, y_1 \wedge y_1 \,\varrho\, y_2 \,) \rightarrow x_0 \,\varrho\, y_2 \,)\,\big).$$

The corresponding $\mathcal{L}_\in$-BSR-sentence $\dot{\Phi}$ defined above is:

$$\begin{aligned}\dot{\Phi} \equiv \exists d\, \exists x_0 \in d\, \exists x_1 \in d\, \forall y_1 \in d\, \forall y_2 \in d\, \big(\, x_0 \neq x_1 \textstyle\bigwedge_{i=0}^{1} x_i \,\in\, x_{1-i} \wedge \\ ((\, x_0 \,\in\, y_1 \wedge y_1 \,\in\, y_2 \,) \rightarrow x_0 \,\in\, y_2 \,)\,\big).\end{aligned}$$

As $\dot{\Phi}$ is satisfied by the HF-sets $\mathbf{d}, \mathbf{x}_0, \mathbf{x}_1$ such that $\mathbf{d} = \{\mathbf{x}_0, \mathbf{x}_1\}$, $\mathbf{x}_0 = \{\mathbf{x}_0, \mathbf{x}_1\}$, $\mathbf{x}_1 = \{\emptyset, \mathbf{x}_0\}$, we conclude that Φ is satisfiable as well. ■

Theorem 6.3.3 *The class $\mathcal{L}_\in$-BSR is strictly more expressive than the class $\mathcal{L}_\varrho$-BSR.*

Proof. On the one hand, the mapping $\Phi \mapsto \dot{\Phi}$ is a satisfiability preserving translation of $\mathcal{L}_\varrho$-BSR into $\mathcal{L}_\in$-BSR.

On the other hand, as seen in this chapter, there is an $\mathcal{L}_\in$-BSR-sentence $\exists a\, \exists b\, \widetilde{\iota\iota}(a, b)$ such that the domain of any model $\mathcal{U}$ of $\mathsf{ZF}^- - \mathsf{FA} + \mathsf{AFA}$ + $\exists a\, \exists b\, \widetilde{\iota\iota}(a, b)$ must have an $\mathbf{a}$ for which $\{x \in \mathcal{U} \,|\, x \in \mathbf{a}\}$ is infinite. This possibility cannot be offered by $\mathcal{L}_\varrho$-BSR because, as observed above, an n'-element structure with $n' \leqslant n$ always suffices to model a satisfiable $\mathcal{L}_\varrho$-BSR-sentence that involves n existentially quantified variables. ■

Our ending remark is relevant to the determination of all sizes of models of a given $\mathcal{L}_\varrho$-BSR-sentence. One can impose that the original formula Φ be satisfied in a domain endowed with at least M elements by simply enriching $\dot{\Phi}$ to the effect that $\#d \geqslant M$:

$$\begin{aligned}\dot{\Phi} \;\; \equiv \;\; & \exists\, d\, (\exists\, x_1 \in d) \cdots (\exists\, x_{n+M} \in d)\; (\forall\, y_1 \in d) \cdots (\forall\, y_m \in d) \\ & \Big(\textstyle\bigwedge_{n<i<j\leqslant n+M} x_i \neq x_j \;\wedge\; \varphi^{\varrho}_{\in}(x_1, \ldots, x_n, y_1, \ldots, y_m) \Big)\;.\end{aligned}$$

Note also that, thanks to the extensionality of set membership, all occurrences of '=' can be eliminated from $\dot{\Phi}$ without leaving the $\mathcal{L}_\in$-BSR class: this ensues from the basic combinatorial fact (cf., e.g., [100]), that for every $N+1$-tuple $\vec{\mathbf{x}}$ of sets there is an N-tuple $\vec{\mathbf{d}}$ of sets such that no two different components of $\vec{\mathbf{x}}$ have the same elements inside $\vec{\mathbf{d}}$.

6.3.2 Considerations on parsimonious sets and finite representability

I dare insinuate the following solution to this ancient problem: The Library is limitless and periodic. If an eternal voyager were to traverse it in any direction, he would find, after many centuries, that the same volumes are repeated in the same disorder (which, repeated, would constitute an order: Order itself). My solitude rejoices in this elegant hope. [21, pp. 87–88]

Looking for a model of a set-theoretic formula φ is looking for a needle in a haystack, because the assignments of sets to the free variables of φ form a *proper class*: a class, in Cantor's metaphor, which is 'too big' to be a set. Chances to achieve success presuppose necessarily, on our part, an ability to contract the search space to a set of reasonable size—ideally, a finite set; or, if we cannot do that, a recursively enumerable infinite set.

In an effort to associate with every potential model of φ a finite representation belonging to a finite or denumerable inventory (a finite digraph, a hereditarily finite hyperset, or the like), we must be able in the first place to single out assignments which carry little or no redundancy. What one shall regard as redundancy depends, of course, on which fragment of the first-order language about sets one is analyzing: in our case this is the collection of all BSR-formulae, which are expressive enough—as we have seen—to specify spirals interlacing two or more infinite hypersets, but, presumably, do not have all sophistication required to enter into the detailed structure of an infinite set beyond the boundary of finite representability.

Definition 6.3.4 provided here below originates from an attempt to reduce significantly the amount of redundancy present in the family $\mathcal{F}$ of set values to be associated with the variables of φ. We expect that further techniques to eliminate redundancy will emerge from forthcoming research; this is why we do not dare to speak of irreducible models as yet, but only of 'succinct' models.

We now know candidate φ which promise to be challenging for our prospective satisfiability algorithm; therefore we can contrast this definition of succinct family with the simplest model we have been able to devise for our $\underline{\iota\iota}_1, \underline{\iota\iota}_2$. One more model has been discussed in Proposition 6.2.15, but definitely it cannot be judged succinct. For expository purposes, we also favor formulae $\underline{\iota\iota}_1, \underline{\iota\iota}_2$ over $\overline{o\iota}$, since their models have been shown to have both 'signs' of non-well-foundedness: finite membership cycles and infinite descending membership chain with no repeated elements. Nevertheless, the framework we propose onwards applies to all formulae given in this chapter.

Definition 6.3.4 (Succinct family) *Let $D = (V, E)$ be a hyper-extensional digraph; let moreover $F \subseteq V$, and let $H = (V', E')$ be a vertex-induced subdigraph of D such that $F \subseteq V'$. If H is hyper-extensional, then we say that H* represents *(F, D).*

If for any digraph H that represents (F, D), we can find a digraph H' which represents (F, H) and is isomorphic to D, then we say that (F, D) is a succinct *family.*

In the ongoing we will exploit the notation $\mathbf{ii}_{\omega}$ for the membership digraph of the hyperset $\{\omega_0, \omega_1\}$, where ω_0, ω_1, depicted in Figure 6.16, is the shared model of $\underline{\iota\iota}_1$ and $\underline{\iota\iota}_2$. In what follows we will show that $(\{\omega_0, \omega_1\}, \mathbf{ii}_{\omega})$ is a *succinct* family. To see this, let us first consider an example.

Let H be the vertex-induced subdigraph of $\mathbf{ii}_{\omega}$ whose vertices are $V(H) = \mathsf{TrCl}(\{\omega_0, \omega_1\}) \setminus \{\omega_{0,1}\}$. Digraph H is hyper-extensional, as the removal of $\omega_{0,1}$ from

$V(H)$ could cause a collision only between $\omega_{1,0}$ and $\omega_{1,1}$. This is not the case, though; in fact $\omega_{1,1} \in \omega_{1,0}$, but no element of $\omega_{1,1}$ is bisimilar to $\omega_{1,0}$. Furthermore, it is easy to see that H is not isomorphic to $\mathbf{ii}_\omega$. However, the removal of $\omega_{1,0}$ from the set of the vertices of H leads us to an H' representing (F, H), isomorphic to $\mathbf{ii}_\omega$.

This example shows why we cannot adopt, in our non-well-founded context, a simplification to the definition of succinctness which would be viable in the classical *well-founded* universe of sets, namely: "We say that (F, D) is a succinct family if any digraph H that represents (F, D) is isomorphic to D".

Besides this annoyance, we have also an unpleasant paradox: unless it has either finite cardinality or a well-founded transitive closure, a succinct family (F, D) admits representing digraphs H_i $(i \in \omega)$ where odd-indexed graphs $H_{2\cdot i+1}$ are not isomorphic to the even-indexed $H_{2\cdot i}$'s; although the (F, H_i) form a chain of strict inclusions, each $(F, H_{2\cdot i+1})$, very much like any $(F, H_{2\cdot i})$, is a succinct family.

The following preparatory lemma states some properties of any H that represents $(\{\omega_0, \omega_1\}, \mathbf{ii}_\omega)$, that will be used in the subsequent theorem to extract from H a digraph isomorphic to $\mathbf{ii}_\omega$.

Lemma 6.3.5 *In any H that represents $(\{\omega_0, \omega_1\}, \mathbf{ii}_\omega)$:*

(1) *both sets $V_{1-b}(H) = V(H) \cap \omega_b$ $(b = 0, 1)$, where $V(H)$ is the overall set of vertices, are infinite;*

(2) *if $l_b = \min\{k \in \omega : \omega_{b,k} \in V(H)\}$ for $b \in \{0, 1\}$, then $l_0 \leqslant l_1$.*

Proof. Begin by observing that $V(H)$ is infinite. If this were not the case, let, for $b \in \{0, 1\}$, $k_b = \max\{k \in \omega : \omega_{b,k} \in V(H)\}$. If $k_0 > k_1$, then ω_{0,k_0} is bisimilar to ω_0; otherwise ω_{1,k_1} is bisimilar to ω_1, contradicting the fact that H is hyper-extensional.
(1) Suppose that $V_b(H)$ is finite, and let $k_b = \max\{k \in \omega : \omega_{b,k} \in V_b(H)\}$. As $V(H)$ is infinite, there exists ω_{1-b,t_1} and ω_{1-b,t_2} in $V_{1-b}(H)$, with $t_1 > t_2 > k_b$. We have reached a contradiction, as ω_{1-b,t_1} and ω_{1-b,t_2} are bisimilar.
(2) If $l_0 > l_1$, then $\emptyset$ does not appear in the decoration of the vertices in $V(H)$, hence all the vertices are bisimilar (and bisimilar to $\Omega = \{\Omega\}$). ■

Theorem 6.3.6 $(\{\omega_0, \omega_1\}, \mathbf{ii}_\omega)$ *is a* succinct *family.*

Proof. Let H represent $(\{\omega_0, \omega_1\}, \mathbf{ii}_\omega)$ without being isomorphic to $\mathbf{ii}_\omega$. We will obtain a digraph H' isomorphic to $\mathbf{ii}_\omega$ and representing $(\{\omega_0, \omega_1\}, H)$ as the outcome of a (possibly infinite) series of 'mending' steps of the following kind:

> As long as $V_0(H)$ has a gap t_1, t_2 for which $t_1 + 1 < t_2$ and t_1, t_2 are consecutive in the sense that $\{\omega_{0,t} : \omega_{0,t} \in V(H) \mid k_1 < t < k_2\} = \emptyset$, observe that the corresponding subset $\mathcal{D} = \{\omega_{1,t} : \omega_{1,t} \in V(H) \mid k_1 \leqslant t < k_2\}$ of $V_1(H)$ is nonull. Keep only one of the vertices $d \in \mathcal{D}$, while removing from $V(H)$ all of $\mathcal{D} \setminus \{d\}$.

Plainly, the digraph H' which results at the end of the above mending process is isomorphic to $\mathbf{ii}_\omega$. ■

But infinity can be 'tamed' in the well-founded case, in the sense that if we invert the arcs of the membership digraph of a succinct family of infinite sets meeting a BSR specification, interpreting each sink as a distinct urelement, only finitely many hypersets will appear in the decoration (cf. [100, Conclusions]). Our next proposition shows that the situation with the hypersets that satisfy our novel formulae is radically different: despite forming a succinct family, our 'twins' have the peculiarity that infinitely many hypersets will occur in the decoration even after we invert membership.

Lemma 6.3.7 *If one regards ω_0 and ω_1 as distinct urelements, the digraph ${\mathbf{ii}_{\omega}}^{-1}$, whose arcs are the ones of $\mathbf{ii}_{\omega}$ but with opposite orientation, is hyper-extensional.*

Proof. At the outset, observe that $\omega_0 \neq \omega_{b,i}$ and $\omega_1 \neq \omega_{b,i}$, for all $b \in \{0,1\}$ and $i \in \omega$, as ω_0 are ω_1 are urelements, while $\omega_{b,i}$ contains either ω_0 or ω_1. Next, suppose that there are $i, j \in \omega$ such that $\omega_{0,i} = \omega_{1,j}$. But $\omega_0 \in \omega_{0,i}$, while no element in $\omega_{1,j}$ can be equal to ω_0. Hence, $\omega_{b,i} = \omega_{1-b,j}$, for all $b \in \{0,1\}$ and $i, j \in \omega$.

We will now prove by induction on $n \geqslant 1$ that $\omega_{1,i} \neq \omega_{1,j}$ for all $i, j \in \{0, \ldots, n\}, i \neq j$. For $n = 1$, $\omega_{1,0} \in \omega_{1,1}$, but $\omega_{1,0} \notin \omega_{1,0}$, as all elements of $\omega_{1,0}$ are different from $\omega_{1,0}$. For $n > 1$, suppose that there is an index $i < n$ such that $\omega_{1,n} = \omega_{1,i}$. We easily reach a contradiction, as $\omega_{1,i} \in \omega_{1,n}$, but, from the induction hypothesis, all elements of $\omega_{1,i}$ differ from $\omega_{1,i}$. Likewise, it can easily be seen that $\omega_{0,i} \neq \omega_{0,j}$ for all $i, j \in \omega, i \neq j$. ■

This lemma is another clue that the collection of satisfiable BSR formulae may require, for hyperset theory—if decidable at all in that context—, a significantly more challenging decision algorithm than for the standard Zermelo-Fraenkel set theory.

Nevertheless, we 'dare insinuate' a framework based on isomorphisms for describing finitely the membership digraph $\mathbf{ii}_{\omega}$, and, possibly, any other succinct family. To illustrate the point on our running example, call *seed* of $\mathbf{ii}_{\omega}$ the set $\{\omega_{0,0}, \omega_{1,0}\}$. Consider then the subdigraph $\mathbf{ii}_{\omega\setminus 1} = \mathbf{ii}_{\omega}[\mathsf{TrCl}(\{\omega_0, \omega_1\}) \setminus \{\omega_{0,0}, \omega_{1,0}\}]$, isomorphic to $\mathbf{ii}_{\omega}$. In the isomorphism, the vertices $\omega_{0,1}$ and $\omega_{1,1}$ correspond to the vertices of the seed, and hence they constitute the seed of the reduced digraph. If we repeat this process of elimination for all $i \in \omega$, every vertex of $\mathbf{ii}_{\omega}$ will become isomorphic, at a certain step $i \in \omega$, to one of the vertices of the original seed. Therefore, in order to represent $\mathbf{ii}_{\omega}$ finitely, we simply have to indicate its (finite) seed, the arcs between the vertices of the seed, and the behavior of the seed with respect to the other vertices (i.e., a finite symbolic representation of the set of arcs between the vertices of the seed and the other vertices of the digraph). This framework is independent of whether the infinite digraph we want to represent contains cycles or not. Indeed, all formulae in this chapter are finitely represented in the same manner.

6.3.3 Bizarre infinitude

> *Under the beneficent influence of the Company, our customs are saturated with chance. The buyer of a dozen amphoras of Damascus wine will not be surprised if one of them contains a talisman or a viper. The scribe who writes a contract almost never fails to introduce some erroneous information. I myself, in this hasty declaration, have falsified some splendor, some atrocity. Perhaps, also, some mysterious monotony [$\cdots$]* [21, pp. 71]

We now briefly discuss two ways in which the models of the BSR-formulae considered in this chapter can differ from the 'standard' ones given earlier.

Definition 6.3.8 *Given a set s, we say that s is* η-incremental *if the following property holds, for all $x, y \in s$:*
if $r = \min\{\mathsf{rank}(x), \mathsf{rank}(y)\}$ satisfies $r = \varrho + \eta$ (for some ordinal ϱ),
then $x^{<(r-\eta)} = y^{<(r-\eta)}$.

Proposition 6.3.9 *Any sets satisfying $\iota\iota^n(x_0, x_1, \ldots, x_{n-1})$ are $(n(n-2))$-incremental, for any $n \geqslant 2$.*

Proof. Consider the case $n = 2$ and let (ω_0, ω_1) be such that $\iota\iota^2(\omega_0, \omega_1)$ is true. Without loss of generality, let $x, y \in \omega_1$ be such that, for $r = \mathsf{rank}(x) \leqslant \mathsf{rank}(y)$, but $x^{<r} \neq y^{<r}$. Assume, again w.l.o.g., that there exists $z \in x^{<r}$ such that $z \notin y^{<r}$. From condition (iii), we get $y \in z$, which contradicts that fact that $\mathsf{rank}(z) < r \leqslant \mathsf{rank}(y)$.

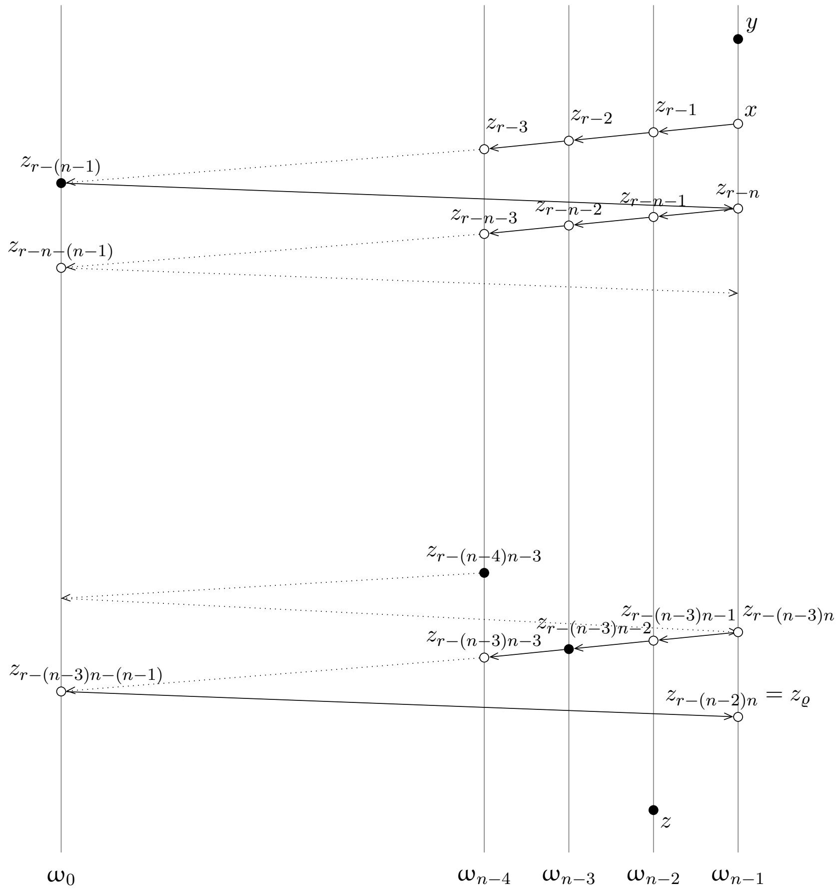


Figure 6.20: A pictorial representation of the proof of Proposition 6.3.9.

Let now $n > 2$ and let $(\omega_0, \omega_1, \ldots, \omega_{n-1})$ be such that $\iota\iota^n(\omega_0, \omega_1, \ldots, \omega_{n-1})$ is true. Without loss of generality, let $x, y \in \omega_{n+1}$ be such that, for $r = \mathsf{rank}(x) \leqslant \mathsf{rank}(y)$, there exists an ordinal ϱ with $r = \varrho + n(n-2)$, but $x^{<(r-n(n-2))} \neq y^{<(r-n(n-2))}$. Again, take w.l.o.g. $z \in \omega_{n-2}$ such that $z \in x^{<(r-n(n-2))}$, but $z \notin y^{<(r-n(n-2))}$.

Since $r = \varrho + n(n-2)$, and, for all $i \in \{0, \ldots, n-1\}$, $\bigcup \omega_{i+1} \subseteq \omega_i$ holds by (ii), there exist elements $z_{r-j} \in \bigcup_{i=0}^{n-1} \omega_i$, $1 \leqslant j \leqslant (n-2)n$, such that each z_{r-j} has rank $r - j$ and such that $x \ni z_{r-1}$, and $z_{r-j} \ni z_{r-(j+1)}$, for every $1 \leqslant j < (n-2)n$. Since this membership chain 'wraps around' the n columns $\omega_0, \ldots, \omega_{n-1}$, to reach a contradiction we can consider its diagonal, that is, elements $z_{r-in-(n-i-1)} \in \omega_i$ $(0 \leqslant i \leqslant n-3)$, of rank $r - in - (n-i-1)$ (see Figure 6.20). The n-tuple

$$z_{r-(n-1)}, z_{r-n-(n-2)}, \ldots, z_{r-(n-4)n-3}, z_{r-(n-3)n-2}, z, y$$

does not satisfy condition

$$(\forall y_0 \in x_0, \ldots, y_{n-1} \in x_{n-1}) \left(\bigvee_{i=0}^{n+1} y_i \in y_{i+1} \right)$$

of $\iota\iota^n$: $z_{r-in-(n-i-1)} \notin z_{r-(i+1)n-(n-i-2)}$, for all $0 \leqslant i \leqslant n-4$, $z_{r-(n-3)n-2} \notin z$, as $\mathsf{rank}(z) < r - n(n-2) \leqslant r - (n-3)n - 2 = \mathsf{rank}(z_{r-(n-3)n-2})$ (for $n \geqslant 2$), $z \notin y$, and $y \notin z_{r-(n-1)}$ as $\mathsf{rank}(z_{r-(n-1)}) < r \leqslant \mathsf{rank}(y)$. ■

When trying to obtain non-succinct models for $\iota\iota^n$, with $n > 2$, one can take the "standard" spiral model denoted $(\omega_0, \ldots, \omega_{n-1})$ of $\iota\iota^n$ and enrich it with spurious elements, which can be later peeled off to reveal the original spiral. As any sets satisfying $\iota\iota^n(x_0, x_1, \ldots, x_{n-1})$ are $n(n-2)$-incremental, the extra elements that can be added to the spiral cannot be very different from the ones already there. Given an element x of the spiral of successor rank $r \geqslant n(n-2)$, we can add an element $\widetilde{x}$ of the same rank r, such that $x^{<(r-n(n-2))} = \widetilde{x}^{<(r-n(n-2))}$, but there exists an element $d \in x$, $r - n(n-2) \leqslant \mathsf{rank}(d) < r$, such that $d \notin \widetilde{x}$. In this sense, the number n of free variables of $\iota\iota^n$ is a parameter of how many superfluous elements can be added to the spiral, and of how 'close' they lie to the original ones. At the extreme end, when $n = 2$, there are no two elements of the same rank in one column of the spiral, hence no extra elements can be added.

For example, for $n = 4$, take the model $(\omega_0, \omega_1, \omega_2, \omega_3)$ of $\iota\iota^4$ and enrich it into $(\widetilde{\omega}_0, \widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3)$, by adding the extra element $\widetilde{\omega}_{1,1} \in \omega_1$ such that

- $\widetilde{\omega}_{1,1} = \{\omega_{0,1}\} \neq \{\omega_{1,0}, \omega_{1,1}\} = \omega_{1,1}$,
- for all $k \geqslant 1$, $\widetilde{\omega}_{0,1} \in \omega_{2,k}$.

To see that this enriched family is a model of $\iota\iota^4$, observe that conditions (i) and (ii) readily hold. Moreover, any possible 4-tuple violating condition (iii) must contain $\omega_{0,0} \in \widetilde{\omega}_0$ and $\widetilde{\omega}_{1,1} \in \widetilde{\omega}_1$. However, any $\omega_{2,j} \in \widetilde{\omega}_2$ either owns $\widetilde{\omega}_{1,1}$ as element (for $j \geqslant 1$), or it belongs to every $\omega_{3,k} \in \omega_3$ (for $j = 0$).

This enriched family is a model of $\iota\iota^4$, but it is not a succinct one, as removing the vertex corresponding to $\widetilde{\omega}_{1,1}$ from the membership digraph of $\mathsf{TrCl}(\widetilde{\omega}_0 \cup \widetilde{\omega}_1 \cup \widetilde{\omega}_2 \cup \widetilde{\omega}_3)$, one finds the membership digraph of the original succinct $\mathsf{TrCl}(\omega_0 \cup \omega_1 \cup \omega_2 \cup \omega_3)$.

Under AFA, the degree of freedom among the models of BSR-formula is markedly more visible. Recall that the proof of Proposition 6.2.15 introduced one such non-succinct spiral ω'_0, ω'_1 in which there were multiple elements of ω'_1 having precisely the same elements in ω_0. Even more generally, it can be proved that by replacing a vertex of ω_1 in the standard model ω_0, ω_1 of $\underline{\iota\iota}_2$ with the membership digraph of any hereditarily finite well-founded set, the resulting digraph continues to be hyper-extensional and to satisfy $\underline{\iota\iota}_2$.

Summary of results

<table>
<tr><th colspan="2">ZF</th></tr>
<tr><th>BSR-formula</th><th>Model</th></tr>
<tr><td>
$\iota\iota(a, b) \equiv$
(i) $a \neq b \wedge a \notin b \wedge b \notin a$
(ii) $\bigcup a \subseteq b \wedge \bigcup b \subseteq a$
(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$
• prenex prefix: $\forall\forall$
• appeared in [112, 113]
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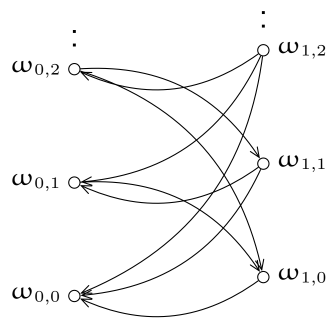

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<tr><td>
$\iota\iota^n(x_0, \dots, x_{n-1}) \equiv$
(i) $x_0 \neq \emptyset \wedge \bigwedge_{i=0}^{n-1} \left(x_{i-1} \notin x_i\right)$
(ii) $\bigwedge_{i=0}^{n-1} \left(\bigcup x_i \subseteq x_{i-1}\right)$
(iii) $(\forall y_0 \in x_0, \dots, y_{n-1} \in x_{n-1})\left(\bigvee_{i=0}^{n-1} y_i \in y_{i+1}\right)$
• prenex prefix: $\forall^n$
• appeared in [100]
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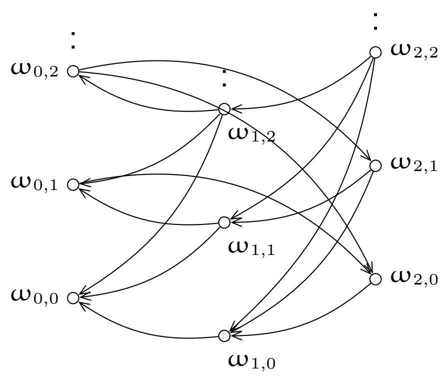
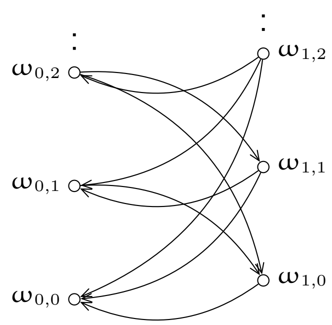


</td></tr>
<tr><th colspan="2">ZF − FA + AFA</th></tr>
<tr><td>
$\widetilde{\iota\iota}(a, b) \equiv$
(i′) $a \neq b \wedge a \notin b \wedge b \notin a \wedge a \cap b = \emptyset$
(ii) $\bigcup a \subseteq b \wedge \bigcup b \subseteq a$
(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$
• prenex prefix: $\forall\forall$
• appeared in [105]
• all models are well-founded
</td><td>


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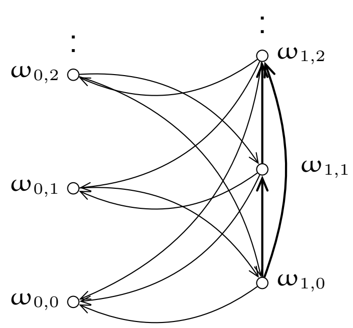
<tr><td>
$\underline{\iota\iota}_1(a, b) \equiv$
(i) $a \neq b \wedge a \notin b \wedge b \notin a$
(ii′) $\bigcup a \subseteq b \wedge \bigcup b \subseteq a \cup b \wedge (\forall y \in b)(y \notin y)$
(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$
(iv′) $(\forall x \in a)(\forall y_1, y_2 \in b)(y_1 \in y_2 \rightarrow y_2 \subseteq y_1)$
(v) $(\forall y_1, y_2 \in b)(y_1 = y_2 \vee y_1 \in y_2 \vee y_2 \in y_1)$
• prenex prefix: $\forall\forall\forall$
• appeared in [102]
• all models have $\in$-cycles and infinite descending $\in$-chains
• $\underline{\iota\iota}_1$ implies $\underline{\iota\iota}_2$
</td><td>


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<table>
<tr><th colspan="2">ZF − FA + AFA (continued)</th></tr>
<tr><td>

$\underline{\iota\iota}_2(a,b) \equiv$

(i) $a \neq b \wedge a \notin b \wedge b \notin a$

(ii′) $\bigcup a \subseteq b \wedge \bigcup b \subseteq a \cup b \wedge (\forall y \in b)(y \notin y)$

(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$

(iv′) $(\forall x \in a)(\forall y_1, y_2 \in b)(y_1 \in y_2 \rightarrow y_2 \subseteq y_1)$

(v′) $(\forall x \in a)(\forall y_1, y_2 \in b)(y_1 \neq y_2 \wedge y_2 \in x \in y_1 \rightarrow y_1 \in y_2)$

- prenex prefix: $\forall\forall\forall$
- appeared in [102, 103]
- all models have $\in$-cycles and infinite descending $\in$-chains

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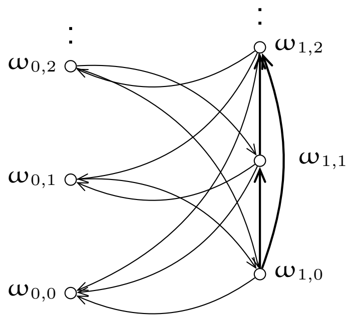

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<tr><td>

$\underline{\iota\iota}^n(x_0, \dots, x_{n-1}) \equiv$

(i) $x_0 \neq \emptyset \wedge \bigwedge_{i=0}^{n-1} \left(x_{i-1} \notin x_i\right)$

(ii′) $\bigwedge_{i=0}^{n-2} \left(\bigcup x_i \subseteq x_{i-1}\right) \wedge \bigcup x_{n-1} \subseteq x_{n-2} \cup x_{n-1}$

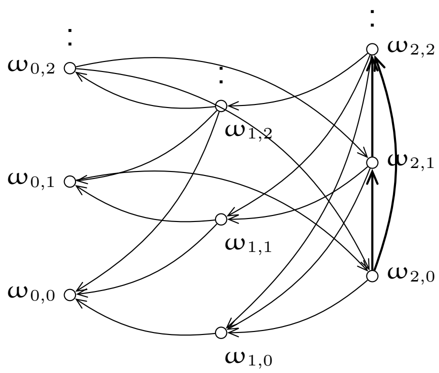
(iii) $(\forall y_0 \in x_0, \dots, y_{n-1} \in x_{n-1})\left(\bigvee_{i=0}^{n-1} y_i \in y_{i+1}\right)$

(iv′) $(\forall y_1, y_2 \in x_{n-1})(y_1 \in y_2 \rightarrow y_2 \subseteq y_1)$

(v) $(\forall y_1, y_2 \in x_{n-1})(y_1 = y_2 \vee y_1 \in y_2 \vee y_2 \in y_1)$

- prenex prefix: $\forall^n$
- appeared in [102]

</td><td>



</td></tr>
<tr><th colspan="2">ZF − FA</th></tr>
<tr><td>

$\overline{\iota\iota}(a,b) \equiv$

(i) $a \neq b \wedge a \notin b \wedge b \notin a$

(ii) $\bigcup a \subseteq b \wedge \bigcup b \subseteq a$

(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$

(iv) $(\forall x_1, x_2 \in a)(\forall y_1, y_2 \in b)(x_2 \in y_2 \in x_1 \in y_1 \rightarrow x_2 \in y_1)$.

- prenex prefix: $\forall\forall\forall\forall$
- appeared in [114]

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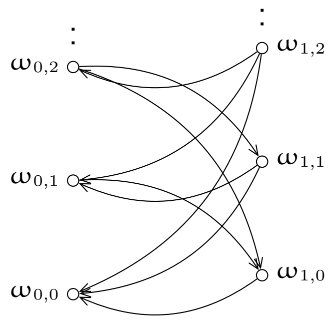

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<tr><td>

$\overline{o\iota}(a,b) \equiv$

(i) $a \neq b \wedge a \notin b \wedge b \notin a$

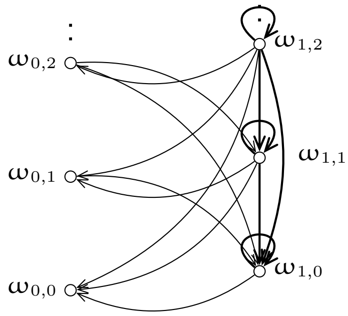
(ii″) $\bigcup(a \setminus b) \subseteq b \wedge \bigcup b \subseteq a \wedge b \subseteq a$

(iii) $(\forall x \in a)(\forall y \in b)(x \in y \vee y \in x)$

(iv″) $(\forall y_1, y_2 \in b)(y_1 \in y_2 \rightarrow y_1 \subseteq y_2)$

- prenex prefix: $\forall\forall\forall$
- appeared in [105]
- all models are non-well-founded

</td><td>



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In the above, all operations on indices are modulo n.

Conclusions

The topics dealt with in this thesis, under the premise "sets as graphs", lie at the intersection of theoretical computer science, discrete mathematics and computational logic.

In Chapter 1 we have set up the main definitions together with the graph theoretic interpretation of a set. In Chapter 2, we first gave recurrence relations for the number of transitive sets with n elements, by borrowing methods from the count of acyclic digraphs. We counted weakly extensional acyclic digraphs, by sources, by vertices of maximum rank, and by arcs. Combinatorial enumeration of hyper-extensional acyclic digraphs remains an open problem. However, we proposed a canonical linear order on hereditarily finite hypersets, which extends Ackermann's order on standard, well-founded sets. This was done by characterizing Ackermann's order in terms of a partition refinement procedure. Finally, we tackled the problem of random generation of sets, by employing a Markov chain algorithm designed for acyclic digraphs. These developments encourage a deeper transfer of results and techniques between (acyclic) digraphs and sets. For example, a direct ranking/unranking procedure for sets, analogous to the Prüfer code of a tree, and its recent extension to acyclic digraphs [137], lies within reach. The notion of extensionality of a digraph can also be expressed in terms of a novel variant of an identifying code in a digraph (as done in Section 4.1.4); accordingly, many code-related questions can be asked for extensional (acyclic) digraphs (see e.g. the extensive bibliography of Antoine Lobstein [80]).

One further direction for research relates sets to descriptive complexity theory. The existence of a logic capturing the complexity class P on finite graphs is an open problem, and a negative answer to it would imply, by Fagin's theorem, that $\mathsf{P} \neq \mathsf{NP}$. One class where P is captured by a logic is the class of hyper-extensional digraphs (thus so is its subclass of extensional acyclic digraphs) [79]. Some questions immediately pop up: Can P be captured on weakly extensional acyclic digraphs with a bounded number of sinks? What about r-extensional acyclic digraphs, for a fixed r?

In Chapter 3 we studied classes of sets well-quasi-ordered by a digraph immersion relation, called strong immersion. We introduced the property of slimness, which basically requires that every membership be necessary to ensure extensionality. Our main result is that slim channeled digraphs with a bounded number of sources are well-quasi-ordered by strong immersion, a claim which also comprises slim hypersets. The relation between sets and digraph immersions that are well-quasi-orders is far from completely understood. One direction for future research could be a study of the more general class of sets in which every *element* is critical for having extensionality. One can also consider other types of immersions, starting, for example, from an adequate generalization to digraphs of the notion of minor for undirected graphs.

In Chapter 4 we introduced set graphs, that is, graphs which admit an extensional acyclic orientation. One motivation for the study of set graphs lies in the graph-theoretic expressive power of sets. We showed that deciding whether a graph is a set graph is an NP-complete problem. This is also the case for the analogous problem of finding a hyper-extensional orientation of a graph. Moreover, their counting variants belong to the complexity class #P-complete.

These complexity results show that it is unlikely that a 'good' characterization of them exists. Instead, one can look for the largest hereditary class of graphs such that every connected member of it is a set graph. It turned out that this class is obtained by forbidding only the smallest connected graph which is not a set graph, the claw, $K_{1,3}$. Moreover, the recognition problem is expressible in monadic second order logic, and is thus solvable in linear time on graphs of bounded tree-width.

The connection between set graphs and claw-freeness is all but superficial. On the one hand, we identified a largest hereditary class of graphs where being a set graph is equivalent to being claw-free. On the other hand, the claw-freeness condition can be generalized in two ways. First, by requiring that all claws of a graph be vertex-disjoint, together with a further polynomially-checkable connectivity condition, another subclass of set graphs was identified. Secondly, we showed that if we forbid $K_{1,r+2}$, $r \geqslant 1$, instead of the claw $K_{1,3}$, r-extensionality can be guaranteed.

The set interpretation of a graph led to a shorter proof of the fact that squares of connected claw-free graphs are vertex-pancyclic, on which we reported in Chapter 5. Our short proof directly shows this, without resorting to the general result that being Hamiltonian is equivalent, for squares of graphs, to being vertex-pancyclic. This framework isolated a mathematical insight that seems to be common in reasoning about claw-free graphs; indeed, almost for free we also got a proof of another classical result regarding the existence of perfect matchings in connected claw-free graphs. We took advantage of the set-theoretic flavor of these proofs to formalize them in an automated proof checker based on sets, Referee. This endeavor turned out to require moderate formal effort: on the one hand, we avoided explicitly defining graphs, together with an entire armamentarium of graph-theoretic concepts; on the other hand, we exploited Referee's built-in set manipulating methods to closely reflect the two proofs. These formal proofs are presented in full in Appendix B.

There are many directions for future work on set graphs. For example, the complexity of set graph recognition must be further elucidated. It is relevant to study whether the problem becomes polynomial on certain restricted inputs or if other connections between set graphs and graphs having a fixed template of graphs as forbidden induced subgraphs exist. What about an approximation algorithm [147] for any of its optimization variants (asking e.g. for a weakly extensional acyclic orientation with a minimum number of sinks, or for an r-extensional acyclic orientation with minimum r)? Moreover, if we bind certain parameters of the input graphs, does the problem become fixed-parameter tractable [55] (as was the case with treewidth)? In this regard, how does set graph recognition relate to the clique-width of a graph?

Given the correspondence between the acyclic orientations and the chromatic number of a graph, and in light of one of our results stating that every graph admits a weakly extensional acyclic orientation, we can analogously introduce a notion of 'set chromatic number' of a graph, and study, for example, 'set perfect graphs'.

Finally, in Chapter 6, we tackled the decidability, over hypersets, of the set-theoretic Bernays-Schönfinkel-Ramsey class consisting of all first-order formulae whose prenex form has a purely universal prefix. Our contribution to this problem lies in a study of the infinite sets expressible by $\forall^*$-formulae and in ways to represent finitely such models. On the one hand, we showed that the simplest possible $\forall\forall$-formula expressing infinite sets under the assumption that membership is well-founded, also expresses infinite hypersets if this assumption is dropped. On the other hand, we proposed novel $\forall\forall\forall$-formulae satisfied exclusively by infinite non-well-founded hypersets. Nevertheless, the decidability problem of the BSR-class over hypersets remains open. One direction for further research concerns a restriction of this problem to $\forall\forall$-formulae, course of action also followed for well-founded sets [16]. If no infinite and 'genuinely' non-well-founded hypersets can be expressed by a $\forall\forall$-formula, which seems to be the case, then the decidability of $\forall\forall$-formulae, over hypersets, should readily ensue from the above-cited result for well-founded sets [16].

A

Finiteness: a Proof-Scenario Checked by Referee

In this appendix a few basic laws of the finiteness property are developed formally with the assistance of the proof-checker Ref. To increase the significance of the proofs that follow, we avoid exploiting von Neumann's Foundation Axiom on the one hand and, on the other hand, we avoid exploiting Ref's built-in ability to handle a predicate Finite tightly akin to our present Fin.

DEF $\mathcal{P}$: [Family of all subsets of a given set] $\mathcal{P}X =_{\text{Def}} \{x : x \subseteq X\}$

THM 23: [Monotonicity of powerset] $S \supseteq X \rightarrow \mathcal{P}X \cup \{\emptyset, X\} \subseteq \mathcal{P}S$. PROOF:

Suppose_not(s_0, x_0) ⇒ AUTO

Set_monot ⇒ $\{x : x \subseteq x_0\} \subseteq \{x : x \subseteq s_0\}$

Use_def$(\mathcal{P})$ ⇒ *Stat1* : $\emptyset \notin \{x : x \subseteq s_0\} \vee x_0 \notin \{x : x \subseteq s_0\}$

$\langle \emptyset, x_0 \rangle \hookrightarrow$ *Stat1* ⇒ false; Discharge ⇒ QED

Traditionally, finiteness is defined through the notion of cardinality of a set: a set is finite if its cardinality precedes the first infinite ordinal. As a shortcut, to begin developing an acceptable formal treatment of finiteness without much preparatory work, we adopt here the following definition (reminiscent of Tarski's 1924 paper "Sur les ensembles fini"): a set F is finite if every non-null family of subsets of F owns an inclusion-minimal element. This notion can be specified very succinctly in terms of the powerset operator.

DEF Fin: [Finiteness property]

$$\text{Fin}(X) \leftrightarrow_{\text{Def}} \langle \forall g \in \mathcal{P}(\mathcal{P}X) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle$$

THM 24: [Monotonicity of finiteness] $Y \supseteq X \,\&\, \text{Fin}(Y) \rightarrow \text{Fin}(X)$. PROOF:

Suppose_not(y_0, x_0) ⇒ AUTO

$\langle y_0, x_0 \rangle \hookrightarrow T23\ (\star)$ ⇒ $\mathcal{P}y_0 \supseteq \mathcal{P}x_0$

Use_def(Fin) ⇒ *Stat1* : $\neg\langle \forall g \in \mathcal{P}(\mathcal{P}x_0) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle \,\&$
$\langle \forall g' \in \mathcal{P}(\mathcal{P}y_0) \setminus \{\emptyset\}, \exists m \mid g' \cap \mathcal{P}m = \{m\} \rangle$

$\langle \mathcal{P}y_0, \mathcal{P}x_0 \rangle \hookrightarrow T23\ (\star)$ ⇒ $\mathcal{P}(\mathcal{P}y_0) \supseteq \mathcal{P}(\mathcal{P}x_0)$

$\langle g_0, g_0 \rangle \hookrightarrow$ *Stat1*(*Stat1*$\star$) ⇒ $\neg\langle \exists m \mid g_0 \cap \mathcal{P}m = \{m\} \rangle \,\&\, \langle \exists m \mid g_0 \cap \mathcal{P}m = \{m\} \rangle$

Discharge ⇒ QED

THM 26a: [Finiteness of the union of two finite sets]
$\mathsf{Fin}(X) \,\&\, \mathsf{Fin}(Y) \rightarrow \mathsf{Fin}(X \cup Y)$. PROOF:

$\mathsf{Suppose\_not}(x_0, y_0) \Rightarrow$ AUTO

Arguing by contradiction, suppose that finite sets x_0 and y_0 exist whose union is not finite. Then a non-null set g_0 formed by subsets of $x_0 \cup y_0$ exists which has no minimal element with respect to inclusion.

$\mathsf{Use\_def(Fin)} \Rightarrow$ $Stat1: \neg\langle \forall g \in \mathcal{P}(\mathcal{P}(x_0 \cup y_0)) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle \,\&$
$Stat2: \langle \forall g' \in \mathcal{P}(\mathcal{P}x_0) \setminus \{\emptyset\}, \exists m \mid g' \cap \mathcal{P}m = \{m\} \rangle \,\&$
$Stat3: \langle \forall gq \in \mathcal{P}(\mathcal{P}y_0) \setminus \{\emptyset\}, \exists m \mid gq \cap \mathcal{P}m = \{m\} \rangle$
$\langle g_0 \rangle \hookrightarrow Stat1 \Rightarrow$ $Stat4: \neg\langle \exists m \mid g_0 \cap \mathcal{P}m = \{m\} \rangle \,\&\, g_0 \in \mathcal{P}(\mathcal{P}(x_0 \cup y_0)) \,\&$
$Stat4a: g_0 \neq \emptyset$

Indicate by g_1 the set of all intersections $x_0 \cap v$ with v ranging over g_0. Since g_0 is non-null, g_1 is non-null either.

$\mathsf{Loc\_def} \Rightarrow$ $Stat5: g_1 = \{x_0 \cap v : v \in g_0\}$
$\langle a \rangle \hookrightarrow Stat4a(Stat4\star) \Rightarrow$ $a \in g_0$
$\mathsf{Suppose} \Rightarrow$ $Stat6: x_0 \cap a \notin \{x_0 \cap v : v \in g_0\}$
$\langle a \rangle \hookrightarrow Stat6(Stat4\star) \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ $Stat7: x_0 \cap a \in g_1$

Therefore, since we have supposed x_0 to be finite and since g_1 is formed by subsets of x_0, g_1 must have a minimal element m_1.

$\mathsf{Suppose} \Rightarrow$ $g_1 \notin \mathcal{P}(\mathcal{P}x_0)$
$\mathsf{Use\_def}(\mathcal{P}) \Rightarrow$ $Stat8: g_1 \notin \{y : y \subseteq \{z : z \subseteq x_0\}\}$
$\langle g_1 \rangle \hookrightarrow Stat8(Stat8\star) \Rightarrow$ $Stat9: g_1 \not\subseteq \{z : z \subseteq x_0\}$
$\langle x_1 \rangle \hookrightarrow Stat9(Stat5, Stat9\star) \Rightarrow$ $Stat10: x_1 \in \{x_0 \cap v : v \in g_0\} \,\&$
$x_1 \notin \{z : z \subseteq x_0\}$
$\langle v_1, x_0 \cap v_1 \rangle \hookrightarrow Stat10(Stat10\star) \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ AUTO
$\langle g_1 \rangle \hookrightarrow Stat2(Stat7\star) \Rightarrow$ $Stat11: \langle \exists m \mid g_1 \cap \mathcal{P}m = \{m\} \rangle$
$\langle m_1 \rangle \hookrightarrow Stat11(Stat11\star) \Rightarrow$ $g_1 \cap \mathcal{P}m_1 = \{m_1\}$

Indicate by g_2 the set of all intersections $y_0 \cap v$, with v ranging over those elements of g_0 whose intersection with x_0 is m_1. Since g_0 must have at least one such element, g_2 is not null.

$\mathsf{Loc\_def} \Rightarrow$ $Stat12: g_2 = \{y_0 \cap v : v \in g_0 \mid x_0 \cap v = m_1\}$
$\mathsf{Suppose} \Rightarrow$ $Stat13: \{y_0 \cap v : v \in g_0 \mid x_0 \cap v = m_1\} = \emptyset$
$\mathsf{ELEM} \Rightarrow$ $Stat14: m_1 \in \{x_0 \cap v : v \in g_0\}$
$\langle v_0 \rangle \hookrightarrow Stat14 \Rightarrow$ AUTO
$\langle v_0 \rangle \hookrightarrow Stat13(Stat14\star) \Rightarrow$ false; $\mathsf{Discharge} \Rightarrow$ AUTO

Therefore, as we have supposed y_0 to be finite, and taking into account that g_2 consists of subsets of y_0, g_2 must have a minimal element m_2.

$\mathsf{Suppose} \Rightarrow$ $Stat15: g_2 \notin \mathcal{P}(\mathcal{P}y_0)$
$\mathsf{Use\_def}(\mathcal{P}) \Rightarrow$ $Stat16: g_2 \notin \{y : y \subseteq \{z : z \subseteq y_0\}\}$

$\langle g_2 \rangle \hookrightarrow Stat16(Stat16\star) \Rightarrow \quad Stat17: \ g_2 \not\subseteq \{z: z \subseteq y_0\}$
$\langle x_2 \rangle \hookrightarrow Stat17(Stat12, Stat17\star) \Rightarrow \quad Stat18: x_2 \in \{y_0 \cap v: v \in g_0 \mid x_0 \cap v = m_1\} \ \&$
$x_2 \notin \{z: z \subseteq y_0\}$
$\langle v_2, y_0 \cap v_2 \rangle \hookrightarrow Stat18(Stat18\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ AUTO
$\langle g_2 \rangle \hookrightarrow Stat3(Stat11\star) \Rightarrow \quad Stat19: \langle \exists m \mid g_2 \cap \mathcal{P}m = \{m\} \rangle$
$\langle m_2 \rangle \hookrightarrow Stat19(Stat19\star) \Rightarrow \quad g_2 \cap \mathcal{P}m_2 = \{m_2\}$

We will prove that $m_1 \cup m_2$ is minimal in g_0, which contradicts our initial assumption. We begin by observing that $m_1 \cup m_2$ belongs to g_0, as it coincides with an element w_0 of g_0 which has intersection m_1 with x_0 and intersection m_2 with y_0.

$(Stat12\star)$ELEM $\Rightarrow \quad Stat20: \ m_2 \in \{y_0 \cap v: v \in g_0 \mid x_0 \cap v = m_1\}$
$\langle w_0 \rangle \hookrightarrow Stat20(Stat20, Stat4\star) \Rightarrow \quad m_2 = y_0 \cap w_0 \ \& \ w_0 \in g_0 \ \& \ x_0 \cap w_0 = m_1 \ \&$
$g_0 \in \mathcal{P}(\mathcal{P}(x_0 \cup y_0))$
Use_def$(\mathcal{P}) \Rightarrow \quad Stat21: \ g_0 \in \{y: y \subseteq \{z: z \subseteq x_0 \cup y_0\}\}$
$\langle y_1 \rangle \hookrightarrow Stat21(Stat20\star) \Rightarrow \quad Stat22: \ w_0 \in \{z: z \subseteq x_0 \cup y_0\}$
$\langle z_1 \rangle \hookrightarrow Stat22(Stat20\star) \Rightarrow \quad w_0 = m_1 \cup m_2$

Since w_0 is not minimal in g_0, indicate by w_1 a strict subset of its that belongs to g_0; accordingly, it will turn out that either $x_0 \cap w_1$ is a strict subset of $x_0 \cap w_0$ or $y_0 \cap w_1$ is a strict subset of $y_0 \cap w_0$.

$\langle w_0, w_0 \rangle \hookrightarrow T23 \ (Stat20\star) \Rightarrow \quad w_0 \in g_0 \cap \mathcal{P}w_0$
$\langle w_0 \rangle \hookrightarrow Stat4(Stat22\star) \Rightarrow \quad Stat23: \ g_0 \cap \mathcal{P}w_0 \not\subseteq \{w_0\}$
Use_def$(\mathcal{P}w_0) \Rightarrow$ AUTO
$\langle w_1 \rangle \hookrightarrow Stat23(Stat23\star) \Rightarrow \quad Stat24: \ w_1 \in \{y: y \subseteq w_0\} \ \& \ w_1 \neq w_0 \ \& \ w_1 \in g_0$
$\langle y_2 \rangle \hookrightarrow Stat24(Stat20\star) \Rightarrow \quad w_1 \subseteq w_0 \ \& \ x_0 \cap w_1 \neq m_1 \lor y_0 \cap w_1 \neq m_2$

Consider first the case $x_0 \cap w_1 \neq x_0 \cap w_0$. One easily sees that such an element violating the minimality of $m_1 = x_0 \cap w_0$ in g_1 would lead us to a contradiction.

Suppose $\Rightarrow \quad x_0 \cap w_1 \neq m_1$
Suppose $\Rightarrow \quad x_0 \cap w_1 \notin g_1$
EQUAL $\langle Stat5 \rangle \Rightarrow \quad Stat25: \ x_0 \cap w_1 \notin \{x_0 \cap v: v \in g_0\}$
$\langle w_1 \rangle \hookrightarrow Stat25(Stat24, Stat24\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ AUTO
Use_def$(\mathcal{P}m_1) \Rightarrow$ AUTO
$(Stat11\star)$ELEM $\Rightarrow \quad Stat26: \ x_0 \cap w_1 \notin \{z: z \subseteq m_1\}$
$\langle x_0 \cap w_1 \rangle \hookrightarrow Stat26(Stat20\star) \Rightarrow \quad$ false
Discharge $\Rightarrow \quad Stat27: \ x_0 \cap w_1 = m_1 \ \& \ y_0 \cap w_1 \neq m_2$

Consider next the case $y_0 \cap w_1 \neq y_0 \cap w_0$ whereas $x_0 \cap w_1 = x_0 \cap w_0$. In this case the minimality of $m_2 = y_0 \cap w_0$ in g_2 would be violated; now there is no way out of the conflict and we get the conclusion we were after with our argument by contradiction.

Suppose $\Rightarrow \quad y_0 \cap w_1 \notin g_2$
EQUAL $\langle Stat12 \rangle \Rightarrow \quad Stat28: \ y_0 \cap w_1 \notin \{y_0 \cap v: v \in g_0 \mid x_0 \cap v = m_1\}$
$\langle w_1 \rangle \hookrightarrow Stat28(Stat24, Stat27\star) \Rightarrow \quad$ false; Discharge $\Rightarrow$ AUTO

Use_def($\mathcal{P}m_2$) $\Rightarrow$ AUTO
($Stat19\star$)ELEM $\Rightarrow$ $Stat29:\ y_0 \cap w_1 \notin \{z:\ z \subseteq m_2\}$
$\langle y_0 \cap w_1 \rangle \hookrightarrow Stat29(Stat20\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM 33: [Every finite non-null set totally ordered by inclusion owns a maximum]
$\mathsf{Fin}(F)\ \&\ F \neq \emptyset\ \&\ \langle \forall x \in F, y \in F \mid x \subseteq y \vee y \subseteq x \rangle \rightarrow \langle \exists m \in F, \forall x \in F \mid x \subseteq m \rangle$. PROOF:
Suppose_not(f) $\Rightarrow$ $Stat0:\ \langle \forall x \in f, y \in f \mid x \subseteq y \vee y \subseteq x \rangle\ \&\ \mathsf{Fin}(f)\ \&\ f \neq \emptyset\ \&$
$\neg \langle \exists m \in f, \forall x \in f \mid x \subseteq m \rangle$

Arguing by contradiction, suppose that f is a counterexample to our claim. Thanks to the finiteness of f, from among all non-null subsets of f which are totally ordered by inclusion but devoid of maximum (one such set being f itself), we can choose a minimal one, f_0.

Loc_def $\Rightarrow$ $Stat1:\ g = \{f' \subseteq f \mid f' \neq \emptyset\ \&\ \langle \forall x \in f', y \in f' \mid x \subseteq y \vee y \subseteq x \rangle\ \&$
$\neg \langle \exists m \in f', \forall x \in f' \mid x \subseteq m \rangle \}$
Suppose $\Rightarrow$ $g \notin \mathcal{P}(\mathcal{P}f) \setminus \{\emptyset\}$
Suppose $\Rightarrow$ $\emptyset = g$
($Stat1\star$)ELEM $\Rightarrow$ $Stat2:\ f \notin \{f' \subseteq f \mid f' \neq \emptyset\ \&$
$\langle \forall x \in f', y \in f' \mid x \subseteq y \vee y \subseteq x \rangle\ \&\ \neg \langle \exists m \in f', \forall x \in f' \mid x \subseteq m \rangle \}$
$\langle f \rangle \hookrightarrow Stat2(\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def($\mathcal{P}$) $\Rightarrow$ $Stat3:\ g \notin \{g':\ g' \subseteq \{f':\ f' \subseteq f\}\}$
Set_monot $\Rightarrow$ $\{f':\ f' \subseteq f \mid f' \neq \emptyset\ \&\ \langle \forall x \in f', y \in f' \mid x \subseteq y \vee y \subseteq x \rangle\ \&$
$\neg \langle \exists m \in f', \forall x \in f' \mid x \subseteq m \rangle \} \subseteq \{f':\ f' \subseteq f\}$
$\langle g \rangle \hookrightarrow Stat3(Stat1\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def(Fin) $\Rightarrow$ $Stat4:\ \langle \forall g \in \mathcal{P}(\mathcal{P}f) \setminus \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle\ \&\ g \in \mathcal{P}(\mathcal{P}f) \setminus \{\emptyset\}$
$\langle g, f_0 \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow$ $Stat5:\ g \cap \mathcal{P}f_0 = \{f_0\}$

Obviously f_0 is not a singleton; hence, arbitrarily picking a_0 from f_0, we will have that $f_1 = f_0 \setminus \{a_0\}$ is non-null and totally ordered by inclusion.

($Stat1, Stat5\star$)ELEM $\Rightarrow$ $Stat6:$
$f_0 \in \{f' \subseteq f \mid f' \neq \emptyset\ \&\ \langle \forall x \in f', y \in f' \mid x \subseteq y \vee y \subseteq x \rangle\ \&\ \neg \langle \exists m \in f', \forall x \in f' \mid x \subseteq m \rangle \}$
$\langle\ \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ $Stat7:\ \langle \forall x \in f_0, y \in f_0 \mid x \subseteq y \vee y \subseteq x \rangle\ \&\ Stat8:\ f_0 \neq \emptyset\ \&$
$Stat9:\ \neg \langle \exists m \in f_0, \forall x \in f_0 \mid x \subseteq m \rangle\ \&\ f_0 \subseteq f$
$\langle a_0, a_0 \rangle \hookrightarrow Stat8(Stat7\star) \Rightarrow$ $Stat10:\ \neg \langle \forall x \in f_0 \mid x \subseteq a_0 \rangle\ \&\ a_0 \in f_0$
$\langle a_2 \rangle \hookrightarrow Stat10(Stat5\star) \Rightarrow$ $Stat11:\ a_2 \in f_0\ \&\ a_2 \not\subseteq a_0\ \&$
$f_0 \setminus \{a_0\} \neq \emptyset\ \&\ f_0 \setminus \{a_0\} \neq f_0\ \&\ f_0 \setminus \{a_0\} \notin g \cap \mathcal{P}f_0$
Suppose $\Rightarrow$ $f_0 \setminus \{a_0\} \notin \mathcal{P}f_0$
Use_def($\mathcal{P}$) $\Rightarrow$ $Stat12:\ f_0 \setminus \{a_0\} \notin \{f':\ f' \subseteq f_0\}$
$\langle f_0 \setminus \{a_0\} \rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
($Stat1\star$)ELEM $\Rightarrow$ $Stat13:\ f_0 \setminus \{a_0\} \notin \{f' \subseteq f \mid f' \neq \emptyset\ \&$
$\langle \forall x \in f', y \in f' \mid x \subseteq y \vee y \subseteq x \rangle\ \&\ \neg \langle \exists m \in f', \forall x \in f' \mid x \subseteq m \rangle \}$
Set_monot $\Rightarrow$ $\langle \forall x \in f_0, y \in f_0 \mid x \subseteq y \vee y \subseteq x \rangle \rightarrow$
$\langle \forall x \in f_0 \setminus \{a_0\}, y \in f_0 \setminus \{a_0\} \mid x \subseteq y \vee y \subseteq x \rangle$

But then, considering that a_1 owns a maximum and that a_0, a_1 can be compared by inclusion, the one of the two which includes the other must be the maximum of f_0: this gives us the desired contradiction.

$\langle f_0 \backslash \{a_0\} \rangle \hookrightarrow Stat13(Stat7\star) \Rightarrow$ $Stat14: \langle \exists m \in f_0 \backslash \{a_0\}, \forall x \in f_0 \backslash \{a_0\} \mid x \subseteq m \rangle$
$\langle a_1 \rangle \hookrightarrow Stat14(Stat14\star) \Rightarrow$ $Stat15: \langle \forall x \in f_0 \backslash \{a_0\} \mid x \subseteq a_1 \rangle \ \& \ a_1 \in f_0 \backslash \{a_0\}$
$\langle a_0, a_1 \rangle \hookrightarrow Stat7(Stat10, Stat15\star) \Rightarrow$ $a_0 \subseteq a_1 \vee a_1 \subseteq a_0$
Suppose $\Rightarrow$ $Stat16: a_1 \subseteq a_0$
$\langle a_2 \rangle \hookrightarrow Stat15(Stat11, Stat16\star) \Rightarrow$ false; Discharge $\Rightarrow$ $Stat17: a_0 \subseteq a_1$
$\langle a_1 \rangle \hookrightarrow Stat9(Stat15, Stat15\star) \Rightarrow$ $Stat18: \neg\langle \forall x \in f_0 \mid x \subseteq a_1 \rangle \ \&$
$\langle \forall x \in f_0 \backslash \{a_0\} \mid x \subseteq a_1 \rangle$
$\langle a_3, a_3 \rangle \hookrightarrow Stat18(Stat17\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM 34: [Every finite non-null set owns a maximal element with respect to inclusion] $\mathsf{Fin}(F) \ \& \ F \neq \emptyset \rightarrow \langle \exists m \in F, \forall x \in F \mid m \subseteq x \rightarrow m = x \rangle$. PROOF:
Suppose_not(f) $\Rightarrow$ $Stat0: f \neq \emptyset \ \& \ \mathsf{Fin}(f) \ \& \ Stat1: \neg\langle \exists m \in f, \forall x \in f \mid m \subseteq x \rightarrow m = x \rangle$

Arguing by contradiction, suppose that f is a counterexample to our claim. Thanks to the finiteness of f, from among all strict subsets y of f whose complements relative to f are totally ordered by inclusion (one such set being any $f \backslash \{a\}$ with $a \in f$), we can choose a minimal one, y_0.

Loc_def $\Rightarrow$ $Stat2: g = \{y \subseteq f \mid y \neq f \ \& \ \langle \forall u \in f \backslash y, v \in f \backslash y \mid u \subseteq v \vee v \subseteq u \rangle\}$
Suppose $\Rightarrow$ $g \notin \mathcal{P}(\mathcal{P}f) \backslash \{\emptyset\}$
Suppose $\Rightarrow$ $\emptyset = g$
$\langle a \rangle \hookrightarrow Stat0(Stat2\star) \Rightarrow$ $Stat3:$
$f \backslash \{a\} \notin \{y \subseteq f \mid y \neq f \ \& \ \langle \forall u \in f \backslash y, v \in f \backslash y \mid u \subseteq v \vee v \subseteq u \rangle\} \ \& \ a \in f$
$\langle f \backslash \{a\} \rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $Stat4:$
$\neg\langle \forall u \in f \backslash (f \backslash \{a\}), v \in f \backslash (f \backslash \{a\}) \mid u \subseteq v \vee v \subseteq u \rangle$
$\langle u, v \rangle \hookrightarrow Stat4(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def($\mathcal{P}$) $\Rightarrow$ $Stat5: g \notin \{g' : g' \subseteq \{y : y \subseteq f\}\}$
Set_monot $\Rightarrow$ $\{y : y \subseteq f \mid y \neq f \ \& \ \langle \forall u \in f \backslash y, v \in f \backslash y \mid u \subseteq v \vee v \subseteq u \rangle\} \subseteq$
$\{y : y \subseteq f\}$
$\langle g \rangle \hookrightarrow Stat5(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def(Fin) $\Rightarrow$ $Stat6: \langle \forall g \in \mathcal{P}(\mathcal{P}f) \backslash \{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\} \rangle \ \& \ g \in \mathcal{P}(\mathcal{P}f) \backslash \{\emptyset\}$
$\langle g, y_0 \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ $Stat7: g \cap \mathcal{P}y_0 = \{y_0\}$

Since $f \backslash y_0$ is a subset of f, it is finite; since it is totally ordered by inclusion, $f \backslash y_0$ must (thanks to our preceding lemma) own a maximum; moreover, since such maximum m_0 cannot be maximal in f, there must exist an $m_1 \in f$ an m_1 which strictly includes m_0.

$(Stat2, Stat7\star)$ELEM $\Rightarrow$ $Stat8:$
$y_0 \in \{y \subseteq f \mid y \neq f \ \& \ \langle \forall u \in f \backslash y, v \in f \backslash y \mid u \subseteq v \vee v \subseteq u \rangle\}$
$\langle\,\rangle \hookrightarrow Stat8(Stat8\star) \Rightarrow$ $Stat9:$
$\langle \forall u \in f \backslash y_0, v \in f \backslash y_0 \mid u \subseteq v \vee v \subseteq u \rangle \ \& \ y_0 \subseteq f \ \& \ y_0 \neq f$
$\langle f, f \backslash y_0 \rangle \hookrightarrow T24\ (Stat0, Stat0\star) \Rightarrow$ $\mathsf{Fin}(f \backslash y_0)$
$\langle f \backslash y_0 \rangle \hookrightarrow T33\ (Stat9\star) \Rightarrow$ $Stat10: \langle \exists m \in f \backslash y_0, \forall x \in f \backslash y_0 \mid x \subseteq m \rangle$
$\langle m_0 \rangle \hookrightarrow Stat10(Stat10\star) \Rightarrow$ $Stat11: \langle \forall x \in f \backslash y_0 \mid x \subseteq m_0 \rangle \ \& \ m_0 \in f \backslash y_0$

$\langle m_0 \rangle \hookrightarrow$ *Stat1*(*Stat11*$\star$) $\Rightarrow$ *Stat12*: $\neg\langle \forall x \in f \mid m_0 \subseteq x \rightarrow m_0 = x\rangle$
$\langle m_1 \rangle \hookrightarrow$ *Stat12*(*Stat12*$\star$) $\Rightarrow$ $m_1 \in f \;\&\; m_0 \subseteq m_1 \;\&\; m_0 \neq m_1$

Obviously m_1 belongs to y_0, and therefore $f \backslash (y_0 \backslash \{m_1\}) = f \backslash y_0 \cup \{m_1\}$ turns out to be totally ordered by inclusion. Consequently, $y_0 \backslash \{m_1\}$ is a strict subset y_0 of f whose complement relative to f is totally ordered by inclusion, which contradicts the supposed minimality of y_0.

Suppose $\Rightarrow$ *Stat13*: $y_0 \backslash \{m_1\} \notin \{z : z \subseteq y_0\}$
$\langle y_0 \backslash \{m_1\} \rangle \hookrightarrow$ *Stat13*(*Stat13*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO
Use_def($\mathcal{P}y_0$) $\Rightarrow$ AUTO
$\langle m_1 \rangle \hookrightarrow$ *Stat11*(*Stat2*$\star$) $\Rightarrow$ *Stat14*:
$y_0 \backslash \{m_1\} \notin \{y \subseteq f \mid y \neq f \;\&\; \langle \forall u \in f \backslash y, v \in f \backslash y \mid u \subseteq v \vee v \subseteq u\rangle\} \;\&$
$f \backslash (y_0 \backslash \{m_1\}) = f \backslash y_0 \cup \{m_1\} \;\&\; f \backslash (y_0 \backslash \{m_1\}) = f \backslash y_0 \cup \{m_1\}$
$\langle y_0 \backslash \{m_1\} \rangle \hookrightarrow$ *Stat14*(*Stat9*$\star$) $\Rightarrow$
$\neg\langle \forall u \in f \backslash (y_0 \backslash \{m_1\}), v \in f \backslash (y_0 \backslash \{m_1\}) \mid u \subseteq v \vee v \subseteq u\rangle$
EQUAL $\langle$*Stat14*$\rangle$ $\Rightarrow$ *Stat15*:
$\neg\langle \forall u \in f \backslash y_0 \cup \{m_1\}, v \in f \backslash y_0 \cup \{m_1\} \mid u \subseteq v \vee v \subseteq u\rangle$
$\langle u_0, v_0 \rangle \hookrightarrow$ *Stat15*(*Stat15*$\star$) $\Rightarrow$ *Stat16*:
$u_0, v_0 \in f \backslash y_0 \cup \{m_1\} \;\&\; u_0 \not\subseteq v_0 \;\&\; v_0 \not\subseteq u_0$
$\langle u_0, v_0 \rangle \hookrightarrow$ *Stat9*(*Stat16*$\star$) $\Rightarrow$ $u_0 = m_1 \vee v_0 = m_1$
$\langle u_0 \rangle \hookrightarrow$ *Stat11*(*Stat12*$\star$) $\Rightarrow$ $u_0 = m_1$
$\langle v_0 \rangle \hookrightarrow$ *Stat11*(*Stat12*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

B

Connected Claw-Free Graphs: a Proof-Scenario Checked by Referee

This scenario contains the formal proofs, checked by J. T. Schwartz's proof-verifier Referee, of two classical results on connected claw-free graphs; namely, that any such graph:

- owns a perfect matching if its number of vertices is even,
- has a Hamiltonian cycle in its square if it owns three or more vertices.

The original proofs (cf. [141, 148] and [81]) referred to undirected graphs, the ones to be presented refer to a special class of digraphs whose vertices are hereditarily finite sets and whose edges reflect the membership relation. Ours is a legitimate change of perspective in the light of [86], as we will briefly explain at the end.

To make our formal development self-contained, we proceed from the bare set-theoretic foundation built into Ref (cf. [133]). The lemmas exploited without proof in what follows are indeed very few, and their full proofs are available in [133].

B.1 Basic laws on the union-set global operation

DEF unionset: [Members of members of a set] $\bigcup X =_{\text{Def}} \{u : v \in X, u \in v\}$

The proof of the following claim, that the union set of a set s is the set-theoretic 'least upper bound' of all its elements, can be found in [133, p. 387].

THM 2: [l.u.b.] $(X \in S \rightarrow X \subseteq \bigcup S)$ & $(\langle \forall y \in S \mid y \subseteq X \rangle \rightarrow \bigcup S \subseteq X)$.

THEORY imageOfDoubleton$(\mathsf{f}(\mathsf{X}), \mathsf{x}_0, \mathsf{x}_1)$
END imageOfDoubleton

ENTER_THEORY imageOfDoubleton

THM imageOfDoubleton: [Image of an 'elementary set']
$\{\mathsf{f}(\mathsf{v}) : \mathsf{v} \in \emptyset\} = \emptyset \,\&\, \{\mathsf{f}(\mathsf{v}) : \mathsf{v} \in \{\mathsf{x}_0\}\} = \{\mathsf{f}(\mathsf{x}_0)\} \,\&$
$\{\mathsf{f}(\mathsf{v}) : \mathsf{v} \in \{\mathsf{x}_0, \mathsf{x}_1\}\} = \{\mathsf{f}(\mathsf{x}_0), \mathsf{f}(\mathsf{x}_1)\}$. PROOF:
Suppose_not() ⇒ AUTO

Ref has the built-in ability to reduce $\{\mathsf{f}(v) : v \in \emptyset\}$ to $\emptyset$ and $\{\mathsf{f}(v) : v \in \{\mathsf{x}_0\}\}$ to $\{\mathsf{f}(\mathsf{x}_0)\}$; hence we are left with only the doubleton to consider. Let c belong to one of $\{\mathsf{f}(v) : v \in \{\mathsf{x}_0, \mathsf{x}_1\}\}$ and $\{\mathsf{f}(\mathsf{x}_0), \mathsf{f}(\mathsf{x}_1)\}$ but not to the other. After excluding, through variable-substitution, the case $\mathsf{c} \notin \{\mathsf{f}(v) : v \in \{\mathsf{x}_0, \mathsf{x}_1\}\}$, we easily exclude both possibilities $\mathsf{c} = \mathsf{f}(\mathsf{x}_0)$ and $\mathsf{c} = \mathsf{f}(\mathsf{x}_1)$, through variable-substitution and equality propagation.

SIMPLF ⇒ *Stat1* : $\{\mathsf{f}(\mathsf{v}) : \mathsf{v} \in \{\mathsf{x}_0, \mathsf{x}_1\}\} \neq \{\mathsf{f}(\mathsf{x}_0), \mathsf{f}(\mathsf{x}_1)\}$
$\langle \mathsf{c} \rangle \hookrightarrow$ *Stat1* ⇒ $\mathsf{c} \in \{\mathsf{f}(\mathsf{v}) : \mathsf{v} \in \{\mathsf{x}_0, \mathsf{x}_1\}\} \neq \mathsf{c} \in \{\mathsf{f}(\mathsf{x}_0), \mathsf{f}(\mathsf{x}_1)\}$
Suppose ⇒ *Stat2* : $\mathsf{c} \notin \{\mathsf{f}(\mathsf{v}) : \mathsf{v} \in \{\mathsf{x}_0, \mathsf{x}_1\}\}$
$\langle \mathsf{x}_0 \rangle \hookrightarrow$ *Stat2* ⇒ AUTO
$\langle \mathsf{x}_1 \rangle \hookrightarrow$ *Stat2* ⇒ AUTO
Discharge ⇒ *Stat3* : $\mathsf{c} \in \{\mathsf{f}(\mathsf{v}) : \mathsf{v} \in \{\mathsf{x}_0, \mathsf{x}_1\}\} \,\&\, \mathsf{c} \notin \{\mathsf{f}(\mathsf{x}_0), \mathsf{f}(\mathsf{x}_1)\}$
$\langle \mathsf{x}' \rangle \hookrightarrow$ *Stat3* ⇒ $\mathsf{x}' \in \{\mathsf{x}_0, \mathsf{x}_1\} \,\&\, \mathsf{f}(\mathsf{x}') \neq \mathsf{f}(\mathsf{x}_0) \,\&\, \mathsf{f}(\mathsf{x}') \neq \mathsf{f}(\mathsf{x}_1)$
Suppose ⇒ $\mathsf{x}' = \mathsf{x}_0$
EQUAL ⇒ false; Discharge ⇒ $\mathsf{x}' = \mathsf{x}_1$
EQUAL ⇒ false; Discharge ⇒ QED

ENTER_THEORY Set_theory
DISPLAY imageOfDoubleton

THEORY imageOfDoubleton$(\mathsf{f}(\mathsf{X}), \mathsf{x}_0, \mathsf{x}_1)$
$\{\mathsf{f}(\mathsf{v}) : \mathsf{v} \in \emptyset\} = \emptyset \,\&\, \{\mathsf{f}(\mathsf{v}) : \mathsf{v} \in \{\mathsf{x}_0\}\} = \{\mathsf{f}(\mathsf{x}_0)\} \,\&$
$\{\mathsf{f}(\mathsf{v}) : \mathsf{v} \in \{\mathsf{x}_0, \mathsf{x}_1\}\} = \{\mathsf{f}(\mathsf{x}_0), \mathsf{f}(\mathsf{x}_1)\}$
END imageOfDoubleton

THM 2*a*: [$\bigcup$ of double-/single-tons] $\mathsf{Z} = \{\mathsf{X}, \mathsf{Y}\} \rightarrow \bigcup \mathsf{Z} = \mathsf{X} \cup \mathsf{Y}$. PROOF:
Suppose_not$(\mathsf{z}_0, \mathsf{x}_0, \mathsf{y}_0)$ ⇒ AUTO

Under the assumption that $\mathsf{z}_0 = \{\mathsf{x}_0, \mathsf{y}_0\} \,\&\, \bigcup \mathsf{z}_0 \neq \mathsf{x}_0 \cup \mathsf{y}_0$ can hold, two citations of Theorem 2 enable us to get $\mathsf{x}_0 \subseteq \bigcup \mathsf{z}_0$ and $\mathsf{y}_0 \subseteq \bigcup \mathsf{z}_0$ from $\mathsf{z}_0 = \{\mathsf{x}_0, \mathsf{y}_0\}$.

$\langle \mathsf{x}_0, \mathsf{z}_0 \rangle \hookrightarrow T2$ ⇒ AUTO
$\langle \mathsf{y}_0, \mathsf{z}_0 \rangle \hookrightarrow T2$ ⇒ AUTO

A third citation of the same Theorem 2 enables us to derive from $\bigcup \mathsf{z}_0 \neq \mathsf{x}_0 \cup \mathsf{y}_0$ that some element of $\mathsf{z}_0 = \{\mathsf{x}_0, \mathsf{y}_0\}$ is not included in $\mathsf{x}_0 \cup \mathsf{y}_0$, which is manifestly absurd.

$\langle \mathsf{x}_0 \cup \mathsf{y}_0, \mathsf{z}_0 \rangle \hookrightarrow T2$ ⇒ *Stat1* : $\neg \langle \forall \mathsf{y} \in \mathsf{z}_0 \mid \mathsf{y} \subseteq \mathsf{x}_0 \cup \mathsf{y}_0 \rangle$
$\langle \mathsf{v} \rangle \hookrightarrow$ *Stat1* ⇒ $\mathsf{v} \in \{\mathsf{x}_0, \mathsf{y}_0\} \,\&\, \mathsf{v} \not\subseteq \mathsf{x}_0 \cup \mathsf{y}_0$

$(\mathit{Stat1}\star)$Discharge $\Rightarrow$ QED

THM 2b: [Union of union] $\bigcup\bigcup X = \bigcup\{\bigcup y : y \in X\}$. PROOF:

Suppose_not(x_0) $\Rightarrow$ AUTO

Use_def$(\bigcup)$ $\Rightarrow$ $\{z : y \in \{u : v \in x_0, u \in v\}, z \in y\} \neq \{s : r \in \{\bigcup y : y \in x_0\}, s \in r\}$

SIMPLF $\Rightarrow$ $\mathit{Stat1}: \{z : v \in x_0, u \in v, z \in u\} \neq \{s : y \in x_0, s \in \bigcup y\}$

$\langle z_0\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathit{Stat2}: z_0 \in \{z : v \in x_0, u \in v, z \in u\} \neq z_0 \in \{s : y \in x_0, s \in \bigcup y\}$

Suppose $\Rightarrow$ $\mathit{Stat3}: z_0 \in \{z : v \in x_0, u \in v, z \in u\}$ & $z_0 \notin \{s : y \in x_0, s \in \bigcup y\}$

Use_def$(\bigcup v_0)$ $\Rightarrow$ AUTO

$\langle v_0, u_0, z, v_0, z_0\rangle \hookrightarrow \mathit{Stat3}(\mathit{Stat2}\star) \Rightarrow$

$\mathit{Stat4}: z_0 \notin \{z : u \in v_0, z \in u\}$ & $v_0 \in x_0$ & $u_0 \in v_0$ & $z_0 \in u_0$

$\langle u_0, z_0\rangle \hookrightarrow \mathit{Stat4}(\mathit{Stat4}\star) \Rightarrow$ false

Discharge $\Rightarrow$ $\mathit{Stat5}: z_0 \in \{s : y \in x_0, s \in \bigcup y\}$

Use_def$(\bigcup y_0)$ $\Rightarrow$ AUTO

$\langle y_0, s_0\rangle \hookrightarrow \mathit{Stat5}(\mathit{Stat5}\star) \Rightarrow$ $\mathit{Stat6}: z_0 \in \{s : u \in y_0, s \in u\}$ & $y_0 \in x_0$

$\langle u_1, s_1\rangle \hookrightarrow \mathit{Stat6}(\mathit{Stat5}, \mathit{Stat2}\star) \Rightarrow$

$\mathit{Stat7}: z_0 \notin \{z : v \in x_0, u \in v, z \in u\}$ & $z_0 \in u_1$ & $u_1 \in y_0$

$\langle y_0, u_1, z_0\rangle \hookrightarrow \mathit{Stat7}(\mathit{Stat6}\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM 2c: [Additivity and monotonicity of monadic union] $\bigcup(X \cup Y) = \bigcup X \cup \bigcup Y$ & $(Y \supseteq X \to \bigcup Y \supseteq \bigcup X)$. PROOF:

Suppose_not(x_0, y_0) $\Rightarrow$ AUTO

Suppose $\Rightarrow$ $\bigcup(x_0 \cup y_0) \neq \bigcup x_0 \cup \bigcup y_0$

$\langle \{x_0, y_0\}\rangle \hookrightarrow T2b \Rightarrow$ $\bigcup\bigcup\{x_0, y_0\} = \bigcup\{\bigcup v : v \in \{x_0, y_0\}\}$

APPLY $\langle\,\rangle$ imageOfDoubleton$(f(X) \mapsto \bigcup X, x_0 \mapsto x_0, x_1 \mapsto y_0) \Rightarrow$

$\{\bigcup v : v \in \{x_0, y_0\}\} = \{\bigcup x_0, \bigcup y_0\}$

$\langle \{x_0, y_0\}, x_0, y_0\rangle \hookrightarrow T2a \Rightarrow$ $\bigcup\{x_0, y_0\} = x_0 \cup y_0$

$\langle \{\bigcup x_0, \bigcup y_0\}, \bigcup x_0, \bigcup y_0\rangle \hookrightarrow T2a \Rightarrow$ $\bigcup\{\bigcup x_0, \bigcup y_0\} = \bigcup x_0 \cup \bigcup y_0$

EQUAL $\Rightarrow$ false

Discharge $\Rightarrow$ $\bigcup(x_0 \cup y_0) = \bigcup x_0 \cup \bigcup y_0$ & $y_0 = x_0 \cup y_0$ & $\bigcup y_0 \not\supseteq \bigcup x_0$

EQUAL $\Rightarrow$ $\bigcup y_0 = \bigcup x_0 \cup \bigcup y_0$

Discharge $\Rightarrow$ QED

THM 2e: [Union of adjunction] $\bigcup(X \cup \{Y\}) = Y \cup \bigcup X$. PROOF:

Suppose_not(x_0, y_0) $\Rightarrow$ $\mathit{Stat0}: \bigcup(x_0 \cup \{y_0\}) \neq y_0 \cup \bigcup x_0$

$\langle a\rangle \hookrightarrow \mathit{Stat0} \Rightarrow$ $a \in \bigcup(x_0 \cup \{y_0\}) \neq a \in y_0 \cup \bigcup x_0$

> Arguing by contradiction, let x_0, y_0 be a counterexample, so that in either one of $\bigcup(x_0 \cup \{y_0\})$ and $y_0 \cup \bigcup x_0$ there is an a not belonging to the other set. Taking the definition of $\bigcup$ into account, by monotonicity we must exclude the possibility that $a \in \bigcup x_0 \setminus \bigcup(x_0 \cup \{y_0\})$; through variable-substitution, we must also discard the possibility that $a \in \bigcup(x_0 \cup \{y_0\}) \setminus \bigcup x_0 \setminus y_0$.

Set_monot $\Rightarrow$ $\{u : v \in x_0, u \in v\} \subseteq \{u : v \in x_0 \cup \{y_0\}, u \in v\}$

Suppose $\Rightarrow$ $\mathit{Stat1}: a \in \{u : v \in x_0 \cup \{y_0\}, u \in v\}$ &

$a \notin \{u : v \in x_0, u \in v\}$ & $a \notin y_0$

$\langle v_0, u_0, v_0, u_0 \rangle \hookrightarrow$ *Stat1* $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Use_def($\bigcup$) $\Rightarrow$ *Stat2* : $a \notin \{u : v \in x_0 \cup \{y_0\}, u \in v\}$ & $a \in y_0$

‖ The only possibility left, namely that $a \in y_0 \setminus \bigcup(x_0 \cup \{y_0\})$, is also manifestly absurd. This contradiction leads us to the desired conclusion.

$\langle y_0, a \rangle \hookrightarrow$ *Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

B.2 Transitive sets

DEF transitivity: [Transitive set] Trans(T) $\leftrightarrow_{\mathrm{Def}}$ $\{y \in T \mid y \not\subseteq T\} = \emptyset$

THM 3*a*: [Transitive sets include their unionsets] Trans(T) $\leftrightarrow T \supseteq \bigcup T$.

Suppose_not(t) $\Rightarrow$ AUTO

Use_def($\bigcup$t) $\Rightarrow$ AUTO

Use_def(Trans(t)) $\Rightarrow$ AUTO

Suppose $\Rightarrow$ *Stat1* : $t \not\supseteq \bigcup t$ & Trans(t)

$\langle c \rangle \hookrightarrow$ *Stat1*($\star$) $\Rightarrow$ *Stat2* : $c \in \{u : v \in t, u \in v\}$ & $\{y \in t \mid y \not\subseteq t\} = \emptyset$ & $c \notin t$

$\langle v, u, v \rangle \hookrightarrow$ *Stat2*(*Stat2*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ $t \supseteq \bigcup t$ & $\neg$Trans(t)

Use_def(Trans) $\Rightarrow$ *Stat3* : $\{y \in t \mid y \not\subseteq t\} \neq \emptyset$ & $t \supseteq \{u : v \in t, u \in v\}$

Loc_def $\Rightarrow$ $a = \mathbf{arb}(d \setminus t)$

$\langle d \rangle \hookrightarrow$ *Stat3*(*Stat3*) $\Rightarrow$

Stat4 : $a \notin \{u : v \in t, u \in v\}$ & $d \in t$ & $a \in d$ & $a \notin t$

$\langle d, a \rangle \hookrightarrow$ *Stat4*(*Stat4*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM 3*b*: [Incomparable elements x, z of a transitive set t]
Trans(T) & $X, Z \in T$ & $X \notin Z$ & $Z \notin X \rightarrow X \subseteq T \setminus \{X, Z\}$. PROOF:

Suppose_not(t, x, z) $\Rightarrow$ AUTO

$\langle t \rangle \hookrightarrow$ *T3a* $\Rightarrow$ *Stat1* : $t = t \cup \{z\} \cup \{x\}$ & $\bigcup t \not\supseteq x \cup (z \cup \bigcup t)$

$\langle t \cup \{z\}, x \rangle \hookrightarrow$ *T2e* $\Rightarrow$ AUTO

$\langle t, z \rangle \hookrightarrow$ *T2e* $\Rightarrow$ AUTO

EQUAL $\langle$*Stat1*$\rangle$ $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM 3*c*: [For a transitive set, elements are also subsets]
Trans(T) & $X \in T \rightarrow X \subseteq T$. PROOF:

Suppose_not(t, x) $\Rightarrow$ AUTO

$\langle t, x, x \rangle \hookrightarrow$ *T3b* $\Rightarrow$ AUTO

Discharge $\Rightarrow$ QED

THM 3*d*: [Trapping phenomenon for trivial sets]
Trans(S) & $X, Z \in S$ & $X \notin Z$ & $Z \notin X$ & $S \setminus \{X, Z\} \subseteq \{\emptyset, \{\emptyset\}\} \rightarrow$
$S \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$. PROOF:

Suppose_not(s, x, z) $\Rightarrow$ AUTO

$\langle s, x, z \rangle \hookrightarrow$ *T3b* $\Rightarrow$ $x \subseteq \{\emptyset, \{\emptyset\}\}$

$\langle s, z, x \rangle \hookrightarrow$ *T3b* $\Rightarrow$ $z \subseteq \{\emptyset, \{\emptyset\}\}$

Discharge $\Rightarrow$ QED

|| Any strict subset of a transitive set t, owns a subset in t which does not belong to it.

THM 4*a*: [Peddicord's lemma] $\mathsf{Trans}(T)\ \&\ Y \subseteq T\ \&$
$Y \neq T\ \&\ A = \mathbf{arb}(T\backslash Y) \rightarrow A \subseteq Y\ \&\ A \in T\backslash Y$. PROOF:

Suppose_not(t, y, a) ⇒ AUTO

$\langle t, a\rangle \hookrightarrow T3c \Rightarrow\ a \subseteq t$

Discharge ⇒ QED

THM 4*b*: [$\emptyset$ belongs to any transitive $t \neq \emptyset$, so does $\{\emptyset\}$ if $t \not\subseteq \{\emptyset\}$, etc.]
$\mathsf{Trans}(T)\ \&\ N \in \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\ \&\ T \not\subseteq N \rightarrow$
$N \subseteq T\ \&\ \big(N \in T \vee (N = \{\emptyset, \{\emptyset\}\}\ \&\ \{\{\emptyset\}\} \in T)\big)$. PROOF:

Suppose_not(t, n) ⇒ AUTO

|| The '($\star$)' context restriction in the following three steps serves to hide the semantics of **arb**: which, to the limited extent necessary here, has been captured by the preceding Peddicord's lemma.

$\langle t, \emptyset, \mathbf{arb}(t\backslash\emptyset)\rangle \hookrightarrow T4a(\star) \Rightarrow\ \emptyset \in t$

$\langle t, \{\emptyset\}, \mathbf{arb}(t\backslash\{\emptyset\})\rangle \hookrightarrow T4a(\star) \Rightarrow\ \{\emptyset\} \in t$

$\langle t, \{\emptyset, \{\emptyset\}\}, \mathbf{arb}(t\backslash\{\emptyset, \{\emptyset\}\})\rangle \hookrightarrow T4a(\star) \Rightarrow$ false; Discharge ⇒ QED

THM 4*c*: [Source removal does not disrupt transitivity]
$\mathsf{Trans}(S)\ \&\ S \supseteq T\ \&\ (S\backslash T) \cap \bigcup S = \emptyset \rightarrow \mathsf{Trans}(T)$. PROOF:

Suppose_not(s, t) ⇒ AUTO

Use_def(Trans) ⇒ *Stat1*: $\{y \in t \mid y \not\subseteq t\} \neq \emptyset\ \&\ \{y \in s \mid y \not\subseteq s\} = \emptyset$

|| Assuming that s is transitive, that t equals s deprived of some sources and that t is not transitive, there must be an element y of t which is not a subset of t, so that a z ∈ y exists which does not belong to t. Due to the transitivity of s, y is included in s and hence z belongs to s; hence, under the assumption that s\t and $\bigcup$s are disjoint, z does not belong to $\bigcup$s.

$\langle y, y\rangle \hookrightarrow Stat1 \Rightarrow$ *Stat2*: $y \not\subseteq t\ \&\ y \in s\ \&\ y \subseteq s$

Use_def($\bigcup$s) ⇒ AUTO

$\langle z\rangle \hookrightarrow Stat2 \Rightarrow$ *Stat3*: $z \notin \{u : v \in s, u \in v\}\ \&\ z \in y$

|| However, this is untenable.

$\langle y, z\rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge ⇒ QED

B.3 Basic laws on the finitude property

|| To begin developing an acceptable treatment of finiteness without much preparatory work, we adopt here the definition (reminiscent of Tarski's 1924 paper "Sur les ensembles fini"): a set F is finite if every non-null family of subsets of F owns an inclusion-minimal element. This notion is readily specified in terms of the power-set operator, as follows:

DEF $\mathcal{P}$: [Family of all subsets of a given set] $\mathcal{P}S =_{\mathrm{Def}} \{x : x \subseteq S\}$

DEF Fin: [Finitude] $\mathsf{Finite}(F) \leftrightarrow_{\mathrm{Def}} \langle \forall g \in \mathcal{P}(\mathcal{P}F)\backslash\{\emptyset\}, \exists m \mid g \cap \mathcal{P}m = \{m\}\rangle$

> The lemma on the monotonicity of finitude and the THEORY of finite induction displayed below are proved in full—together with various other laws on finiteness which we will not need here—in [133, pp. 405–407].

THM 24: [Monotonicity of finitude] $\mathsf{Y} \supseteq \mathsf{X}\ \&\ \mathsf{Finite}(\mathsf{Y}) \rightarrow \mathsf{Finite}(\mathsf{X})$.

THEORY finiteInduction$(\mathsf{s}_0, \mathsf{P}(\mathsf{S}))$
$\quad \mathsf{Finite}(\mathsf{s}_0)\ \&\ \mathsf{P}(\mathsf{s}_0)$
$\Rightarrow (\mathsf{fin}_\Theta)$
$\quad \langle \forall \mathsf{S} \mid \mathsf{S} \subseteq \mathsf{fin}_\Theta \rightarrow \mathsf{Finite}(\mathsf{S})\ \&\ (\mathsf{P}(\mathsf{S}) \leftrightarrow \mathsf{S} = \mathsf{fin}_\Theta)\rangle$
END finiteInduction

B.4 Some combinatorics of the union-set operation

THM 31*d*: [Unionset of $\emptyset$ and $\{\emptyset\}$] $\mathsf{Y} \subseteq \{\emptyset\} \leftrightarrow \bigcup \mathsf{Y} = \emptyset$. PROOF:

Suppose_not(x_0) $\Rightarrow$ AUTO
- Use_def$(\bigcup \mathsf{x}_0)$ $\Rightarrow$ AUTO
- Suppose $\Rightarrow$ $\mathsf{x}_0 \subseteq \{\emptyset\}$
  - ELEM $\Rightarrow$ *Stat1* : $\{\mathsf{z} : \mathsf{y} \in \mathsf{x}_0, \mathsf{z} \in \mathsf{y}\} \neq \emptyset$
  - $\langle \mathsf{y}_0, \mathsf{z}_1 \rangle \hookrightarrow$*Stat1* $\Rightarrow$ false
- Discharge $\Rightarrow$ *Stat2* : $\mathsf{x}_0 \not\subseteq \{\emptyset\}\ \&\ \{\mathsf{z} : \mathsf{y} \in \mathsf{x}_0, \mathsf{z} \in \mathsf{y}\} = \emptyset$

$\langle \mathsf{y}_1, \mathsf{y}_1, \mathbf{arb}(\mathsf{y}_1) \rangle \hookrightarrow$*Stat2* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM 31*e*: [Unionset of a set obtained through single removal] $\bigcup(\mathsf{X} \setminus \{\mathsf{Y}\}) \supseteq \bigcup \mathsf{X} \setminus \mathsf{Y}\ \&\ \bigcup \mathsf{X} \supseteq \bigcup(\mathsf{X} \setminus \{\mathsf{Y}\})$. PROOF:

- Suppose_not(x, y) $\Rightarrow$ AUTO
- $\langle \mathsf{x} \setminus \{\mathsf{y}\}, \mathsf{x} \rangle \hookrightarrow T2c(\star)$ $\Rightarrow$ *Stat1* : $\bigcup(\mathsf{x} \setminus \{\mathsf{y}\}) \not\supseteq \bigcup \mathsf{x} \setminus \mathsf{y}$
- $\langle \mathsf{c} \rangle \hookrightarrow$*Stat1(Stat1⋆)* $\Rightarrow$ *Stat2* : $\mathsf{c} \in \bigcup \mathsf{x} \setminus \mathsf{y}\ \&\ \mathsf{c} \notin \bigcup(\mathsf{x} \setminus \{\mathsf{y}\})$
- Use_def$(\bigcup)$ $\Rightarrow$ *Stat3* : $\mathsf{c} \in \{\mathsf{u} : \mathsf{v} \in \mathsf{x}, \mathsf{u} \in \mathsf{v}\}\ \&$ $\mathsf{c} \notin \{\mathsf{u} : \mathsf{v} \in \mathsf{x} \setminus \{\mathsf{y}\}, \mathsf{u} \in \mathsf{v}\}\ \&\ \mathsf{c} \notin \mathsf{y}$

$\langle \mathsf{v}_0, \mathsf{u}_0, \mathsf{v}_0, \mathsf{u}_0 \rangle \hookrightarrow$*Stat3(Stat3⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM 31*f*: [Unionset, after a removal followed by two adjunctions] $\bigcup \mathsf{M} \supseteq \mathsf{P}\ \&\ \mathsf{Q} \cup \mathsf{R} = \mathsf{P} \cup \mathsf{S} \rightarrow \bigcup(\mathsf{M} \setminus \{\mathsf{P}\} \cup \{\mathsf{Q}, \mathsf{R}\}) = \bigcup \mathsf{M} \cup \mathsf{S}$. PROOF:

Suppose_not$(\mathsf{m}, \mathsf{p}, \mathsf{q}, \mathsf{r}, \mathsf{s})$ $\Rightarrow$ AUTO
- TELEM $\Rightarrow$ $\mathsf{m} \setminus \{\mathsf{p}\} \cup \{\mathsf{q}\} \cup \{\mathsf{r}\} = \mathsf{m} \setminus \{\mathsf{p}\} \cup \{\mathsf{q}, \mathsf{r}\}$
- EQUAL $\Rightarrow$ $\bigcup(\mathsf{m} \setminus \{\mathsf{p}\} \cup \{\mathsf{q}\} \cup \{\mathsf{r}\}) = \bigcup(\mathsf{m} \setminus \{\mathsf{p}\} \cup \{\mathsf{q}, \mathsf{r}\})$
- $\langle \mathsf{m} \setminus \{\mathsf{p}\}, \mathsf{q} \rangle \hookrightarrow T2e$ $\Rightarrow$ AUTO
- $\langle \mathsf{m} \setminus \{\mathsf{p}\} \cup \{\mathsf{q}\}, \mathsf{r} \rangle \hookrightarrow T2e(\star)$ $\Rightarrow$ $\bigcup(\mathsf{m} \setminus \{\mathsf{p}\} \cup \{\mathsf{q}, \mathsf{r}\}) = \bigcup(\mathsf{m} \setminus \{\mathsf{p}\}) \cup (\mathsf{p} \cup \mathsf{s})$

$\langle \mathsf{m}, \mathsf{p} \rangle \hookrightarrow T31e(\star)$ $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM 31*g*: [Incomparability of pre-pivotal elements] $\mathsf{Y} \in \mathsf{X}\ \&\ \mathsf{X} \in \mathsf{Z}\ \&\ \mathsf{X}, \mathsf{Z} \in \mathsf{S} \rightarrow \mathsf{Y} \in \bigcup(\mathsf{S} \cap \bigcup \mathsf{S})$. PROOF:

Suppose_not$(\mathsf{y}, \mathsf{x}, \mathsf{z}, \mathsf{s})$ $\Rightarrow$ $\mathsf{y} \in \mathsf{x}\ \&\ \mathsf{x} \in \mathsf{z}\ \&\ \mathsf{x}, \mathsf{z} \in \mathsf{s}\ \&\ \mathsf{y} \notin \bigcup(\mathsf{s} \cap \bigcup \mathsf{s})$
- Use_def$(\bigcup)$ $\Rightarrow$ *Stat1* : $\mathsf{y} \notin \{\mathsf{v} : \mathsf{u} \in \mathsf{s} \cap \bigcup \mathsf{s}, \mathsf{v} \in \mathsf{u}\}$
- Use_def$(\bigcup \mathsf{s})$ $\Rightarrow$ AUTO
- $\langle \mathsf{x}, \mathsf{y} \rangle \hookrightarrow$*Stat1*$(\star)$ $\Rightarrow$ *Stat2* : $\mathsf{x} \notin \{\mathsf{t} : \mathsf{w} \in \mathsf{s}, \mathsf{t} \in \mathsf{w}\}$

$\langle z, x \rangle \hookrightarrow Stat2(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

Preparatory to a technique to which we will resort for extending perfect matchings, we introduce the following trivial combinatorial lemma:

THM 31*h*: [Less-one lemma for unionset]
$\bigcup M = T \setminus \{C\} \ \& \ S = T \cup X \cup \{V\} \ \& \ (Y = V \vee (C = Y \ \& \ Y \in S)) \rightarrow$
$\langle \exists d \mid \bigcup (M \cup \{X \cup \{Y\}\}) = S \setminus \{d\} \rangle$. PROOF:
Suppose_not$(m, t, c, s, x, v, y) \Rightarrow$ $Stat0: \neg \langle \exists d \mid \bigcup (m \cup \{x \cup \{y\}\}) = s \setminus \{d\} \rangle \&$
$\bigcup m = t \setminus \{c\} \ \& \ s = t \cup x \cup \{v\} \ \& \ (y = v \vee (c = y \ \& \ y \in s))$

For, supposing the contrary, $\bigcup (m \cup \{x \cup \{y\}\})$ would differ from each of $s \setminus \{s\}$, $s \setminus \{c\}$, and $s \setminus \{v\}$, the first of which equals s. Thanks to Theorem 2e, we can rewrite $\bigcup (m \cup \{x \cup \{y\}\})$ as $x \cup \{y\} \cup \bigcup m$; but then the decision algorithm for a fragment of set theory known as 'multi-level syllogistic with singleton' yields an immediate contradiction.

$\langle s \rangle \hookrightarrow Stat0 \Rightarrow$ $\bigcup (m \cup \{x \cup \{y\}\}) \neq s$
$\langle c \rangle \hookrightarrow Stat0 \Rightarrow$ $\bigcup (m \cup \{x \cup \{y\}\}) \neq s \setminus \{c\}$
$\langle v \rangle \hookrightarrow Stat0 \Rightarrow$ $\bigcup (m \cup \{x \cup \{y\}\}) \neq s \setminus \{v\}$
$\langle m, x \cup \{y\} \rangle \hookrightarrow T2e \Rightarrow$ AUTO
EQUAL $\Rightarrow$ $Stat1: x \cup \{y\} \cup \bigcup m \neq s \setminus \{c\} \ \&$
$x \cup \{y\} \cup \bigcup m \neq s \setminus \{v\} \ \& \ x \cup \{y\} \cup \bigcup m \neq s$
$(Stat0, Stat1)$Discharge $\Rightarrow$ QED

THM 32: [Finite, non-null sets own sources] $(\mathsf{Finite}(F) \ \& \ F \neq \emptyset, \rightarrow (F \setminus \bigcup F \neq \emptyset))$. PROOF:
Suppose_not$(f_1) \Rightarrow$ AUTO

Arguing by contradiction, suppose that there are counterexamples to the claim. Then, exploiting finite induction, we can pick a minimal counterexample, f_0.

APPLY $\langle \mathsf{fin}_\Theta : f_0 \rangle$ finiteInduction$(s_0 \mapsto f_1, P(S) \mapsto (S \neq \emptyset \ \& \ S \setminus \bigcup S = \emptyset)) \Rightarrow$
$Stat0: \langle \forall s \mid s \subseteq f_0 \rightarrow \mathsf{Finite}(s) \ \& \ (s \neq \emptyset \ \& \ s \setminus \bigcup s = \emptyset \leftrightarrow s = f_0) \rangle$
Loc_def $\Rightarrow$ $a = \mathbf{arb}(f_0)$
$\langle f_0 \rangle \hookrightarrow Stat0 \Rightarrow$ $Stat1: \mathsf{Finite}(f_0) \ \& \ a \in f_0 \ \& \ f_0 \setminus \bigcup f_0 = \emptyset$

Momentarily supposing that $f_0 = \{a\}$, one gets $\bigcup f_0 \not\subseteq a$, because $\bigcup f_0 \subseteq a$ would imply $f_0 \setminus \bigcup f_0 \supseteq \{a\} \setminus a$ and hence would imply the emptiness of $\{a\} \setminus a$, whence the manifest absurdity $a \in a$ follows. But, on the other hand, $\bigcup \{a\} \subseteq a$ trivially holds; therefore we must exclude that f_0 is a singleton $\{a\}$.

Suppose $\Rightarrow$ $f_0 = \{a\} \ \& \ \bigcup f_0 \not\subseteq a$
EQUAL $\Rightarrow$ $\bigcup \{a\} \not\subseteq a$
Use_def$(\bigcup) \Rightarrow$ $\{u : v \in \{a\}, u \in v\} \not\subseteq a$
SIMPLF $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

Due to our minimality assumption, the strict non-null subset $f_0 \setminus \{\mathbf{arb}(f_0)\}$ of f_0 cannot be a counterexample to the claim; therefore it has sources and hence $f_0 \setminus \bigcup (f_0 \setminus \{\mathbf{arb}(f_0)\}) \neq \emptyset$.

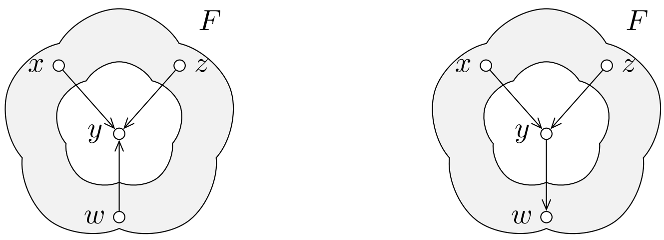

Figure B.1: The forbidden orientations of a claw in a claw-free set.

$\langle f_0 \setminus \{a\}, a\rangle \hookrightarrow T2e(\star) \Rightarrow \quad \bigcup(f_0 \setminus \{a\} \cup \{a\}) = \bigcup(f_0 \setminus \{a\}) \cup a \; \& \; f_0 \setminus \{a\} \cup \{a\} = f_0$
$\langle f_0 \setminus \{a\}\rangle \hookrightarrow Stat0(\star) \Rightarrow \quad f_0 \setminus \bigcup(f_0 \setminus \{a\}) \neq \emptyset$

Since $\mathbf{arb}(f_0)$ does not intersect f_0, the inequality just found conflicts with the equality $f_0 \setminus (\bigcup(f_0 \setminus \{\mathbf{arb}(f_0)\}) \cup \mathbf{arb}(f_0)) = \emptyset$ which one gets from Theorem 2e through equality propagation.

EQUAL $\Rightarrow \quad f_0 \setminus (\bigcup(f_0 \setminus \{a\}) \cup a) = \emptyset \; \& \; a = \mathbf{arb}(f_0)$
Discharge $\Rightarrow$ QED

B.5 Claw-free, transitive sets and their pivots

A *claw* is defined to be a pair Y, F such that (1) F has at least three elements, (2) no element of F belongs to any other element of F, (3) either Y belongs to all elements of F or there is a W in Y such that Y belongs to all elements of $F \setminus \{W\}$.

DEF claw: [Pair forming a claw, perhaps endowed with more than 3 el'ts]

$$\mathsf{Claw}(Y,F) \leftrightarrow_{Def} F \cap \bigcup F = \emptyset \; \& \; \langle \exists x,z,w \mid F \supseteq \{x,z,w\} \; \& \; x \neq z \; \& \; w \notin \{x,z\} \; \& \; \{w\} \cap Y \supseteq \{v \in F \mid Y \notin v\}\rangle$$

To really interest us, a claw-free set must be transitive: we omit this requirement in the definition given here below, but we will make it explicit in the major theorems pertaining to claw-freeness.

DEF clawFreeness: [Claw-freeness in a membership digraph]

$$\mathsf{ClawFree}(S) \leftrightarrow_{Def} \langle \forall y \in S, e \subseteq S \mid \neg \mathsf{Claw}(y,e)\rangle$$

THM $\mathsf{clawFreeness}_a$: [Subsets of claw-free sets are claw-free]
$\mathsf{ClawFree}(S) \; \& \; T \subseteq S \rightarrow \mathsf{ClawFree}(T)$. PROOF:
Suppose_not(s, t) $\Rightarrow$ AUTO
Use_def(ClawFree) $\Rightarrow$ $Stat1: \neg\langle \forall y \in t, e \subseteq t \mid \neg \mathsf{Claw}(y,e)\rangle \; \& \; \langle \forall y \in s, e \subseteq s \mid \neg \mathsf{Claw}(y,e)\rangle$
$\langle y,e,y,e\rangle \hookrightarrow Stat1(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM $\mathsf{clawFreeness}_b$: [Any potential claw must have a bypass]
$\mathsf{ClawFree}(S) \; \& \; S \supseteq \{Y,X,Z,W\} \; \& \; Y \in X \cap Z \; \&$
$W \in Y \; \& \; X \notin Z \cup \{Z\} \; \& \; Z \notin X \rightarrow W \in X \cup Z$. PROOF:

$\mathsf{Suppose\_not}(s,y,x,z,w) \Rightarrow$ AUTO
$\mathsf{Use\_def}(\mathsf{ClawFree}) \Rightarrow$ $\mathit{Stat0}:\ \langle \forall y \in s, e \subseteq s \,|\, \neg\mathsf{Claw}(y,e)\rangle\ \&$
$x \notin w\ \&\ z \notin w\ \&\ x \notin z\ \&\ w \notin x\ \&\ w \notin z\ \&$
$z \notin x\ \&\ x \neq z\ \&\ w \in y\ \&\ y \in x \cap z$
$\mathsf{Loc\_def} \Rightarrow$ $\mathit{Stat1}:\ e = \{x,z,w\}$
$\mathsf{Use\_def}(\mathsf{Claw}(y,e)) \Rightarrow$ AUTO
$\langle y,e\rangle \hookrightarrow \mathit{Stat0}(\mathit{Stat1}\star) \Rightarrow$ $\neg\Big(e \cap \bigcup e = \emptyset\ \&\ \langle \exists x,z,w \,|\, e \supseteq \{x,z,w\}\ \&$
$x \neq z\ \&\ w \notin \{x,z\}\ \&\ \{w\} \cap y \supseteq \{v \in e \,|\, y \notin v\}\rangle\Big)$
$\mathsf{EQUAL} \Rightarrow$ $\bigcup e = \bigcup\{x,z,w\}$
$\mathsf{Suppose} \Rightarrow$ $\mathit{Stat2}:\ e \cap \bigcup e \neq \emptyset$
$\mathsf{Use\_def}(\bigcup e) \Rightarrow$ AUTO
$\langle c\rangle \hookrightarrow \mathit{Stat2}(\star) \Rightarrow$ $\mathit{Stat3}:\ c \in \{u :\ v \in e, u \in v\}\ \&\ c \in e$
$\langle v_0,u_0\rangle \hookrightarrow \mathit{Stat3}(\mathit{Stat1},\mathit{Stat1}\star) \Rightarrow$ $\mathit{Stat4}:\ v_0, c \in \{x,z,w\}\ \&\ c \in v_0$
$(\mathit{Stat0},\mathit{Stat4}\star)\mathsf{Discharge} \Rightarrow$ $\mathit{Stat5}:\ \langle \exists x,z,w \,|\, e \supseteq \{x,z,w\}\ \&\ x \neq z\ \&$
$w \notin \{x,z\}\ \&\ \{w\} \cap y \supseteq \{v \in e \,|\, y \notin v\}\rangle$
$\langle x,z,w\rangle \hookrightarrow \mathit{Stat5}(\mathit{Stat0}\star) \Rightarrow$ $\mathit{Stat6}:\ \{w\} \cap y \not\supseteq \{v \in e \,|\, y \notin v\}$
$\langle d\rangle \hookrightarrow \mathit{Stat6}(\mathit{Stat6}\star) \Rightarrow$ $\mathit{Stat7}:\ d \in \{v \in e \,|\, y \notin v\}\ \&\ d \notin \{w\} \cap y$
$\langle\,\rangle \hookrightarrow \mathit{Stat7}(\mathit{Stat1},\mathit{Stat1}\star) \Rightarrow$ $\mathit{Stat8}:\ d \in \{x,z,w\}\ \&\ y \notin d$
$(\mathit{Stat0},\mathit{Stat8},\mathit{Stat7}\star)\mathsf{Discharge} \Rightarrow$ QED

THEORY pivotsForClawFreeness(s_0)
$\mathsf{ClawFree}(s_0)\ \&\ \mathsf{Finite}(s_0)\ \&\ \mathsf{Trans}(s_0)$
$s_0 \not\subseteq \{\emptyset\}$
END pivotsForClawFreeness

ENTER_THEORY pivotsForClawFreeness

By way of first approximation, we want to select from each finite transitive set s not included in $\{\emptyset\}$ a 'pivotal pair' consisting of an element x of maximum rank in s and an element y of maximum rank in x. To avoid introducing the recursive notion of rank of a set, we slightly generalize the idea: for any set s (not necessarily finite or transitive) we define the *frontier* of s to consist of those elements x of s which own elements y belonging to s such that the length of no membership chain issuing from y, ending in s, and contained in s ever exceeds 2. Any element y which is thus related to an element x of the frontier of s will be called a *pivot* of s.

DEF frontier: [Frontier of a set] $\mathsf{front}(S) =_{\mathrm{Def}} \{x \in S \,|\, x \cap S \backslash \bigcup(S \cap \bigcup S) \neq \emptyset\}$

THM frontier$_1$: [Non-trivial finite sets have a non-null frontier]
$\mathsf{Finite}(S \cap \bigcup S)\ \&\ S \cap \bigcup S \neq \emptyset \rightarrow \mathsf{front}(S) \neq \emptyset$. PROOF:
$\mathsf{Suppose\_not}(s) \Rightarrow$ AUTO
$\langle s \cap \bigcup s\rangle \hookrightarrow T32 \Rightarrow$ $\mathit{Stat1}:\ s \cap \bigcup s \backslash \bigcup(s \cap \bigcup s) \neq \emptyset$
$\mathsf{Use\_def}(\bigcup s) \Rightarrow$ AUTO
$\langle y\rangle \hookrightarrow \mathit{Stat1} \Rightarrow$ $\mathit{Stat2}:\ y \in \{u :\ v \in s, u \in v\}\ \&\ y \in s\ \&\ y \notin \bigcup(s \cap \bigcup s)$
$\mathsf{Use\_def}(\mathsf{front}(s)) \Rightarrow$ AUTO

$\langle \mathsf{x}, \mathsf{u} \rangle \hookrightarrow Stat2 \Rightarrow$

$Stat3: \mathsf{x} \notin \{\mathsf{x}_1 \in \mathsf{s} \mid \mathsf{x}_1 \cap \mathsf{s} \setminus \bigcup(\mathsf{s} \cap \bigcup \mathsf{s}) \neq \emptyset\}$ & $\mathsf{x} \in \mathsf{s}$ & $\mathsf{y} \in \mathsf{x}$

$\langle \mathsf{x} \rangle \hookrightarrow Stat3 \Rightarrow$ false; Discharge $\Rightarrow$ QED

Our next claim is that if we choose a pivot element y of a transitive set s from an element of the frontier of s, then removal of all predecessors of y from s leads to a transitive set t such that y is a source of t.

THM frontier$_2$: [Transitivity-preserving reduction of a transitive set]
$\mathsf{Trans}(\mathsf{S})$ & $\mathsf{X} \in \mathsf{front}(\mathsf{S})$ & $\mathsf{Y} \in \mathsf{X} \setminus \bigcup\bigcup \mathsf{S}$ & $T = \{\mathsf{z} \in \mathsf{S} \mid \mathsf{Y} \notin \mathsf{z}\} \rightarrow$
$\mathsf{Trans}(T)$ & $T \subseteq \mathsf{S}$ & $\mathsf{X} \notin T$ & $\mathsf{Y} \in T \setminus \bigcup T$. PROOF:

Suppose_not(s, x, y, t) $\Rightarrow$ AUTO

Arguing by contradiction, let s, x, y, t be a counterexample to the claim. Taking the definition of t into account to exploit monotonicity, we readily get $\mathsf{t} \subseteq \mathsf{s}$ and $\mathsf{x} \in \mathsf{t}$.

Set_monot $\Rightarrow$ $\{\mathsf{z} \in \mathsf{s} \mid \mathsf{y} \notin \mathsf{z}\} \subseteq \{\mathsf{z} : \mathsf{z} \in \mathsf{s}\}$

Suppose $\Rightarrow$ $Stat0: \mathsf{x} \in \{\mathsf{z} \in \mathsf{s} \mid \mathsf{y} \notin \mathsf{z}\}$

$\langle\,\rangle \hookrightarrow Stat0 \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{x} \notin \mathsf{t}$

Now taking the definition of front into account, we can simplify our initial assumption to the following:

Use_def(front) $\Rightarrow$ $Stat1: \mathsf{x} \in \{\mathsf{x}' \in \mathsf{s} \mid \mathsf{x}' \cap \mathsf{s} \setminus \bigcup(\mathsf{s} \cap \bigcup \mathsf{s}) \neq \emptyset\}$

$\langle\,\rangle \hookrightarrow Stat1 \Rightarrow$ $\mathsf{Trans}(\mathsf{s})$ & $\mathsf{x} \in \mathsf{s}$ & $\mathsf{x} \cap \mathsf{s} \setminus \bigcup(\mathsf{s} \cap \bigcup \mathsf{s}) \neq \emptyset$ & $\mathsf{y} \in \mathsf{x} \setminus \bigcup\bigcup \mathsf{s}$ &
$\mathsf{t} = \{\mathsf{z} \in \mathsf{s} \mid \mathsf{y} \notin \mathsf{z}\}$ & $\mathsf{Trans}(\mathsf{s})$ & $\neg(\mathsf{Trans}(\mathsf{t})$ & $\mathsf{y} \in \mathsf{t} \setminus \bigcup \mathsf{t})$

Since s is transitive, if t were not transitive then by Theorem 4c $\mathsf{s} \setminus \mathsf{t}$ would have an element z not being a source of s. But then y would belong to $\mathsf{z} \in \bigcup \mathsf{s}$, which conflicts with y being a pivot.

Suppose $\Rightarrow$ $\neg\mathsf{Trans}(\mathsf{t})$

$\langle \mathsf{s}, \mathsf{t} \rangle \hookrightarrow T4c \Rightarrow$ $Stat2: (\mathsf{s} \setminus \mathsf{t}) \cap \bigcup \mathsf{s} \neq \emptyset$

Use_def($\bigcup \mathsf{s}$) $\Rightarrow$ AUTO

$\langle \mathsf{z} \rangle \hookrightarrow Stat2 \Rightarrow$ $Stat3: \mathsf{z} \in \{\mathsf{u}' : \mathsf{w}' \in \mathsf{s}, \mathsf{u}' \in \mathsf{w}'\}$ &
$\mathsf{z} \notin \{\mathsf{z}' \in \mathsf{s} \mid \mathsf{y} \notin \mathsf{z}'\}$ & $\mathsf{z} \in \mathsf{s}$

$\langle \mathsf{v}, \mathsf{a}, \mathsf{z} \rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $\mathsf{y} \in \mathsf{z}$ & $\mathsf{z} \in \mathsf{v}$ & $\mathsf{v} \in \mathsf{s}$

Use_def($\bigcup\bigcup \mathsf{s}$) $\Rightarrow$ AUTO

EQUAL $\langle Stat1 \rangle \Rightarrow$ $\mathsf{y} \notin \{\mathsf{u} : \mathsf{w} \in \{\mathsf{u}' : \mathsf{w}' \in \mathsf{s}, \mathsf{u}' \in \mathsf{w}'\}, \mathsf{u} \in \mathsf{w}\}$

SIMPLF $\Rightarrow$ $Stat4: \mathsf{y} \notin \{\mathsf{u} : \mathsf{w}' \in \mathsf{s}, \mathsf{w} \in \mathsf{w}', \mathsf{u} \in \mathsf{w}\}$

$\langle \mathsf{v}, \mathsf{z}, \mathsf{y} \rangle \hookrightarrow Stat4(Stat1\star) \Rightarrow$ false; Discharge $\Rightarrow$ $\mathsf{y} \in \bigcup \mathsf{t} \vee \mathsf{y} \notin \mathsf{t}$

Now knowing that Trans(t), we must consider the other possibility, namely that $\mathsf{y} \notin \mathsf{t} \setminus \bigcup \mathsf{t}$. However, after expanding t and $\bigcup \mathsf{t}$ according to their definitions, ...

Use_def($\bigcup \mathsf{t}$) $\Rightarrow$ AUTO

Use_def(Trans(s)) $\Rightarrow$ AUTO

EQUAL $\Rightarrow$ $Stat5: \{\mathsf{y} \in \mathsf{s} \mid \mathsf{y} \not\subseteq \mathsf{s}\} = \emptyset$ &
$(\mathsf{y} \in \{\mathsf{u} : \mathsf{v} \in \{\mathsf{z} \in \mathsf{s} \mid \mathsf{y} \notin \mathsf{z}\}, \mathsf{u} \in \mathsf{v}\} \vee \mathsf{y} \notin \{\mathsf{z} \in \mathsf{s} \mid \mathsf{y} \notin \mathsf{z}\})$

... we see that neither one of the possibilities $\mathsf{y} \in \bigcup \mathsf{t}$, $\mathsf{y} \notin \mathsf{t}$ is tenable.

$\langle x\rangle \hookrightarrow Stat5(Stat1\star) \Rightarrow \quad y \in s$
SIMPLF $\Rightarrow$ $Stat6: \ y \in \{u : v \in s, u \in v \mid y \notin v\} \lor y \notin \{z \in s \mid y \notin z\}$
$\langle w, u, y\rangle \hookrightarrow Stat6(Stat5\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

DEF $\mathsf{clawFreeness_{frontEl}}$: [Frontier el't of a claw-free transitive non-trivial set]
$$x_\Theta =_{\mathrm{Def}} \mathbf{arb}(\mathsf{front}(s_0))$$

DEF $\mathsf{clawFreeness_{pivotEl}}$: [Pivotal el't of a claw-free transitive non-trivial set]
$$y_\Theta =_{\mathrm{Def}} \mathbf{arb}(x_\Theta \backslash \bigcup\bigcup s_0)$$

THM $\mathsf{clawFreeness_c}$: [x_Θ truly belongs to the frontier of s_0]
$x_\Theta \in \mathsf{front}(s_0) \ \& \ x_\Theta \backslash \bigcup\bigcup s_0 \neq \emptyset \ \& \ x_\Theta \in s_0$. PROOF:
Suppose_not() $\Rightarrow$ AUTO
Assump $\Rightarrow$ $Stat0: \ \mathsf{ClawFree}(s_0) \ \& \ \mathsf{Finite}(s_0 \cap \bigcup s_0) \ \& \ \mathsf{Trans}(s_0) \ \& \ s_0 \not\subseteq \{\emptyset\}$
$\langle s_0\rangle \hookrightarrow T3a \Rightarrow \quad s_0 \cap \bigcup s_0 = \bigcup s_0$
$\langle s_0\rangle \hookrightarrow T31d \Rightarrow \quad s_0 \cap \bigcup s_0 \neq \emptyset$
$\langle s_0\rangle \hookrightarrow T\mathsf{frontier}_1 \Rightarrow \quad Stat1: \ \mathsf{front}(s_0) \neq \emptyset$
Use_def(front(s_0)) $\Rightarrow$ AUTO
Use_def(x_Θ) $\Rightarrow$ $Stat2: \ x_\Theta \in \{x \in s_0 \mid x \cap s_0 \backslash \bigcup(s_0 \cap \bigcup s_0) \neq \emptyset\} \ \& \ x_\Theta \in \mathsf{front}(s_0)$
$\langle\ \rangle \hookrightarrow Stat2(Stat2\star) \Rightarrow \quad x_\Theta \in s_0 \ \& \ x_\Theta \backslash \bigcup(s_0 \cap \bigcup s_0) \neq \emptyset$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

> Pivotal elements, in a transitive claw-free set such as the one treated in this THEORY, own at most two predecessors.

THM $\mathsf{clawFreeness_0}$: [Pivots own at most two predecessors]
$Y \in X \backslash \bigcup\bigcup s_0 \ \& \ X \in s_0 \rightarrow \langle \exists z \in s_0 \mid \{v \in s_0 \mid Y \in v\} = \{X, z\} \ \& \ Y \in z\rangle$.
Suppose_not(y, x) $\Rightarrow$ $Stat1: \neg\langle \exists z \in s_0 \mid \{v \in s_0 \mid y \in v\} = \{x, z\} \ \& \ y \in z\rangle \ \& \ x \in s_0 \ \& \ y \in x \backslash \bigcup\bigcup s_0$

> Suppose that y, x constitute a counter-example, so that y has, in addition to x, at least two predecessors z and w in s_0.

Suppose $\Rightarrow$ $Stat2: \ x \notin \{v \in s_0 \mid y \in v\}$
$\langle x\rangle \hookrightarrow Stat2(\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle x\rangle \hookrightarrow Stat1(\star) \Rightarrow \quad Stat3: \ \{v \in s_0 \mid y \in v\} \neq \{x\}$
$\langle z\rangle \hookrightarrow Stat3(\star) \Rightarrow \quad Stat4: \ z \in \{v \in s_0 \mid y \in v\} \ \& \ x \neq z$
$\langle\ \rangle \hookrightarrow Stat4(Stat4\star) \Rightarrow \quad Stat5: \ z \in s_0 \ \& \ y \in z$
$\langle z\rangle \hookrightarrow Stat1(\star) \Rightarrow \quad Stat6: \ \{v \in s_0 \mid y \in v\} \neq \{z, x\}$
$\langle w\rangle \hookrightarrow Stat6(\star) \Rightarrow \quad Stat7: \ w \in \{v \in s_0 \mid y \in v\} \ \& \ w \notin \{x, z\}$
$\langle\ \rangle \hookrightarrow Stat7(Stat7\star) \Rightarrow \quad Stat8: \ w \in s_0 \ \& \ y \in w$
Loc_def $\Rightarrow$ $e = \{x, z, w\}$
Suppose $\Rightarrow$ $Stat9: \ \{v \in e \mid y \notin v\} \neq \emptyset$
$\langle v\rangle \hookrightarrow Stat9(\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO

> The transitivity of s_0, since $y \in x$ and $x \in s_0$, implies that $y \in s_0$; therefore, in view of the claw-freeness of s_0, y and $e = \{x, z, w\}$ do not form a claw.

Assump $\Rightarrow$ $\mathsf{ClawFree}(s_0) \ \& \ \mathsf{Trans}(s_0)$
Use_def(ClawFree) $\Rightarrow$ $Stat10: \ \langle \forall y \in s_0, e \subseteq s_0 \mid \neg\mathsf{Claw}(y, e)\rangle$

$\langle s_0, x\rangle \hookrightarrow T3c\ (\star) \Rightarrow \quad y \in s_0$
$\langle y, e\rangle \hookrightarrow Stat10(\star) \Rightarrow \quad \neg \mathsf{Claw}(y, e)$

|| It readily follows from the definition of claw that $\{x, z, w\}$ and $\bigcup\{x, z, w\}$ intersect; therefore, we can pick an element a common to the two.

$\mathsf{Use\_def}(\mathsf{Claw}) \Rightarrow \quad Stat11 : \neg\langle \exists x, z, w \mid e \supseteq \{x, z, w\}\ \&\ x \neq z\ \&\ w \notin \{x, z\}$
$\&\ \{w\} \cap y \supseteq \{v \in e \mid y \notin v\}\rangle \vee e \cap \bigcup e \neq \emptyset$
$\langle x, z, w\rangle \hookrightarrow Stat11(Stat4\star) \Rightarrow \quad e \cap \bigcup e \neq \emptyset$
$\mathsf{EQUAL}\ \langle Stat8\rangle \Rightarrow \quad Stat12 : \{x, z, w\} \cap \bigcup\{x, z, w\} \neq \emptyset$
$\langle a\rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow \quad Stat13 : a \in \{x, z, w\}\ \&\ a \in \bigcup\{x, z, w\}$

|| But then $y \in a$, $a \subseteq \bigcup\bigcup\{x, z, w\}$, and $\bigcup\bigcup\{x, z, w\} \subseteq \bigcup\bigcup s_0$ must hold, implying that $y \in \bigcup\bigcup s_0$; but we have started with the assumption that $y \notin \bigcup\bigcup s_0$. This contradiction proves the claim.

$\langle \{x, z, w\}, s_0\rangle \hookrightarrow T2c\ (Stat1, Stat5, Stat8, Stat13\star) \Rightarrow \quad y \in a$
$\&\ \bigcup s_0 \supseteq \bigcup\{x, z, w\}$
$\langle \bigcup\{x, z, w\}, \bigcup s_0\rangle \hookrightarrow T2c\ (Stat13\star) \Rightarrow \quad \bigcup\bigcup s_0 \supseteq \bigcup(\bigcup\{x, z, w\})$
$\langle a, \bigcup\{x, z, w\}\rangle \hookrightarrow T2\ (Stat13\star) \Rightarrow \quad Stat15 : y \in \bigcup\bigcup s_0$
$(Stat1, Stat15\star)\mathsf{Discharge} \Rightarrow$ QED

THM clawFreeness$_d$: [Shape of the frontier at a pivotal pair]
$\langle \exists z \mid \{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z\}\ \&\ y_\Theta \in z\rangle$. PROOF:
$\mathsf{Suppose\_not}() \Rightarrow$ AUTO
$\langle\ \rangle \hookrightarrow T\mathsf{clawFreeness_c} \Rightarrow \quad Stat1 : x_\Theta \backslash \bigcup\bigcup s_0 \neq \emptyset\ \&\ x_\Theta \in s_0$
$\mathsf{Use\_def}(y_\Theta) \Rightarrow \quad y_\Theta \in x_\Theta \backslash \bigcup\bigcup s_0$
$\langle y_\Theta, x_\Theta\rangle \hookrightarrow T\mathsf{clawFreeness_0} \Rightarrow$
$Stat2 : \langle \exists z \in s_0 \mid \{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z\}\ \&\ y_\Theta \in z\rangle\ \&$
$\neg\langle \exists z \mid \{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z\}\ \&\ y_\Theta \in z\rangle$
$\langle z_0, z_0\rangle \hookrightarrow Stat2 \Rightarrow \quad \mathsf{false};$ $\quad \mathsf{Discharge} \Rightarrow$ QED

|| Via Skolemization, we give a name to the third item in a pivotal tripleton:

APPLY $\langle v1_\Theta : z_\Theta\rangle$ Skolem$\Rightarrow$
THM clawFreeness$_e$. $\{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z_\Theta\}\ \&\ y_\Theta \in z_\Theta$.

THM clawFreeness$_f$: [Tripleton pivot in claw-free, transitive set]
$\{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z_\Theta\}\ \&\ \{x_\Theta, y_\Theta, z_\Theta\} \subseteq s_0\ \&$
$y_\Theta \in x_\Theta \cap z_\Theta \backslash \bigcup\bigcup s_0\ \&\ x_\Theta \notin z_\Theta\ \&\ z_\Theta \notin x_\Theta$. PROOF:
$\mathsf{Suppose\_not}() \Rightarrow$ AUTO
$\langle\ \rangle \hookrightarrow T\mathsf{clawFreeness_c} \Rightarrow \quad Stat3 : x_\Theta \backslash \bigcup\bigcup s_0 \neq \emptyset$
$\mathsf{Use\_def}(y_\Theta) \Rightarrow \quad y_\Theta \notin \bigcup\bigcup s_0\ \&\ y_\Theta \in x_\Theta$
$\langle\ \rangle \hookrightarrow T\mathsf{clawFreeness_e} \Rightarrow \quad Stat1 : x_\Theta \in \{v : v \in s_0 \mid y_\Theta \in v\}\ \&$
$z_\Theta \in \{v \in s_0 \mid y_\Theta \in v\}\ \&$
$\{v : v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z_\Theta\}\ \&\ y_\Theta \in z_\Theta$
$\langle v_0, v_1\rangle \hookrightarrow Stat1 \Rightarrow \quad x_\Theta \in s_0\ \&\ z_\Theta \in s_0$
$\mathsf{Assump} \Rightarrow \quad \mathsf{Trans}(s_0)$
$\langle s_0, z_\Theta\rangle \hookrightarrow T3c \Rightarrow \quad y_\Theta \in s_0$

$\langle s_0 \rangle \hookrightarrow T3a \Rightarrow \quad s_0 \cap \bigcup s_0 = \bigcup s_0$
EQUAL $\Rightarrow \quad y_\Theta \notin \bigcup(s_0 \cap \bigcup s_0)$
$\langle y_\Theta, x_\Theta, z_\Theta, s_0 \rangle \hookrightarrow T31g \Rightarrow \quad x_\Theta \notin z_\Theta$
$\langle y_\Theta, z_\Theta, x_\Theta, s_0 \rangle \hookrightarrow T31g \Rightarrow$ false; Discharge $\Rightarrow$ QED

DEF clawFreeness$_{\text{rmv}}$: [Removing el'ts above pivot] $\quad t_\Theta =_{\text{Def}} \{v \in s_0 \mid y_\Theta \notin v\}$

> The removal of the predecessors of a pivot from a claw-free, transitive non-trivial set such as the one treated by this THEORY does not disrupt transitivity.

THM clawFreeness$_{\text{g}}$: [Removing el'ts above pivot preserves transitivity]
$\text{Trans}(t_\Theta)\ \&\ \text{ClawFree}(t_\Theta)\ \&\ t_\Theta \subseteq s_0\ \&\ x_\Theta \notin t_\Theta\ \&\ y_\Theta \in t_\Theta \setminus \bigcup t_\Theta\ \&$
$t_\Theta = s_0 \setminus \{x_\Theta, z_\Theta\}$. PROOF:
Suppose_not() $\Rightarrow$ AUTO
Use_def(t_Θ) $\Rightarrow$ *Stat1* : $t_\Theta = \{v \in s_0 \mid y_\Theta \notin v\}$
Set_monot $\Rightarrow$ $\{v \in s_0 \mid y_\Theta \notin v\} \subseteq \{v : v \in s_0\}$
Assump $\Rightarrow$ $\text{Trans}(s_0)\ \&\ \text{ClawFree}(s_0)$
$\langle s_0, t_\Theta \rangle \hookrightarrow T\text{clawFreeness}_a(Stat1\star) \Rightarrow$ $\text{ClawFree}(t_\Theta)$
$\langle\ \rangle \hookrightarrow T\text{clawFreeness}_c \Rightarrow$ $x_\Theta \in \text{front}(s_0)$
$\langle\ \rangle \hookrightarrow T\text{clawFreeness}_f \Rightarrow$ *Stat2* : $\{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z_\Theta\}\ \&$
$y_\Theta \in x_\Theta \setminus \bigcup\bigcup s_0\ \&\ y_\Theta \notin \bigcup\bigcup s_0$
$\langle s_0, x_\Theta, y_\Theta, t_\Theta \rangle \hookrightarrow T\text{frontier}_2(\star) \Rightarrow$ *Stat3* : $t_\Theta \neq s_0 \setminus \{x_\Theta, z_\Theta\}\ \&$
$\text{Trans}(t_\Theta)\ \&\ t_\Theta \subseteq s_0\ \&\ x_\Theta \notin t_\Theta\ \&\ y_\Theta \in t_\Theta \setminus \bigcup t_\Theta$
$\langle e \rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow$ $e \in t_\Theta \neq e \in s_0 \setminus \{x_\Theta, z_\Theta\}$
Suppose $\Rightarrow$ *Stat4* : $e \in \{v \in s_0 \mid y_\Theta \notin v\}\ \&\ e \notin s_0 \setminus \{x_\Theta, z_\Theta\}$
$\langle\ \rangle \hookrightarrow Stat4(Stat2\star) \Rightarrow$ *Stat5* : $e \in \{v \in s_0 \mid y_\Theta \in v\}\ \&\ e \in s_0\ \&\ y_\Theta \notin e$
$\langle\ \rangle \hookrightarrow Stat5(Stat5\star) \Rightarrow$ false; Discharge $\Rightarrow$
Stat6 : $e \notin \{v \in s_0 \mid y_\Theta \in v\}\ \&\ e \notin \{v \in s_0 \mid y_\Theta \notin v\}\ \&\ e \in s_0$
$\langle e, e \rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

ENTER_THEORY Set_theory
DISPLAY pivotsForClawFreeness

THEORY pivotsForClawFreeness(s_0)
$\text{ClawFree}(s_0)\ \&\ \text{Finite}(s_0)\ \&\ \text{Trans}(s_0)$
$s_0 \not\subseteq \{\emptyset\}$
$\Rightarrow (x_\Theta, y_\Theta, z_\Theta, t_\Theta)$
$\{v \in s_0 \mid y_\Theta \in v\} = \{x_\Theta, z_\Theta\}\ \&\ \{x_\Theta, y_\Theta, z_\Theta\} \subseteq s_0\ \&\ y_\Theta \in x_\Theta \cap z_\Theta \setminus \bigcup\bigcup s_0\ \&$
$x_\Theta \notin z_\Theta\ \&\ z_\Theta \notin x_\Theta$
$t_\Theta = \{v \in s_0 \mid y_\Theta \notin v\}$
$\text{ClawFree}(t_\Theta)\ \&\ \text{Trans}(t_\Theta)\ \&\ t_\Theta \subseteq s_0\ \&\ x_\Theta \notin t_\Theta\ \&\ y_\Theta \in t_\Theta \setminus \bigcup t_\Theta\ \&$
$t_\Theta = s_0 \setminus \{x_\Theta, z_\Theta\}$
END pivotsForClawFreeness

B.6 Hanks, cycles, and Hamiltonian cycles

The following notion approximately models the concept of a graph where every vertex has at least two incident edges. However, we neither require that (1) edges be doubletons, nor that (2) the set H of edges and the one of vertices—which is understood to be $\bigcup\mathsf{H}$—be disjoint.

DEF cycle_0: [Collection of edges whose endpoints have degree greater than 1]

$$\mathsf{Hank(H)} \leftrightarrow_{\mathrm{Def}} \emptyset \notin \mathsf{H} \;\&\; \langle \forall \mathsf{e} \in \mathsf{H} \,|\, \mathsf{e} \subseteq \bigcup(\mathsf{H} \setminus \{\mathsf{e}\}) \rangle$$

DEF cycle_1: [Cycle (unless null)]

$$\mathsf{Cycle(C)} \leftrightarrow_{\mathrm{Def}} \mathsf{Hank(C)} \;\&\; \langle \forall \mathsf{d} \subseteq \mathsf{C} \,|\, \mathsf{Hank(d)} \;\&\; \mathsf{d} \neq \emptyset \rightarrow \mathsf{d} = \mathsf{C} \rangle$$

THM hank_0: [Alternative characterization of a hank]
$\mathsf{Hank(H)} \leftrightarrow \big(\emptyset \notin \mathsf{H} \;\&\; \langle \forall \mathsf{e} \in \mathsf{H}, \mathsf{x} \in \mathsf{e}, \exists \mathsf{q} \in \mathsf{H} \,|\, \mathsf{q} \neq \mathsf{e} \;\&\; \mathsf{x} \in \mathsf{q} \rangle\big)$. PROOF:
Suppose_not(h) ⇒ AUTO
Suppose ⇒ *Stat1* : $\neg\langle \forall \mathsf{e} \in \mathsf{h}, \mathsf{x} \in \mathsf{e}, \exists \mathsf{q} \in \mathsf{h} \,|\, \mathsf{q} \neq \mathsf{e} \;\&\; \mathsf{x} \in \mathsf{q} \rangle \;\&$
$\langle \forall \mathsf{e} \in \mathsf{h} \,|\, \mathsf{e} \subseteq \bigcup(\mathsf{h} \setminus \{\mathsf{e}\}) \rangle$
Use_def $(\bigcup(\mathsf{h} \setminus \{\mathsf{e}_0\}))$ ⇒ AUTO
$\langle \mathsf{e}_0, \mathsf{x}_0, \mathsf{e}_0 \rangle \hookrightarrow$ *Stat1* ⇒ *Stat2* : $\mathsf{x}_0 \in \{\mathsf{v} : \mathsf{u} \in \mathsf{h} \setminus \{\mathsf{e}_0\}, \mathsf{v} \in \mathsf{u}\} \;\&$
$\neg\langle \exists \mathsf{q} \in \mathsf{h} \,|\, \mathsf{q} \neq \mathsf{e}_0 \;\&\; \mathsf{x}_0 \in \mathsf{q} \rangle \;\&\; \mathsf{e}_0 \in \mathsf{h} \;\&\; \mathsf{x}_0 \in \mathsf{e}_0$
$\langle \mathsf{q}_0, \mathsf{v}_0, \mathsf{q}_0 \rangle \hookrightarrow$ *Stat2* ⇒ false; Discharge ⇒ AUTO
Use_def(Hank) ⇒ *Stat3* : $\neg\langle \forall \mathsf{e} \in \mathsf{h} \,|\, \mathsf{e} \subseteq \bigcup(\mathsf{h} \setminus \{\mathsf{e}\}) \rangle \;\&$
$\langle \forall \mathsf{e} \in \mathsf{h}, \mathsf{x} \in \mathsf{e}, \exists \mathsf{q} \in \mathsf{h} \,|\, \mathsf{q} \neq \mathsf{e} \;\&\; \mathsf{x} \in \mathsf{q} \rangle$
$\langle \mathsf{e}_1 \rangle \hookrightarrow$ *Stat3* ⇒ *Stat4* : $\mathsf{e}_1 \not\subseteq \bigcup(\mathsf{h} \setminus \{\mathsf{e}_1\}) \;\&$
$\langle \forall \mathsf{e} \in \mathsf{h}, \mathsf{x} \in \mathsf{e}, \exists \mathsf{q} \in \mathsf{h} \,|\, \mathsf{q} \neq \mathsf{e} \;\&\; \mathsf{x} \in \mathsf{q} \rangle \;\&\; \mathsf{e}_1 \in \mathsf{h}$
Use_def $(\bigcup(\mathsf{h} \setminus \{\mathsf{e}_1\}))$ ⇒ AUTO
$\langle \mathsf{x}_1, \mathsf{e}_1, \mathsf{x}_1 \rangle \hookrightarrow$ *Stat4* ⇒ *Stat5* : $\langle \exists \mathsf{q} \in \mathsf{h} \,|\, \mathsf{q} \neq \mathsf{e}_1 \;\&\; \mathsf{x}_1 \in \mathsf{q} \rangle \;\&$
$\mathsf{x}_1 \notin \{\mathsf{v} : \mathsf{u} \in \mathsf{h} \setminus \{\mathsf{e}_1\}, \mathsf{v} \in \mathsf{u}\} \;\&\; \mathsf{x}_1 \in \mathsf{e}_1$
$\langle \mathsf{q}_1, \mathsf{q}_1, \mathsf{x}_1 \rangle \hookrightarrow$ *Stat5* ⇒ false; Discharge ⇒ QED

THM hank_1: [No singleton-or-doubleton of non-null sets is a cycle]
$\mathsf{H} \subseteq \{\mathsf{X}, \mathsf{U}\} \;\&\; \mathsf{Hank(H)} \rightarrow \mathsf{H} = \emptyset$. PROOF:
Suppose_not $(\mathsf{h}_0, \mathsf{x}_0, \mathsf{u}_0)$ ⇒ *Stat0* : $\mathsf{h}_0 \neq \emptyset \;\&\; \mathsf{h}_0 \subseteq \{\mathsf{x}_0, \mathsf{u}_0\} \;\&\; \mathsf{Hank}(\mathsf{h}_0)$

For, assuming that h_0 is a hank, non-null, and a subset of a doubleton $\{\mathsf{x}_0, \mathsf{u}_0\}$, we will reach a contradiction arguing as follows. If a is one of the (at most two) elements of h_0, since $\emptyset \neq \mathsf{a} \;\&\; \mathsf{a} \subseteq \bigcup(\mathsf{h}_0 \setminus \{\mathsf{a}\})$ ensues from the definition of hank, $\bigcup(\mathsf{h}_0 \setminus \{\mathsf{a}\})$ must be non-null; hence $\mathsf{h}_0 = \{\mathsf{a}, \mathsf{b}\}$, where $\mathsf{b} \neq \mathsf{a}$. But then $\bigcup(\mathsf{h}_0 \setminus \{\mathsf{a}\}) = \bigcup\{\mathsf{b}\}$ and $\bigcup(\mathsf{h}_0 \setminus \{\mathsf{b}\}) = \bigcup\{\mathsf{a}\}$, i.e., $\bigcup(\mathsf{h}_0 \setminus \{\mathsf{a}\}) = \mathsf{b}$ and $\bigcup(\mathsf{h}_0 \setminus \{\mathsf{b}\}) = \mathsf{a}$; therefore $\mathsf{a} \subseteq \mathsf{b}$ and $\mathsf{b} \subseteq \mathsf{a}$ ensue from the definition of hank, leading us to the identity $\mathsf{a} = \mathsf{b}$, which contradicts an earlier inequality.

$\langle \mathsf{a} \rangle \hookrightarrow$ *Stat0(Stat0⋆)* ⇒ *Stat1* : $\mathsf{a} \in \mathsf{h}_0$
Use_def(Hank) ⇒ *Stat2* : $\langle \forall \mathsf{e} \in \mathsf{h}_0 \,|\, \mathsf{e} \subseteq \bigcup(\mathsf{h}_0 \setminus \{\mathsf{e}\}) \rangle \;\&\; \emptyset \notin \mathsf{h}_0$
$\langle \mathsf{a} \rangle \hookrightarrow$ *Stat2* ⇒ $\mathsf{a} \subseteq \bigcup(\mathsf{h}_0 \setminus \{\mathsf{a}\})$
$\langle \mathsf{h}_0 \setminus \{\mathsf{a}\} \rangle \hookrightarrow$ *T31d* ⇒ *Stat3* : $\mathsf{h}_0 \neq \{\mathsf{a}\}$
$\langle \mathsf{b} \rangle \hookrightarrow$ *Stat3* ⇒ *Stat4* : $\mathsf{b} \in \mathsf{h}_0 \;\&\; \mathsf{b} \neq \mathsf{a}$
$\langle \mathsf{b} \rangle \hookrightarrow$ *Stat2* ⇒ $\mathsf{b} \subseteq \bigcup(\mathsf{h}_0 \setminus \{\mathsf{b}\})$

$\langle \{a\}, a, a\rangle \hookrightarrow T2a \Rightarrow \quad \bigcup\{a\} = a$
$\langle \{b\}, b, b\rangle \hookrightarrow T2a \Rightarrow \quad \bigcup\{b\} = b$
$(Stat0, Stat1, Stat4\star)$ELEM $\Rightarrow \quad h_0 \setminus \{a\} = \{b\} \ \& \ h_0 \setminus \{b\} = \{a\}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

The following is the basic case of a general theorem scheme where the length of the chain can be any number > 2.

THM hank$_2$: [A membership chain and an extra edge form a hank]
$X \in Y \ \& \ Y \in Z \rightarrow \mathsf{Hank}(\{\{X,Y\}, \{Y,Z\}, \{Z,X\}\})$. PROOF:
Suppose_not(x_0, y_0, z_0) $\Rightarrow$ AUTO
Use_def(Hank) $\Rightarrow$ $Stat0: \ \neg\langle \forall e \in \{\{x_0,y_0\}, \{y_0,z_0\}, \{z_0,x_0\}\} \mid e \subseteq \bigcup(\{\{x_0,y_0\}, \{y_0,z_0\}, \{z_0,x_0\}\} \setminus \{e\})\rangle \ \& \ x_0 \in y_0 \ \& \ y_0 \in z_0$
$\langle e_0\rangle \hookrightarrow Stat0 \Rightarrow \quad e_0 \in \{\{x_0,y_0\}, \{y_0,z_0\}, \{z_0,x_0\}\} \ \& \ e_0 \not\subseteq \bigcup(\{\{x_0,y_0\}, \{y_0,z_0\}, \{z_0,x_0\}\} \setminus \{e_0\})$
Suppose $\Rightarrow \quad e_0 = \{x_0,y_0\} \ \& \ \{\{x_0,y_0\}, \{y_0,z_0\}, \{z_0,x_0\}\} \setminus \{e_0\} = \{\{y_0,z_0\}, \{z_0,x_0\}\}$
$\langle \{\{y_0,z_0\}, \{z_0,x_0\}\}, \{y_0,z_0\}, \{z_0,x_0\}\rangle \hookrightarrow T2a \Rightarrow$
$\bigcup\{\{y_0,z_0\}, \{z_0,x_0\}\} = \{y_0,z_0,x_0\}$
EQUAL $\Rightarrow \quad \{x_0,y_0\} \not\subseteq \{y_0,z_0,x_0\}$; Discharge $\Rightarrow$ AUTO
Suppose $\Rightarrow \quad e_0 = \{y_0,z_0\} \ \& \ \{\{x_0,y_0\}, \{y_0,z_0\}, \{z_0,x_0\}\} \setminus \{e_0\} = \{\{x_0,y_0\}, \{z_0,x_0\}\}$
$\langle \{\{x_0,y_0\}, \{z_0,x_0\}\}, \{x_0,y_0\}, \{z_0,x_0\}\rangle \hookrightarrow T2a \Rightarrow$
$\bigcup\{\{x_0,y_0\}, \{z_0,x_0\}\} = \{x_0,y_0,z_0\}$
EQUAL $\Rightarrow \quad \{y_0,z_0\} \not\subseteq \{x_0,y_0,z_0\}$; Discharge $\Rightarrow$ AUTO
$\langle \{\{x_0,y_0\}, \{y_0,z_0\}\}, \{x_0,y_0\}, \{y_0,z_0\}\rangle \hookrightarrow T2a \Rightarrow$
$\bigcup\{\{x_0,y_0\}, \{y_0,z_0\}\} = \{x_0,y_0,z_0\} \ \& \ e_0 = \{z_0,x_0\} \ \& \ \{\{x_0,y_0\}, \{y_0,z_0\}, \{z_0,x_0\}\} \setminus \{e_0\} = \{\{x_0,y_0\}, \{y_0,z_0\}\}$
EQUAL $\Rightarrow \quad \{z_0,x_0\} \not\subseteq \{x_0,y_0,z_0\}$; Discharge $\Rightarrow$ QED

The following is the basic case of a general theorem scheme where the length of the path can be any number > 1: Replacing an edge by a path with the same endpoints does not disrupt a hank.

THM hank$_3$: [Hank enrichment] $(\mathsf{Hank}(H) \ \& \ \{W,Y\} \in H \ \& \ W \neq Y \ \& \ X \notin \bigcup H \ \& \ H' = H \setminus \{\{W,Y\}\} \cup \{\{W,X\}, \{X,Y\}\}) \rightarrow \mathsf{Hank}(H')$. PROOF:
Suppose_not$(h_0, w_0, y_0, x_1, h_1)$ $\Rightarrow$ $Stat0: \ (\mathsf{Hank}(h_0) \ \& \ \{w_0,y_0\} \in h_0 \ \& \ w_0 \neq y_0 \ \& \ x_1 \notin \bigcup h_0 \ \& \ h_1 = h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\}, \{x_1,y_0\}\}) \ \& \ \neg\mathsf{Hank}(h_1)$

Suppose that the premisses are met by h_0, w_0, y_0, x_1, and h_1. In order to prove $\mathsf{Hank}(h_1)$, we assume it to be false, so that the definition of hank implies the existence of an $e_1 \in h_1$ and a $z_1 \in e_1$ such that $z_1 \notin \bigcup(h_1 \setminus \{e_1\})$.

Use_def(Hank) $\Rightarrow$ $Stat1: \ \neg\langle \forall e \in h_1 \mid e \subseteq \bigcup(h_1 \setminus \{e\})\rangle \ \&$
$Stat2: \ \langle \forall e \in h_0 \mid e \subseteq \bigcup(h_0 \setminus \{e\})\rangle$
$\langle e_1, e_1\rangle \hookrightarrow Stat1 \Rightarrow \quad Stat3: \ e_1 \not\subseteq \bigcup(h_1 \setminus \{e_1\}) \ \& \ e_1 \in h_1 \ \& \ (e_1 \in h_0 \rightarrow e_1 \subseteq \bigcup(h_0 \setminus \{e_1\}))$
Use_def$(\bigcup(h_1 \setminus \{e_1\}))$ $\Rightarrow$ AUTO
$\langle z_1\rangle \hookrightarrow Stat3(Stat3\star) \Rightarrow \quad Stat4: \ z_1 \notin \{v : u \in h_1 \setminus \{e_1\}, v \in u\} \ \& \ z_1 \in e_1$

|| Since x_1 belongs to the distinct edges $\{w_0, x_1\}, \{x_1, y_0\}$ of h_1, clearly $z_1 \neq x_1$.

Suppose ⇒ $z_1 = x_1$
$\langle \{w_0, x_1\}, x_1 \rangle \hookrightarrow$ *Stat4* ⇒ $\{w_0, x_1\} = e_1$
$\langle \{x_1, y_0\}, x_1 \rangle \hookrightarrow$ *Stat4* ⇒ false; Discharge ⇒ Auto

|| Moreover, e_1 cannot be one of the edges of $h_1 \backslash h_0$.

Suppose ⇒ $e_1 \in \{\{w_0, x_1\}, \{x_1, y_0\}\}$
Use_def$(\bigcup(h_0 \backslash \{\{w_0, y_0\}\}))$ ⇒ Auto
$\langle \{w_0, y_0\} \rangle \hookrightarrow$ *Stat2* ⇒ *Stat5* : $z_1 \in \{v : u \in h_0 \backslash \{\{w_0, y_0\}\}, v \in u\}$
$\langle e', z' \rangle \hookrightarrow$ *Stat5* ⇒ $e' \in h_0 \backslash \{\{w_0, y_0\}\}$ & $z_1 \in e'$
Use_def$(\bigcup h_0)$ ⇒ Auto
$\langle e', z_1 \rangle \hookrightarrow$ *Stat4* ⇒ $x_1 \in e'$ & *Stat6* : $x_1 \notin \{v : u \in h_0, v \in u\}$
$\langle e', x_1 \rangle \hookrightarrow$ *Stat6* ⇒ false; Discharge ⇒ Auto
Use_def$(\bigcup(h_0 \backslash \{e_1\}))$ ⇒ Auto

|| We know, at this point, that $e_1 \in h_0 \backslash \{\{w_0, y_0\}\}$. Since h_0 is a hank, z_1 has in h_0 at least one incident edge different from e_1; since the latter is no longer available in $h_1 \backslash \{e_1\}$, it must be $\{w_0, y_0\}$, and either $z_1 = w_0$ or $z_1 = y_0$ hence holds. Both cases lead to a contradiction, though; in fact $\{w_0, x_1\}, \{x_1, y_0\}$ differ from e_1 and these edges of h_1 are, respectively, incident to w_0 and to y_0.

(*Stat0*⋆)ELEM ⇒ $e_1 \in h_0$ &
Stat7 : $z_1 \in \{v : u \in h_0 \backslash \{e_1\}, v \in u\}$ & $z_1 \notin \{v : u \in h_1 \backslash \{e_1\}, v \in u\}$
$\langle e_0, z_0, e_0, z_1 \rangle \hookrightarrow$ *Stat7* ⇒ *Stat8* : $z_1 \in \{w_0, y_0\}$
$\langle \{w_0, x_1\}, w_0 \rangle \hookrightarrow$ *Stat4* ⇒ $z_1 \neq w_0$
$\langle \{x_1, y_0\}, y_0 \rangle \hookrightarrow$ *Stat4* ⇒ $z_1 \neq y_0$
(*Stat8*⋆)Discharge ⇒ Qed

Thm cycle$_0$: [A membership 2-chain and an extra edge make a cycle]
$X \in Y$ & $Y \in Z \to \mathsf{Cycle}(\{\{X, Y\}, \{Y, Z\}, \{Z, X\}\})$. Proof:
Suppose_not(x_0, y_0, z_0) ⇒ Auto
Use_def$(\mathsf{Cycle}(\{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}))$ ⇒ Auto
$\langle x_0, y_0, z_0 \rangle \hookrightarrow$ Thank$_2(\star)$ ⇒ *Stat0* : $\neg\langle \forall d \subseteq \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\} \mid$
Hank(d) & $d \neq \emptyset \to d = \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\} \rangle$
$\langle d \rangle \hookrightarrow$ *Stat0* ⇒ *Stat2* : Hank(d) & $d \neq \emptyset$ &
$d \neq \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}$ & $d \subseteq \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}$
$\langle d, \{y_0, z_0\}, \{z_0, x_0\} \rangle \hookrightarrow$ Thank$_1$(*Stat2*⋆) ⇒ $d \neq \{\{y_0, z_0\}, \{z_0, x_0\}\}$ &
$d \neq \{\{y_0, z_0\}\}$ & $d \neq \{\{z_0, x_0\}\}$
$\langle d, \{x_0, y_0\}, \{z_0, x_0\} \rangle \hookrightarrow$ Thank$_1$(*Stat2*, *Stat2*⋆) ⇒ $d \neq \{\{x_0, y_0\}, \{z_0, x_0\}\}$ &
$d \neq \{\{x_0, y_0\}\}$
$\langle d, \{x_0, y_0\}, \{y_0, z_0\} \rangle \hookrightarrow$ Thank$_1$(*Stat2*, *Stat2*⋆) ⇒ $d \neq \{\{x_0, y_0\}, \{y_0, z_0\}\}$
(*Stat2*⋆)ELEM ⇒ false; Discharge ⇒ Qed

|| The replacement of an edge by a 2-path with the same endpoints does not disrupt a cycle.

Thm cycle$_1$: [Cycle enrichment] Cycle(C) & $\{W, Y\} \in C$ & $W \neq Y$ &
$X \notin \bigcup C$ & $C' = C \backslash \{\{W, Y\}\} \cup \{\{W, X\}, \{X, Y\}\} \to \mathsf{Cycle}(C')$. Proof:
Suppose_not$(h_0, w_0, y_0, x_1, h_1)$ ⇒ Auto

Supposing that h_0, w_0, y_0, x_1, h_1 constitute a counter-example to the claim, observe that $\mathsf{Hank}(h_1)$ must hold; hence we can consider a strictly smaller hank d_1 than h_1. It readily turns out that either $\{w_0, x_1\} \in d_1$ or $\{x_1, y_0\} \in d_1$; for otherwise h_0 would strictly include d_1, since $d_1 \neq h_0$ follows from $\{w_0, y_0\} \in h_0 \setminus h_1$.

$\langle h_0, w_0, y_0, x_1, h_1 \rangle \hookrightarrow Thank_3 \Rightarrow$ AUTO
Use_def(Cycle) $\Rightarrow$ $Stat0:\ \{w_0, y_0\} \in h_0\ \&\ w_0 \neq y_0\ \&$
$x_1 \notin \bigcup h_0\ \&\ h_1 = h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_1\}, \{x_1, y_0\}\}\ \&$
$Stat1:\ \neg\langle \forall d \subseteq h_1 \mid \mathsf{Hank}(d)\ \&\ d \neq \emptyset \rightarrow d = h_1 \rangle\ \&$
$Stat2:\ \langle \forall d \subseteq h_0 \mid \mathsf{Hank}(d)\ \&\ d \neq \emptyset \rightarrow d = h_0 \rangle\ \&\ \mathsf{Hank}(h_0)\ \&\ \mathsf{Hank}(h_1)$
$\langle d_1, d_1 \rangle \hookrightarrow Stat1(Stat0\star) \Rightarrow$ $Stat3:\ d_1 \subseteq h_1\ \&\ d_1 \neq h_1\ \&\ d_1 \neq \emptyset\ \&\ \mathsf{Hank}(d_1)\ \&$
$\neg(\{w_0, x_1\} \notin d_1\ \&\ \{x_1, y_0\} \notin d_1)$
Use_def($\bigcup h_0$) $\Rightarrow$ AUTO
$\langle d_1 \rangle \hookrightarrow Thank_0 \Rightarrow$ $Stat4:\ \langle \forall e \in d_1, x \in e, \exists q \in d_1 \mid q \neq e\ \&\ x \in q \rangle\ \&$
$Stat4a:\ x_1 \notin \{v :\ u \in h_0, v \in u\}\ \&\ \emptyset \notin d_1$

Should one of $\{w_0, x_1\}, \{x_1, y_0\}$, but not the other, belong to d_1, we would easily get a contradiction: the two cases are treated symmetrically. At this point we have derived that both $\{w_0, x_1\}$ and $\{x_1, y_0\}$ belong to d_1.

$\langle \{w_0, x_1\}, x_1, q_0, q_0, x_1 \rangle \hookrightarrow Stat4(Stat0\star) \Rightarrow$ $\neg(\{w_0, x_1\} \in d_1\ \&\ \{x_1, y_0\} \notin d_1)$
$\langle \{x_1, y_0\}, x_1, q_1, q_1, x_1 \rangle \hookrightarrow Stat4(Stat0\star) \Rightarrow$ $Stat5:\ \{w_0, x_1\}, \{x_1, y_0\} \in d_1$

We will show that the set d_0 obtained by replacing $\{w_0, x_1\}$ and $\{x_1, y_0\}$ by $\{w_0, y_0\}$ in d_1 is non-null and is a cycle strictly included in h_0. Obviously $\{w_0, x_1\} \neq \{w_0, y_0\}\ \&\ \{x_1, y_0\} \neq \{w_0, y_0\}$, because x_1, y_0, and w_0 are distinct.

$\langle \{w_0, y_0\}, x_1 \rangle \hookrightarrow Stat4a(Stat0\star) \Rightarrow$ $Stat6:\ x_1 \neq w_0\ \&\ x_1 \neq y_0\ \&\ w_0 \neq y_0\ \&$
$\{w_0, y_0\} \notin d_1$
Loc_def $\Rightarrow$ $Stat7:\ d_0 = d_1 \cup \{\{w_0, y_0\}\} \setminus \{\{w_0, x_1\}, \{x_1, y_0\}\}$
($Stat5, Stat7, Stat6, Stat3, Stat0, Stat4\star$)ELEM $\Rightarrow$ $d_0 \subseteq h_0\ \&\ d_0 \neq \emptyset\ \&$
$d_0 \neq h_0\ \&\ \emptyset \notin d_0$
Use_def($\bigcup d_0$) $\Rightarrow$ AUTO
$\langle d_0, h_0 \rangle \hookrightarrow T2c\ (Stat0\star) \Rightarrow$ $Stat8:\ x_1 \notin \{u :\ v \in d_0, u \in v\}$

Despite us having assumed at the beginning that h_0 contains no proper cycle, so that in particular $\mathsf{Hank}(d_0)$ cannot hold, due to an edge e_0 of d_0 and to an endpoint x_0 of e_0 which is not properly covered in d_0, ...

$\langle d_0 \rangle \hookrightarrow Thank_0 \Rightarrow$ AUTO
$\langle d_0 \rangle \hookrightarrow Stat2(Stat7\star) \Rightarrow$ $Stat9:\ \neg\langle \forall e \in d_0, x \in e, \exists q \in d_0 \mid q \neq e\ \&\ x \in q \rangle$
$\langle e_0, x_0 \rangle \hookrightarrow Stat9 \Rightarrow$
$Stat10:\ \neg\langle \exists q \in d_0 \mid q \neq e_0\ \&\ x_0 \in q \rangle\ \&\ e_0 \in d_0\ \&\ x_0 \in e_0$

... we now aim at showing that this offending edge e_0 of d_0 will also offend d_1, which contradicts a fact noted at the beginning. Here we shortly digress to prove that $e_0 = \{w_0, y_0\}$ must hold, else e_0 would offend d_1.

Suppose $\Rightarrow$ $Stat11:\ e_0 \neq \{w_0, y_0\}$

Indeed, assuming $e_0 \neq \{w_0, y_0\}$, e_0 would also belong to d_1, and each one of its endpoints must have edges distinct from e_0 incident to it in d_1. However, it will turn out that this cannot be the case for the endpoint x_0, which hence is not properly covered in d_1.

$\langle e_0, x_0\rangle \hookrightarrow$ *Stat4(Stat7, Stat10, Stat11⋆)* $\Rightarrow$ *Stat12*: $\langle \exists q \in d_1 \mid q \neq e_0 \,\&\, x_0 \in q\rangle$
$\langle q_2\rangle \hookrightarrow$ *Stat12(Stat12⋆)* $\Rightarrow$ *Stat13*: $x_0 \in q_2 \,\&\, q_2 \neq e_0 \,\&\, q_2 \in d_1$

|| To see this, let $q_2 \neq e_0$ be the edge that covers x_0 in d_1.

Suppose $\Rightarrow$ *Stat14*: $q_2 = \{w_0, x_1\} \vee q_2 = \{x_1, y_0\}$

|| If this edge q_2 were one of the two which have been removed, the edge $\{w_0, y_0\}$ would satisfactorily cover x_0 in d_0.

$\langle \{w_0, y_0\}\rangle \hookrightarrow$ *Stat10(Stat7, Stat6, Stat11⋆)* $\Rightarrow$ *Stat15*: $x_0 \notin \{w_0, y_0\}$
$\langle e_0, x_0\rangle \hookrightarrow$ *Stat8* $\Rightarrow$ AUTO
(*Stat10⋆*)Discharge $\Rightarrow$ *Stat16*: $\neg(q_2 = \{w_0, x_1\} \vee q_2 = \{x_1, y_0\})$

|| q_2 must hence belong to d_0; but again, this implies that q_2 would satisfactorily cover x_0 in d_0. Therefore, $e_0 = \{w_0, y_0\}$ must hold.

$\langle q_2\rangle \hookrightarrow$ *Stat10(Stat7, Stat16, Stat13⋆)* $\Rightarrow$ false;
Discharge $\Rightarrow$ *Stat17*: $e_0 = \{w_0, y_0\}$

|| The only remaining possibility, $e_0 = \{w_0, y_0\}$, is also untenable. Indeed, d_1 has two edges incident to w_0, one of which is $\{w_0, x_1\}$; likewise, d_1 has two edges incident to y_0, one of which is $\{x_1, y_0\}$. For either one of the endpoints w_0, y_0 of e_0, the second incident edge belongs to d_1 and differs from $\{w_0, y_0\}$, so it must belong to d_0 as well; since d_0 also owns the edge $\{w_0, y_0\}$ incident to either endpoint, it is not true that e_0 is an offending edge for d_0, a fact that contradicts one of the assumptions made.

$\langle \{w_0, x_1\}, w_0, q_4\rangle \hookrightarrow$ *Stat4(Stat5, Stat7, Stat6, Stat17⋆)* $\Rightarrow$
Stat19: $q_4 \neq e_0 \,\&\, q_4 \in d_0 \,\&\, w_0 \in q_4$
$\langle \{x_1, y_0\}, y_0, q_5\rangle \hookrightarrow$ *Stat4(Stat5, Stat5⋆)* $\Rightarrow$
Stat20: $y_0 \in q_5 \,\&\, q_5 \neq \{x_1, y_0\} \,\&\, q_5 \in d_1$
(*Stat20, Stat7, Stat6, Stat17⋆*)ELEM $\Rightarrow$ *Stat21*: $q_5 \neq e_0 \,\&\, q_5, e_0 \in d_0$
$\langle q_5\rangle \hookrightarrow$ *Stat10(Stat21, Stat17, Stat20⋆)* $\Rightarrow$ *Stat22*: $x_0 \notin q_5 \,\&\, x_0 = w_0$
$\langle q_4\rangle \hookrightarrow$ *Stat10(Stat19, Stat22⋆)* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

DEF hamiltonian$_1$: [Hamiltonian cycle, in graph devoid of isolated vertices]
$$\mathsf{Hamiltonian}(H, S, E) \leftrightarrow_{\text{Def}} \mathsf{Cycle}(H) \,\&\, \bigcup H = S \,\&\, H \subseteq E$$

|| In our specialized context, where edges are 2-sets whose elements satisfy a peculiar membership constraint, we do not simply require that a Hamiltonian cycle H touches every vertex, but also that every source has an incident membership edge in H.

DEF hamiltonian$_2$: [Edges in squared membership]
$$\mathsf{sqEdges}(S) =_{\text{Def}} \{\{x, y\} : x \in S, y \in S \setminus \{x\}, z \in S \cap x \mid y = z \vee y \in z \vee z \in y\}$$

DEF hamiltonian$_3$: [Restraining condition for Hamiltonian cycles]
$$\mathsf{SqHamiltonian}(H, S) \leftrightarrow_{\text{Def}} \mathsf{Hamiltonian}\big(H, S, \mathsf{sqEdges}(S)\big) \,\&\, \langle \forall x \in S \setminus \bigcup S, \exists y \in x \mid \{x, y\} \in H\rangle$$

THM hamiltonian$_1$: [Enriched Hamiltonian cycles]
$$S = T \cup \{X\} \,\&\, X \notin T \,\&\, Y \in X \,\&\, \mathsf{SqHamiltonian}(H, T) \,\&\, \{W, Y\} \in H \,\&\, \big(W \in Y \vee (Y \in W \,\&\, K \neq Y \,\&\, \{W, K\} \in H \,\&\, K \in W)\big) \rightarrow$$

$\mathsf{SqHamiltonian}(H\backslash\{\{W,Y\}\}\cup\{\{W,X\},\{X,Y\}\},S)$. PROOF:

Suppose_not$(s_0,t_0,x_1,y_0,h_0,w_0,k_0)\Rightarrow$ AUTO

Use_def(SqHamiltonian) $\Rightarrow$ *Stat0*: $\langle\forall x\in t_0\backslash\bigcup t_0,\exists y\in x\mid\{x,y\}\in h_0\rangle$ & $\mathsf{Hamiltonian}(h_0,t_0,\mathsf{sqEdges}(t_0))$ & $\neg\Big(\mathsf{Hamiltonian}(h_0\backslash\{\{w_0,y_0\}\}\cup\{\{w_0,x_1\},\{x_1,y_0\}\},s_0,\mathsf{sqEdges}(s_0))$ & $\langle\forall x\in s_0\backslash\bigcup s_0,\exists y\in x\mid\{x,y\}\in h_0\backslash\{\{w_0,y_0\}\}\cup\{\{w_0,x_1\},\{x_1,y_0\}\}\rangle\Big)$

Suppose $\Rightarrow$ *Stat1*: $\neg\langle\forall x\in s_0\backslash\bigcup s_0,\exists y\in x\mid\{x,y\}\in h_0\backslash\{\{w_0,y_0\}\}\cup\{\{w_0,x_1\},\{x_1,y_0\}\}\rangle$

Suppose that $s_0,t_0,x_1,y_0,h_0,w_0,k_0$ make a counterexample to the claim. One reason why

$$\mathsf{SqHamiltonian}(h_0\backslash\{\{w_0,y_0\}\}\cup\{\{w_0,x_1\},\{x_1,y_0\}\},s_0)$$

is violated might be

$$\neg\langle\forall x\in s_0\backslash\bigcup s_0,\exists y\in x\mid\{x,y\}\in h_0\backslash\{\{w_0,y_0\}\}\cup\{\{w_0,x_1\},\{x_1,y_0\}\}\rangle;$$

if this is the case, we can choose an x' witnessing this fact.

$\langle x'\rangle\hookrightarrow$*Stat1* $\Rightarrow$ *Stat2*: $\neg\langle\exists k\in x'\mid\{x',k\}\in h_0\backslash\{\{w_0,y_0\}\}\cup\{\{w_0,x_1\},\{x_1,y_0\}\}\rangle$ & $x'\in s_0\backslash\bigcup s_0$

To see that $x'\in t_0\backslash\bigcup t_0$ follows from the constraint $x'\in s_0\backslash\bigcup s_0$, we assume the contrary and argue as follows: (1) Unless x' belongs to t_0, we must have $x'=x_1$, which however has an incident membership edge, namely $\{x_1,y_0\}$, in $h_0\backslash\{\{w_0,y_0\}\}\cup\{\{w_0,x_1\},\{x_1,y_0\}\}$. (2) Thus, since $x'\in t_0$, we have $x'\in t_0\cap\bigcup t_0$ and hence $x'\in s_0\cap\bigcup s_0$, contradicting the constraint on x'.

Suppose $\Rightarrow$ $x'\notin t_0\backslash\bigcup t_0$

$\langle y_0\rangle\hookrightarrow$*Stat2*$(\star)\Rightarrow$ $x'\in t_0$ & $s_0=t_0\cup\{x_1\}$ & $x_1\notin t_0$ & $y_0\in x_1$

$\langle t_0,s_0\rangle\hookrightarrow$*T2c* (*Stat2*$\star$) $\Rightarrow$ $\bigcup s_0\supseteq\bigcup t_0$

(*Stat2*$\star$)Discharge $\Rightarrow$ AUTO

Knowing that $x'\in t_0\backslash\bigcup t_0$, we can find a $y_1\in x'$ such that $\{x',y_1\}\in h_0$. Since this membership edge is no longer available after the modification of h_0, it must be $\{w_0,y_0\}$; therefore, $x'=w_0$, for otherwise $x'=y_0$ would (in view of the fact $y_0\in x_1$) contradict the assumption $x'\in s_0\backslash\bigcup s_0$.

$\langle x',y_1\rangle\hookrightarrow$*Stat0*$(\star)\Rightarrow$ *Stat3*: $y_1\in x'$ & $\{x',y_1\}\in h_0$ & $x_1\in s_0$ & $y_0\in x_1$

Use_def$(\bigcup s_0)\Rightarrow$ AUTO

$\langle y_1\rangle\hookrightarrow$*Stat2*(*Stat3*$\star$) $\Rightarrow$ *Stat4*: $x'\notin\{u:v\in s_0,u\in v\}$ & $\{x',y_1\}=\{w_0,y_0\}$

$\langle x_1,y_0\rangle\hookrightarrow$*Stat4*(*Stat2*$\star$) $\Rightarrow$ $x'=w_0$

If $x'\in y_0$, the assumption $x'\in s_0\backslash\bigcup s_0$ would be contradicted similarly: but then, by the initial assumption, we must have $\{x',k_0\}\in h_0$ & $k_0\neq y_0$ & $k_0\in x'$, conflicting with Stat2, because $\{x',k_0\}=\{w_0,y_0\}$ would imply $k_0=y_0$.

$\langle x_1,y_0\rangle\hookrightarrow$*Stat4*$(\star)\Rightarrow$ $x'\notin y_0$ & $\{x',k_0\}\in h_0$ & $k_0\in x'$ & $k_0\neq y_0$

$\langle k_0\rangle\hookrightarrow$*Stat2*(*Stat4*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$

Stat5: $\mathsf{Hamiltonian}(h_0,t_0,\mathsf{sqEdges}(t_0))$ & $\neg\mathsf{Hamiltonian}(h_0\backslash\{\{w_0,y_0\}\}\cup\{\{w_0,x_1\},\{x_1,y_0\}\},s_0,\mathsf{sqEdges}(s_0))$

At this point the reason why

$$\mathsf{SqHamiltonian}(h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\},\{x_1,y_0\}\},s_0)$$

is false must be that

$$\mathsf{Hamiltonian}\big(h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\},\{x_1,y_0\}\},s_0,\mathsf{sqEdges}(s_0)\big)$$

is false; however, we will derive a contradiction also in this case.

Use_def$\Big(\mathsf{Hamiltonian}\big(h_0,t_0,\mathsf{sqEdges}(t_0)\big)\Big) \Rightarrow$ AUTO

ELEM $\Rightarrow$ *Stat6* : $s_0 = t_0 \cup \{x_1\} \ \& \ x_1 \notin t_0 \ \& \ y_0 \in x_1 \ \& \ \{w_0,y_0\} \in h_0 \ \&$ $\big(w_0 \in y_0 \vee (y_0 \in w_0 \ \& \ k_0 \neq y_0 \ \& \ \{w_0,k_0\} \in h_0 \ \& \ k_0 \in w_0)\big)$

Loc_def $\Rightarrow$ *Stat7* : $h_1 = h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\},\{x_1,y_0\}\}$

Use_def$\Big(\mathsf{Hamiltonian}\big(h_1,s_0,\mathsf{sqEdges}(s_0)\big)\Big) \Rightarrow$ AUTO

EQUAL $\langle$*Stat5*$\rangle \Rightarrow$ *Stat8* : $\big(\mathsf{Cycle}(h_0) \ \& \ \bigcup h_0 = t_0 \ \& \ h_0 \subseteq \mathsf{sqEdges}(t_0)\big) \ \&$ $\neg\big(\mathsf{Cycle}(h_1) \ \& \ \bigcup h_1 = s_0 \ \& \ h_1 \subseteq \mathsf{sqEdges}(s_0)\big)$

In fact, after observing that $\{w_0,y_0\} \subseteq \bigcup h_0$ must hold, we will be able to discard one by one each potential reason why $\mathsf{Hamiltonian}(h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\},\{x_1,y_0\}\},s_0,\mathsf{sqEdges}(s_0))$ should be false.

$\langle h_0,w_0,y_0,x_1,h_1\rangle \hookrightarrow T\mathsf{cycle}_1$ (*Stat6*$\star$) $\Rightarrow$ *Stat9* : $\big(\mathsf{Cycle}(h_0) \ \& \ \bigcup h_0 = t_0 \ \&$ $h_0 \subseteq \mathsf{sqEdges}(t_0)\big) \ \& \ \neg\big(\bigcup h_1 = s_0 \ \& \ h_1 \subseteq \mathsf{sqEdges}(s_0)\big)$

Suppose $\Rightarrow$ *Stat10* : $\{w_0,y_0\} \not\subseteq \bigcup h_0$

Use_def$(\bigcup h_0) \Rightarrow$ AUTO

$\langle b\rangle \hookrightarrow$ *Stat10*(*Stat10*$\star$) $\Rightarrow$ *Stat11* : $b \notin \{u : v \in h_0, u \in v\} \ \& \ b \in \{w_0,y_0\}$

$\langle \{w_0,y_0\},b\rangle \hookrightarrow$ *Stat11*(*Stat11*, *Stat6*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$

Stat12 : $w_0,y_0 \in t_0$

Suppose $\Rightarrow$ *Stat13* : $h_1 \not\subseteq \mathsf{sqEdges}(s_0)$

Use_def$(\mathsf{sqEdges}(s_0)) \Rightarrow$ AUTO

$\langle e\rangle \hookrightarrow$ *Stat13*(*Stat7*$\star$) $\Rightarrow$ $(e \in h_0 \vee e = \{w_0,x_1\} \vee e = \{x_1,y_0\}) \ \&$ *Stat14* : $e \notin \{\{x,y\} : x \in s_0, y \in s_0 \setminus \{x\}, z \in s_0 \cap x \mid y = z \vee y \in z \vee z \in y\}$

(*Stat6*, *Stat12*$\star$)ELEM $\Rightarrow$ *Stat15* : $x_1,y_0,w_0 \in s_0 \ \& \ y_0 \in x_1 \ \& \ (w_0 \in y_0 \vee y_0 \in w_0) \ \& \ x_1 \neq w_0$

$\langle x_1,y_0,y_0\rangle \hookrightarrow$ *Stat14*(*Stat15*$\star$) $\Rightarrow$ $e \neq \{x_1,y_0\}$

$\langle x_1,w_0,y_0\rangle \hookrightarrow$ *Stat14*(*Stat15*$\star$) $\Rightarrow$ $e \neq \{w_0,x_1\}$

Use_def$(\mathsf{sqEdges}(t_0)) \Rightarrow$ AUTO

(*Stat8*$\star$)ELEM $\Rightarrow$ *Stat16* : $e \in \{\{x,y\} : x \in t_0, y \in t_0 \setminus \{x\}, z \in t_0 \cap x \mid y = z \vee y \in z \vee z \in y\}$

$\langle x_2,y_2,z_2\rangle \hookrightarrow$ *Stat16*(*Stat16*$\star$) $\Rightarrow$ *Stat17* : $e = \{x_2,y_2\} \ \& \ x_2,y_2,z_2 \in t_0 \ \&$ $x_2 \neq y_2 \ \& \ z_2 \in x_2 \ \& \ (y_2 = z_2 \vee y_2 \in z_2 \vee z_2 \in y_2)$

(*Stat6*$\star$)ELEM $\Rightarrow$ $s_0 \supseteq t_0$

$\langle x_2,y_2,z_2\rangle \hookrightarrow$ *Stat14*(*Stat17*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ *Stat18* : $\bigcup h_1 \neq s_0$

We prove first that $\bigcup h_1 \subseteq s_0$.

$\langle \{\{w_0,x_1\},\{x_1,y_0\}\},\{w_0,x_1\},\{x_1,y_0\}\rangle \hookrightarrow T2a$(*Stat18*$\star$) $\Rightarrow$ $\bigcup\{\{w_0,x_1\},\{x_1,y_0\}\} = \{w_0,x_1,y_0\}$

$\langle h_0 \setminus \{\{w_0,y_0\}\},\{\{w_0,x_1\},\{x_1,y_0\}\}\rangle \hookrightarrow T2c$(*Stat18*$\star$) $\Rightarrow$ $\bigcup(h_0 \setminus \{\{w_0,y_0\}\} \cup \{\{w_0,x_1\},\{x_1,y_0\}\})$ $= \bigcup(h_0 \setminus \{\{w_0,y_0\}\}) \cup \bigcup\{\{w_0,x_1\},\{x_1,y_0\}\}$

$\langle h_0 \setminus \{\{w_0,y_0\}\},h_0\rangle \hookrightarrow T2c$(*Stat8*$\star$) $\Rightarrow$

$\bigcup h_0 \supseteq \bigcup(h_0 \setminus \{\{w_0, y_0\}\})$ & $\{w_0, y_0\} \subseteq \bigcup h_0$
EQUAL ⟨*Stat7*⟩ ⇒ $\bigcup h_1 = \bigcup(h_0 \setminus \{\{w_0, y_0\}\}) \cup \{w_0, x_1, y_0\}$
(*Stat6*⋆)ELEM ⇒ $\bigcup h_1 \subseteq s_0$

‖ The remaining case is $s_0 \not\subseteq \bigcup h_1$, which entails that we can find an $a \in t_0 \setminus \bigcup h_1$.

Use_def($\bigcup h_1$) ⇒ AUTO
⟨a⟩↪*Stat18*(*Stat18*⋆) ⇒ *Stat23* : $a \notin \{u : v \in h_1, u \in v\}$ & $a \in s_0 \setminus \{x_1\}$

‖ Since $a \in t_0$ and $t_0 = \bigcup h_0$, we can find an $e' \in h_0$ such that $a \in e'$.

Use_def($\bigcup$) ⇒ *Stat24* : $a \in \{u : v \in h_0, u \in v\}$
⟨e′, u′⟩↪*Stat24*(*Stat25*⋆) ⇒ *Stat25* : $e' \in h_0$ & $a \in e'$

‖ Since $h_1 = h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_1\}, \{x_1, y_0\}\}$, we conclude that $e' = \{w_0, y_0\}$ must hold. Hence, either $a = w_0$ or $a = y_0$ must hold, both of which yield a contradiction.

⟨e′, a⟩↪*Stat23*(*Stat7*, *Stat25*⋆) ⇒ $e' = \{w_0, y_0\}$
⟨{w₀, x₁}, w₀⟩↪*Stat23*(*Stat7*⋆) ⇒ $a = y_0$
⟨{x₁, y₀}, y₀⟩↪*Stat23*(*Stat7*⋆) ⇒ false; Discharge ⇒ QED

THM hamiltonian$_2$: [Doubly enriched Hamiltonian cycles]
$S = T \cup \{X, Z\}$ & $\{X, Z\} \cap T = \emptyset$ & $X \neq Z$ & $Y \in X \cap Z$ &
SqHamiltonian(H, T) & $\{W, Y\} \in H$ & $W \in Y \cap X \rightarrow$
SqHamiltonian($H \setminus \{\{W, Y\}\} \cup \{\{W, X\}, \{X, Z\}, \{Z, Y\}\}, S$). PROOF:
Suppose_not($s_0, t_0, x_0, z_0, y_0, h_0, w_0$) ⇒ AUTO
⟨$t_0 \cup \{x_0\}, t_0, x_0, y_0, h_0, w_0, \emptyset$⟩↪*T*hamiltonian$_1$ ⇒
SqHamiltonian($h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_0\}, \{x_0, y_0\}\}, t_0 \cup \{x_0\}$)
Loc_def ⇒ *Stat1* :
$h_1 = h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_0\}, \{x_0, y_0\}\}$ & $t_1 = t_0 \cup \{x_0\}$
ELEM ⇒ *Stat2* : $s_0 = t_1 \cup \{z_0\}$ & $x_0 \notin t_0$
EQUAL ⇒ SqHamiltonian(h_1, t_1)
Suppose ⇒ $\{x_0, y_0\} \in h_0$

‖ Notice that since h_0 is a Hamiltonian path in t_0, its unionset must equal t_0; since x_0 does not belong to t_0, but it belongs to $\{x_0, y_0\}$, it follows that $\{x_0, y_0\}$ cannot belong to h_0.

Use_def(SqHamiltonian(h_0, t_0)) ⇒ AUTO
Use_def(Hamiltonian(h_0, t_0, sqEdges(t_0))) ⇒ AUTO
Use_def($\bigcup$) ⇒ *Stat3* : $x_0 \notin \{u : v \in h_0, u \in v\}$
⟨{x₀, y₀}, x₀⟩↪*Stat3*(*Stat2*⋆) ⇒ false; Discharge ⇒ AUTO
ELEM ⇒ $\{x_0, y_0\} \neq \{w_0, x_0\}$ & $z_0 \notin t_1$ &
$y_0 \in z_0$ & $y_0 \in x_0$ & $w_0 \in y_0$ & $w_0 \in x_0$
⟨$t_1 \cup \{z_0\}, t_1, z_0, y_0, h_1, x_0, w_0$⟩↪*T*hamiltonian$_1$ (*Stat1*⋆) ⇒
SqHamiltonian($h_1 \setminus \{\{x_0, y_0\}\} \cup \{\{x_0, z_0\}, \{z_0, y_0\}\}, t_1 \cup \{z_0\}$)
(*Stat1*⋆)ELEM ⇒ $h_1 \setminus \{\{x_0, y_0\}\} = h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_0\}\}$
(*Stat1*⋆)ELEM ⇒ $h_1 \setminus \{\{x_0, y_0\}\} \cup \{\{x_0, z_0\}, \{z_0, y_0\}\}$
$= h_0 \setminus \{\{w_0, y_0\}\} \cup \{\{w_0, x_0\}, \{x_0, z_0\}, \{z_0, y_0\}\}$
EQUAL ⇒ false; Discharge ⇒ QED

THM hamiltonian$_3$: [Trivial Hamiltonian cycles]

$S = \{X, Y, Z\} \ \& \ X \in Y \ \& \ Y \in Z \rightarrow$
$\quad \mathsf{SqHamiltonian}(\{\{X, Y\}, \{Y, Z\}, \{Z, X\}\}, S)$. PROOF:

$\mathsf{Suppose\_not}(s, x_0, y_0, z_0) \Rightarrow$ AUTO

> Arguing by contradiction, assume that $\{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}$ is not a 'square Hamiltonian' cycle for $s = \{x_0, y_0, z_0\}$, where $x_0 \in y_0$ and $y_0 \in z_0$ holds. We will first exclude the possibility that $\{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}$ is not a Hamiltonian cycle in the 'square edges' of s; after discarding this, we will also exclude that this cycle may fail to satisfy the restraining condition that it has a genuine membership edge incident into each source of s.

$\mathsf{Use\_def}(\mathsf{SqHamiltonian}(\{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}, s)) \Rightarrow$ AUTO

$\mathsf{Use\_def}\Big(\mathsf{Hamiltonian}(\{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}, s, \mathsf{sqEdges}(s))\Big) \Rightarrow$ AUTO

$\langle x_0, y_0, z_0 \rangle \hookrightarrow T\mathsf{cycle}_0 \Rightarrow$ AUTO

$\mathsf{ELEM} \Rightarrow \quad Stat1: \ s = \{x_0, y_0, z_0\} \ \& \ x_0 \in y_0 \ \& \ y_0 \in z_0$

$\mathsf{Suppose} \Rightarrow \quad Stat8: \ \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\} \not\subseteq \mathsf{sqEdges}(s)$

$\quad \mathsf{Use\_def}(\mathsf{sqEdges}(s)) \Rightarrow$ AUTO

$\quad \langle e_0 \rangle \hookrightarrow Stat8(Stat8\star) \Rightarrow \quad Stat9:$

$\quad\quad e_0 \notin \{\{x, y\} : x \in s, y \in s \setminus \{x\}, z \in s \cap x \mid y = z \vee y \in z \vee z \in y\} \ \&$

$\quad\quad e_0 \in \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}$

$\quad \langle z_0, y_0, y_0 \rangle \hookrightarrow Stat9(Stat1, Stat1\star) \Rightarrow \quad e_0 \neq \{y_0, z_0\}$

$\quad \langle z_0, x_0, y_0 \rangle \hookrightarrow Stat9(Stat1, Stat9\star) \Rightarrow \quad e_0 \neq \{z_0, x_0\}$

$\quad \langle y_0, x_0, x_0 \rangle \hookrightarrow Stat9(Stat1, Stat1\star) \Rightarrow \quad e_0 \neq \{x_0, y_0\}$

$(Stat9\star)\mathsf{Discharge} \Rightarrow$ AUTO

$\mathsf{Suppose} \Rightarrow \quad Stat4: \ \bigcup \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\} \neq s$

$\quad (Stat1, Stat1\star)\mathsf{ELEM} \Rightarrow \quad s = \{x_0, y_0, z_0\} \ \&$

$\quad\quad \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\} = \{\{x_0, y_0\}, \{y_0, z_0\}\} \cup \{\{z_0, x_0\}\} \ \&$

$\quad\quad \{x_0, y_0, z_0\} \cup \{z_0, x_0\} = \{x_0, y_0, z_0\}$

$\quad \langle \{\{x_0, y_0\}, \{y_0, z_0\}\}, \{\{z_0, x_0\}\} \rangle \hookrightarrow T2c \ (Stat5\star) \Rightarrow \quad Stat5:$

$\quad\quad \bigcup(\{\{x_0, y_0\}, \{y_0, z_0\}\} \cup \{\{z_0, x_0\}\})$

$\quad\quad = \bigcup \{\{x_0, y_0\}, \{y_0, z_0\}\} \cup \bigcup \{\{z_0, x_0\}\}$

$\quad \langle \{\{x_0, y_0\}, \{y_0, z_0\}\}, \{x_0, y_0\}, \{y_0, z_0\} \rangle \hookrightarrow T2a \ (Stat6\star) \Rightarrow \quad Stat6:$

$\quad\quad \bigcup \{\{x_0, y_0\}, \{y_0, z_0\}\} = \{x_0, y_0, z_0\}$

$\quad \langle \{\{z_0, x_0\}, \{z_0, x_0\}\}, \{z_0, x_0\}, \{z_0, x_0\} \rangle \hookrightarrow T2a \ (Stat7\star) \Rightarrow \quad Stat7:$

$\quad\quad \bigcup \{\{z_0, x_0\}, \{z_0, x_0\}\} = \{z_0, x_0\} \ \& \ \{\{z_0, x_0\}, \{z_0, x_0\}\} = \{\{z_0, x_0\}\}$

$\mathsf{EQUAL} \ \langle Stat4 \rangle \Rightarrow \quad \mathsf{false}; \qquad \mathsf{Discharge} \Rightarrow \quad Stat10:$

$\quad\quad \neg \langle \forall z \in s \setminus \bigcup s, \exists y \in z \mid \{z, y\} \in \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\} \rangle$

> We conclude by checking that $\{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}$ owns a genuine membership edge incident into each source of s; as a matter of fact, z_0 is the only source of s and $\{y_0, z_0\}$ is a membership edge.

$\langle z' \rangle \hookrightarrow Stat10(Stat10\star) \Rightarrow \quad Stat11:$

$\quad\quad \neg \langle \exists y \in z' \mid \{z', y\} \in \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\} \rangle \ \& \ z' \in s \setminus \bigcup s$

$\langle y_0 \rangle \hookrightarrow Stat11(Stat11\star) \Rightarrow \quad Stat12:$

$\quad\quad y_0 \notin z' \vee \{z', y_0\} \notin \{\{x_0, y_0\}, \{y_0, z_0\}, \{z_0, x_0\}\}$

$\mathsf{Use\_def}(\bigcup) \Rightarrow \quad Stat13: \ z' \notin \{u : v \in s, u \in v\} \ \& \ z' \in s$

$\langle z_0, y_0 \rangle \hookrightarrow Stat13(Stat12, Stat1\star) \Rightarrow \quad Stat14: \ z' = x_0$

$\langle y_0, x_0\rangle \hookrightarrow Stat13(Stat14, Stat13, Stat12, Stat1\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

Any non-trivial transitive set whose square is devoid of Hamiltonian cycles must strictly comprise certain sets.

THM hamiltonian$_4$: [Potential revealers of non-Hamiltonicity]

$$\mathsf{Trans}(S) \,\&\, S \not\subseteq \{\emptyset,\{\emptyset\}\} \,\&\, \neg\langle\exists h \mid \mathsf{SqHamiltonian}(h,S)\rangle \rightarrow S \neq \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\} \,\&\, S \neq \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\} \,\&\, S \neq \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\} \,\&\, S \supseteq \{\emptyset,\{\emptyset\}\} \,\&\, \big(\{\{\emptyset\}\} \in S \vee \{\emptyset,\{\emptyset\}\} \in S\big).$$

PROOF:

Suppose_not(t) $\Rightarrow$ AUTO

Indeed, any set satisfying the premises of our present claim must, due to its transitivity and non-triviality, include either one of the Hamiltonian cycles endowed with the vertices $\emptyset,\{\emptyset\},\{\{\emptyset\}\}$ and $\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}$, respectively; but it must also own additional elements, else the last premise would be falsified. Moreover, it cannot have exactly the elements $\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}$, as these form a Hamiltonian cycle.

$\langle t,\{\emptyset,\{\emptyset\}\}\rangle \hookrightarrow T4b \Rightarrow$ $Stat1:\ \neg\langle\exists h \mid \mathsf{SqHamiltonian}(h,t)\rangle \,\&\, (t \supseteq \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\} \vee t \supseteq \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\})$

The cases $t = \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}$ and $t = \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$ must be excluded, because we have the respective Hamiltonian cycles

$$\{\{\emptyset,\{\emptyset\}\},\{\{\emptyset\},\{\{\emptyset\}\}\},\{\{\{\emptyset\}\},\emptyset\}\},$$
$$\{\{\emptyset,\{\emptyset\}\},\{\{\emptyset\},\{\emptyset,\{\emptyset\}\}\},\{\{\emptyset,\{\emptyset\}\},\emptyset\}\}.$$

$\langle t,\emptyset,\{\emptyset\},\{\{\emptyset\}\}\rangle \hookrightarrow T\mathsf{hamiltonian}_3 \Rightarrow$ AUTO

$\langle \{\{\emptyset,\{\emptyset\}\},\{\{\emptyset\},\{\{\emptyset\}\}\},\{\{\{\emptyset\}\},\emptyset\}\}\rangle \hookrightarrow Stat1(\star) \Rightarrow$ $t \neq \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}$

$\langle t,\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\rangle \hookrightarrow T\mathsf{hamiltonian}_3 \Rightarrow$ AUTO

$\langle \{\{\emptyset,\{\emptyset\}\},\{\{\emptyset\},\{\emptyset,\{\emptyset\}\}\},\{\{\emptyset,\{\emptyset\}\},\emptyset\}\}\rangle \hookrightarrow Stat1(\star) \Rightarrow$

$Stat2:\ t = \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}$

Having thus established that $t = \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\}$, we can now exploit Theorem hamiltonian$_2$ to enrich the Hamiltonian cycle for $\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\}$ into one which does to our case.

$\langle \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\},\emptyset,\{\emptyset\},\{\{\emptyset\}\}\rangle \hookrightarrow T\mathsf{hamiltonian}_3(Stat2\star) \Rightarrow$

$\mathsf{SqHamiltonian}(\{\{\emptyset,\{\emptyset\}\},\{\{\emptyset\},\{\{\emptyset\}\}\},\{\{\{\emptyset\}\},\emptyset\}\},\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\})$

$\langle \{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\},\emptyset,\{\emptyset\},\{\{\emptyset\}\}\rangle \hookrightarrow T\mathsf{hamiltonian}_3 \Rightarrow$

$\mathsf{SqHamiltonian}(\{\{\emptyset,\{\emptyset\}\},\{\{\emptyset\},\{\{\emptyset\}\}\},\{\{\{\emptyset\}\},\emptyset\}\},\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\})$

$\langle \{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\},\{\emptyset,\{\emptyset\},\{\{\emptyset\}\}\},\{\emptyset,\{\emptyset\}\},\{\emptyset\},$
$\{\{\emptyset,\{\emptyset\}\},\{\{\emptyset\},\{\{\emptyset\}\}\},\{\{\{\emptyset\}\},\emptyset\}\},\emptyset\rangle \hookrightarrow T\mathsf{hamiltonian}_1(Stat2\star) \Rightarrow$

$\mathsf{SqHamiltonian}(\{\{\emptyset,\{\emptyset\}\},\{\{\emptyset\},\{\{\emptyset\}\}\},\{\{\{\emptyset\}\},\emptyset\}\} \setminus \{\{\emptyset,\{\emptyset\}\}\} \cup \{\{\emptyset,\{\emptyset,\{\emptyset\}\}\},\{\{\emptyset,\{\emptyset\}\},\{\emptyset\}\}\},$
$\{\emptyset,\{\emptyset\},\{\{\emptyset\}\},\{\emptyset,\{\emptyset\}\}\})$

EQUAL $\langle Stat2\rangle \Rightarrow$

$\mathsf{SqHamiltonian}(\{\{\emptyset,\{\emptyset\}\},\{\{\emptyset\},\{\{\emptyset\}\}\},\{\{\{\emptyset\}\},\emptyset\}\} \setminus \{\{\emptyset,\{\emptyset\}\}\} \cup \{\{\emptyset,\{\emptyset,\{\emptyset\}\}\},\{\{\emptyset,\{\emptyset\}\},\{\emptyset\}\}\},t)$

$\langle \{\{\emptyset,\{\emptyset\}\},\{\{\emptyset\},\{\{\emptyset\}\}\},\{\{\{\emptyset\}\},\emptyset\}\} \setminus \{\{\emptyset,\{\emptyset\}\}\} \cup \{\{\emptyset,\{\emptyset,\{\emptyset\}\}\},\{\{\emptyset,\{\emptyset\}\},\{\emptyset\}\}\}\rangle$
$\hookrightarrow Stat1(Stat2\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

B.7 Hamiltonicity of squared claw-free sets

|| Non-trivial, claw-free, finite transitive sets have Hamiltonian squares.

THM clawFreeness$_1$: [Hamiltonicity of non-trivial, claw-free sets]
$\mathsf{Finite}(S)\ \&\ \mathsf{Trans}(S)\ \&\ \mathsf{ClawFree}(S)\ \&\ S \not\subseteq \{\emptyset, \{\emptyset\}\} \rightarrow$
$\langle \exists h \mid \mathsf{SqHamiltonian}(h, S)\rangle$. PROOF:
Suppose_not(s_1) ⇒ AUTO

For, assuming the opposite, there would exist an inclusion-minimal, finite transitive non-trivial claw-free set whose square lacks a Hamiltonian cycle.

APPLY $\langle \mathsf{fin}_\Theta : s_0\rangle$ finiteInduction$\big(s_0 \mapsto s_1, P(S) \mapsto (\mathsf{Trans}(S)\ \&\ \mathsf{ClawFree}(S)\ \&$
$S \not\subseteq \{\emptyset, \{\emptyset\}\}\ \&\ \neg\langle \exists h \mid \mathsf{SqHamiltonian}(h, S)\rangle)\big)$ ⇒
Stat1 : $\langle \forall s \mid s \subseteq s_0 \rightarrow \mathsf{Finite}(s)\ \&\ \big(\mathsf{Trans}(s)\ \&\ \mathsf{ClawFree}(s)\ \&$
$s \not\subseteq \{\emptyset, \{\emptyset\}\}\ \&\ \neg\langle \exists h \mid \mathsf{SqHamiltonian}(h, s)\rangle \leftrightarrow s = s_0\big)\rangle$
$\langle s_0\rangle \hookrightarrow$ *Stat1(Stat1⋆)* ⇒ *Stat2* : $\neg\langle \exists h \mid \mathsf{SqHamiltonian}(h, s_0)\rangle\ \&\ \mathsf{Finite}(s_0)\ \&$
$\mathsf{Trans}(s_0)\ \&\ \mathsf{ClawFree}(s_0)\ \&\ s_0 \not\subseteq \{\emptyset, \{\emptyset\}\}$

Thanks to the finiteness of such an s_0, the THEORY pivotsForClawFreeness can be applied to s_0. We thereby pick an element x from the frontier of s_0, and an element y of x which is pivotal relative to s_0. This y will have at most two in-neighbors (one of the two being x) in s_0. We denote by z an in-neighbor of y in s_0, such that z differs from x if possible. Observe, among others, that neither one of x, z can belong to the other.

APPLY $\langle x_\Theta : x, y_\Theta : y, z_\Theta : z, t_\Theta : t\rangle$ pivotsForClawFreeness$(s_0 \mapsto s_0)$ ⇒
Stat3 : $\{v \in s_0 \mid y \in v\} = \{x, z\}\ \&\ z \in s_0\ \&\ y \in z\ \&\ y \in x\ \&\ y, x \in s_0\ \&$
$y \notin \bigcup\bigcup s_0\ \&\ t = \{u \in s_0 \mid y \notin u\}\ \&\ \mathsf{Trans}(t)\ \&\ \mathsf{ClawFree}(t)\ \&$
$s_0 \supseteq t\ \&\ x \notin t\ \&\ y \in t \setminus \bigcup t\ \&\ t = s_0 \setminus \{x, z\}\ \&\ x \notin z\ \&\ z \notin x$

Thus it turns out readily that removal of x, z from s_0 leads to a set t to which, unless t is 'trivial' (i.e. a subset of $\{\emptyset, \{\emptyset\}\}$), the inductive hypothesis applies; hence, by that hypothesis, there is a Hamiltonian cycle h_0 for t.

Suppose ⇒ $t \subseteq \{\emptyset, \{\emptyset\}\}$

In order that t be trivial, we should have $s_0 \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$; but then, as already shown in the proof of Theorem hamiltonian$_4$, we have the ability, either directly, or by enrichment of a Hamiltonian cycle for $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$, to construct a Hamiltonian cycle for s_0: thus, if we suppose $t \subseteq \{\emptyset, \{\emptyset\}\}$ then we get a contradiction.

$\langle s_0, x, z\rangle \hookrightarrow$ *T3d(Stat2⋆)* ⇒ *Stat7* : $s_0 \subseteq \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
$\langle s_0\rangle \hookrightarrow$ Thamiltonian$_4$*(Stat2, Stat7⋆)* ⇒ false; Discharge ⇒ AUTO
$\langle t\rangle \hookrightarrow$ *Stat1(Stat3⋆)* ⇒ *Stat9* : $\langle \exists h \mid \mathsf{SqHamiltonian}(h, t)\rangle\ \&\ t \not\subseteq \{\emptyset, \{\emptyset\}\}$
$\langle h_0\rangle \hookrightarrow$ *Stat9(Stat9⋆)* ⇒ *Stat10* : $\mathsf{SqHamiltonian}(h_0, t)$
Use_def$\big(\mathsf{Hamiltonian}(h_0, t, \mathsf{sqEdges}(t))\big)$ ⇒ AUTO
Use_def(SqHamiltonian) ⇒
Stat11 : $\langle \forall x \in t \setminus \bigcup t, \exists y \in x \mid \{x, y\} \in h_0\rangle\ \&\ \mathsf{Cycle}(h_0)\ \&$
$\bigcup h_0 = t\ \&\ h_0 \subseteq \mathsf{sqEdges}(t)$

It follows from y being a source of $t = \bigcup h_0$ that there is an edge $\{y, w\}$, with $w \in y$, in h_0, If $x = z$, to get a Hamiltonian cycle h_1 for s_0 (a fact conflicting with the inductive hypothesis) it will suffice to take $h_1 = h_0 \setminus \{\{y, w\}\} \cup \{\{x, y\}, \{x, w\}\}$, where $\{x, w\}$ is a square edge because $w \in y$ and $y \in x$ both hold. On the other hand, if $x \neq z$, claw-freeness implies that either $w \in x$ or $w \in z$. Assume the former for definiteness, and put $h_2 = h_0 \setminus \{\{y, w\}\} \cup \{\{y, z\}, \{z, x\}, \{x, w\}\}$, where $\{x, z\}$ is a square edge and $\{x, w\}$ and $\{y, z\}$ are genuine edges incident in the sources x, z. We are again facing a contradiction, because h_2 is a Hamiltonian cycle for s_0.

$\langle y, w\rangle \hookrightarrow$ *Stat11(Stat3⋆)* $\Rightarrow$ *Stat12* : $w \in y \;\&\; \{w, y\} \in h_0$

Suppose $\Rightarrow$ $x = z$

$\langle s_0, t, x, y, h_0, w, \emptyset\rangle \hookrightarrow T$hamiltonian$_1$*(Stat3⋆)* $\Rightarrow$

$\mathsf{SqHamiltonian}(h_0 \setminus \{\{w, y\}\} \cup \{\{w, x\}, \{x, y\}\}, s_0)$

$\langle h_0 \setminus \{\{w, y\}\} \cup \{\{w, x\}, \{x, y\}\}\rangle \hookrightarrow$ *Stat2(Stat2⋆)* $\Rightarrow$ false;

Discharge $\Rightarrow$ *Stat13* : $x \neq z$

*(Stat2, Stat3⋆)*ELEM $\Rightarrow$ *Stat14* : $\mathsf{ClawFree}(s_0) \;\&\; y \in x \cap z \;\&\; s_0 \supseteq \{y, x, z\}$

$\langle s_0, y\rangle \hookrightarrow$ *T3c(Stat2, Stat3, Stat12⋆)* $\Rightarrow$ $w \in s_0$

$\langle s_0, y, x, z, w\rangle \hookrightarrow T$clawFreeness$_b$*(Stat3⋆)* $\Rightarrow$ $w \in x \cup z$

Suppose $\Rightarrow$ *Stat15* : $w \in x$

$\langle s_0, t, x, z, y, h_0, w\rangle \hookrightarrow T$hamiltonian$_2$*(Stat3,Stat13,Stat14,Stat10,Stat12,Stat15⋆)* $\Rightarrow$

$\mathsf{SqHamiltonian}(h_0 \setminus \{\{w, y\}\} \cup \{\{w, x\}, \{x, z\}, \{z, y\}\}, s_0)$

$\langle h_0 \setminus \{\{w, y\}\} \cup \{\{w, x\}, \{x, z\}, \{z, y\}\}\rangle \hookrightarrow$ *Stat2(Stat2⋆)* $\Rightarrow$ false;

Discharge $\Rightarrow$ *Stat16* : $w \in z$

$\langle s_0, t, z, x, y, h_0, w\rangle \hookrightarrow T$hamiltonian$_2$*(Stat3,Stat13,Stat14,Stat10,Stat12,Stat16⋆)* $\Rightarrow$

$\mathsf{SqHamiltonian}(h_0 \setminus \{\{w, y\}\} \cup \{\{w, z\}, \{z, x\}, \{x, y\}\}, s_0)$

$\langle h_0 \setminus \{\{w, y\}\} \cup \{\{w, z\}, \{z, x\}, \{x, y\}\}\rangle \hookrightarrow$ *Stat2(Stat2⋆)* $\Rightarrow$ false;

Discharge $\Rightarrow$ QED

B.8 Perfect matchings

Next we introduce the notion of *perfect matching*. This is a partition consisting of doubletons one of whose elements belongs to the other. Special cases of a perfect matching are: the empty set and, more generally, all subsets of a perfect matching.

DEF perfect_matching: [Set of disjoint membership pairs]

$\mathsf{PerfectMatching}(M) \quad \leftrightarrow_{\text{Def}}$

$\langle \forall p \in M, \exists x \in p, y \in x, \forall q \in M \mid x \in q \vee y \in q \rightarrow \{x, y\} = q\rangle$

THM perfectMatching$_0$: [The null set is a perfect matching]

$\mathsf{PerfectMatching}(\emptyset)$. PROOF:

Suppose_not() $\Rightarrow$ AUTO

Use_def(PerfectMatching) $\Rightarrow$ *Stat0* : $\neg\langle \forall p \in \emptyset, \exists x \in p, y \in x, \forall q \in \emptyset \mid$

$x \in q \vee y \in q \rightarrow \{x, y\} = q\rangle$

$\langle p_1\rangle \hookrightarrow$ *Stat0* $\Rightarrow$ false; Discharge $\Rightarrow$ QED

THM perfectMatching$_1$: [Perfect matchings consist of true doubletons]

$\mathsf{PerfectMatching}(M) \;\&\; P \in M \rightarrow P \notin \{\emptyset, \{X\}\}$. PROOF:

$\mathsf{Suppose\_not}(m, p_0, x_0) \Rightarrow$ AUTO
$\mathsf{Use\_def}(\mathsf{PerfectMatching}) \Rightarrow$ *Stat1* : $\langle \forall p \in m, \exists x \in p, y \in x, \forall q \in m \mid x \in q \vee y \in q \rightarrow \{x, y\} = q\rangle$
$\langle p_0, x, y, p_0\rangle \hookrightarrow$ *Stat1*$(\star) \Rightarrow$ false; Discharge $\Rightarrow$ QED

THM perfectMatching$_2$: [All subsets of a perfect matching are perfect]
$\mathsf{PerfectMatching}(M) \;\&\; M \supseteq N \rightarrow \mathsf{PerfectMatching}(N)$. PROOF:
$\mathsf{Suppose\_not}(m, n) \Rightarrow$ AUTO
$\mathsf{Set\_monot} \Rightarrow$
$\langle \forall p \in m, \exists x \in p, y \in x, \forall q \in m \mid x \in q \vee y \in q \rightarrow \{x, y\} = q\rangle \rightarrow \langle \forall p \in n, \exists x \in p, y \in x, \forall q \in n \mid x \in q \vee y \in q \rightarrow \{x, y\} = q\rangle$
$\mathsf{Use\_def}(\mathsf{PerfectMatching}) \Rightarrow$ false; Discharge $\Rightarrow$ QED

> By adjoining a pair $\{x, y\}$ with $y \in x$ to a perfect matching none of whose blocks has either x or y as an element, we always obtain a perfect matching.

THM perfectMatching$_3$: [Bottom-up assembly of a perfect matching]
$\mathsf{PerfectMatching}(M) \;\&\; X \notin \bigcup M \;\&\; Y \notin \bigcup M \;\&\; Y \in X \rightarrow \mathsf{PerfectMatching}(M \cup \{\{X, Y\}\})$. PROOF:
$\mathsf{Suppose\_not}(m, x_0, y_0) \Rightarrow$ *Stat2* : $\mathsf{PerfectMatching}(m) \;\&\; x_0 \notin \bigcup m \;\&\; y_0 \notin \bigcup m \;\&\; y_0 \in x_0 \;\&\; \neg\mathsf{PerfectMatching}(m \cup \{\{x_0, y_0\}\})$
Suppose $\Rightarrow$ *Stat3* : $\neg\langle \forall q \in m \mid x_0 \notin q \;\&\; y_0 \notin q\rangle$
$\mathsf{Use\_def}(\bigcup) \Rightarrow$ *Stat4* : $x_0 \notin \{v : u \in m, v \in u\} \;\&\; y_0 \notin \{v : u \in m, v \in u\}$
$\langle q_2\rangle \hookrightarrow$ *Stat3*(*Stat3*$\star) \Rightarrow$ $q_2 \in m \;\&\; x_0 \in q_2 \vee y_0 \in q_2$
$\langle q_2, x_0, q_2, y_0\rangle \hookrightarrow$ *Stat4*(*Stat4*$\star) \Rightarrow$ false; Discharge $\Rightarrow$
Stat5 : $\langle \forall q \in m \mid x_0 \notin q \;\&\; y_0 \notin q\rangle$
$\mathsf{Use\_def}(\mathsf{PerfectMatching}) \Rightarrow$
Stat6 : $\neg\langle \forall p \in m \cup \{\{x_0, y_0\}\}, \exists x \in p, y \in x, \forall q \in m \cup \{\{x_0, y_0\}\} \mid x \in q \vee y \in q \rightarrow \{x, y\} = q\rangle \;\&$
Stat7 : $\langle \forall p \in m, \exists x \in p, y \in x, \forall q \in m \mid x \in q \vee y \in q \rightarrow \{x, y\} = q\rangle$
$\langle p_0\rangle \hookrightarrow$ *Stat6*(*Stat6*$\star) \Rightarrow$
Stat8 : $\neg\langle \exists x \in p_0, y \in x, \forall q \in m \cup \{\{x_0, y_0\}\} \mid x \in q \vee y \in q \rightarrow \{x, y\} = q\rangle \;\&\; p_0 \in m \cup \{\{x_0, y_0\}\}$
Suppose $\Rightarrow$ *Stat9* : $p_0 = \{x_0, y_0\}$
$\langle x_0, y_0\rangle \hookrightarrow$ *Stat8*(*Stat2*, *Stat9*$\star) \Rightarrow$
Stat10 : $\neg\langle \forall q \in m \cup \{\{x_0, y_0\}\} \mid x_0 \in q \vee y_0 \in q \rightarrow \{x_0, y_0\} = q\rangle$
$\langle q_1\rangle \hookrightarrow$ *Stat9*(*Stat9*$\star) \Rightarrow$ $q_1 \in m \;\&\; x_0 \in q_1 \vee y_0 \in q_1$
$\langle q_1\rangle \hookrightarrow$ *Stat5*(*Stat10*$\star) \Rightarrow$ false; Discharge $\Rightarrow$ AUTO
$\langle p_0, x_1, y_1\rangle \hookrightarrow$ *Stat7* $\Rightarrow$ AUTO
$\langle x_1, y_1\rangle \hookrightarrow$ *Stat8*(*Stat8*$\star) \Rightarrow$
Stat13 : $\neg\langle \forall q \in m \cup \{\{x_0, y_0\}\} \mid x_1 \in q \vee y_1 \in q \rightarrow \{x_1, y_1\} = q\rangle \;\&$
Stat12 : $\langle \forall q \in m \mid x_1 \in q \vee y_1 \in q \rightarrow \{x_1, y_1\} = q\rangle \;\&\; p_0 \in m \;\&\; x_1 \in p_0 \;\&\; y_1 \in x_1$
$\langle q_0, q_0\rangle \hookrightarrow$ *Stat13*(*Stat13*$\star) \Rightarrow$ *Stat14* : $q_0 = \{x_0, y_0\} \;\&\; x_1 \in q_0 \vee y_1 \in q_0$
$\langle p_0\rangle \hookrightarrow$ *Stat5*$(\star) \Rightarrow$ $x_0 \notin p_0 \;\&\; y_0 \notin p_0$

$\langle p_0 \rangle \hookrightarrow Stat12(Stat13, Stat13\star) \Rightarrow \quad \{x_1, y_1\} = p_0$

$(Stat14\star)$Discharge $\Rightarrow$ QED

If, in a perfect matching m, we replace one block $\{y, w\}$ by pairs $\{y, z\}, \{x, w\}$, then, under suitable conditions ensuring disjointness between blocks and membership within each block, we get a perfect matching again.

THM perfectMatching$_4$: [Deviated perfect matching]

$\mathsf{PerfectMatching}(M) \;\&\; \{Y, W\} \in M \;\&\; X \notin \bigcup M \;\&\; Z \notin \bigcup M \;\&\; Y \in Z \;\&$
$\quad Y \neq X \;\&\; X \neq Z \;\&\; W \in X \rightarrow$
$\qquad \mathsf{PerfectMatching}(M \setminus \{\{Y, W\}\} \cup \{\{Y, Z\}, \{X, W\}\})$. PROOF:

Suppose_not$(m, y_0, w_0, x_0, z_0) \Rightarrow$ AUTO

For assuming that m, y_0, w_0, x_0, z_0 are a counterexample to the claim, we could get a contradiction arguing as follows. Begin by observing that neither y_0 nor w_0 can belong to the unionset of the perfect submatching $m \setminus \{\{y_0, w_0\}\}$ of m.

Suppose $\Rightarrow \quad Stat1: \{y_0, w_0\} \cap \bigcup(m \setminus \{\{y_0, w_0\}\}) \neq \emptyset$

$\quad$ Use_def(PerfectMatching) $\Rightarrow \quad Stat2:$

$\qquad \langle \forall p \in m, \exists x \in p, y \in x, \forall q \in m \mid x \in q \lor y \in q \rightarrow \{x, y\} = q \rangle$

$\quad$ Use_def$(\bigcup(m \setminus \{\{y_0, w_0\}\})) \Rightarrow$ AUTO

$\quad \langle w_1 \rangle \hookrightarrow Stat1 \Rightarrow \quad Stat3:$

$\qquad w_1 \in \{u : v \in m \setminus \{\{y_0, w_0\}\}, u \in v\} \;\&\; w_1 \in \{y_0, w_0\}$

$\quad \langle p_0, w_2 \rangle \hookrightarrow Stat3 \Rightarrow \quad p_0 \in m \setminus \{\{y_0, w_0\}\} \;\&\; w_1 \in p_0$

$\quad \langle p_0, x_2, y_2 \rangle \hookrightarrow Stat2 \Rightarrow \quad Stat4:$

$\qquad \langle \forall q \in m \mid x_2 \in q \lor y_2 \in q \rightarrow \{x_2, y_2\} = q \rangle \;\&\; x_2 \in p_0 \;\&\; y_2 \in x_2$

$\quad \langle p_0 \rangle \hookrightarrow Stat4 \Rightarrow \quad p_0 = \{x_2, y_2\}$

$\langle \{y_0, w_0\} \rangle \hookrightarrow Stat4(\star) \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow$ AUTO

$\langle m, m \setminus \{\{y_0, w_0\}\} \rangle \hookrightarrow T$perfectMatching$_2 \Rightarrow \quad \mathsf{PerfectMatching}(m \setminus \{\{y_0, w_0\}\})$

Thus, taking into account that $w_0 \in x_0$ and that $x_0 \notin \bigcup m$ which is a superset of $\bigcup(m \setminus \{\{y_0, w_0\}\})$, we can extend the perfect matching $m \setminus \{\{y_0, w_0\}\}$ with the doubleton $\{x_0, w_0\}$.

$\langle m \setminus \{\{y_0, w_0\}\}, m \rangle \hookrightarrow T2c \Rightarrow \quad x_0 \notin \bigcup(m \setminus \{\{y_0, w_0\}\})$

$\langle m \setminus \{\{y_0, w_0\}\}, x_0, w_0 \rangle \hookrightarrow T$perfectMatching$_3 \Rightarrow$

$\qquad \mathsf{PerfectMatching}(m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\})$

Observe next that $x_0 \neq y_0$ and $z_0 \neq w_0$, because $x_0 \notin \bigcup m$ and $z_0 \notin \bigcup m$ whereas $y_0 \in \bigcup m$ and $w_0 \in \bigcup m$. It then follows from $z_0 \neq w_0$, thanks to the assumption $z_0 \neq x_0$, that z_0 does not belong to the unionset of the perfect matching $m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\}$.

Suppose $\Rightarrow \quad x_0 = w_0 \lor z_0 = w_0$

$\quad$ Use_def$(\bigcup) \Rightarrow \quad Stat5: z_0 \notin \{u : v \in m, u \in v\} \;\&\; x_0 \notin \{u : v \in m, u \in v\}$

$\langle \{y_0, w_0\}, w_0, \{y_0, w_0\}, y_0 \rangle \hookrightarrow Stat5 \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow$ AUTO

Suppose $\Rightarrow \quad z_0 \in \bigcup(m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\})$

$\quad \langle m \setminus \{\{y_0, w_0\}\}, m \rangle \hookrightarrow T2c \Rightarrow \quad z_0 \notin \bigcup(m \setminus \{\{y_0, w_0\}\})$

$\langle m \setminus \{\{y_0, w_0\}\}, \{x_0, w_0\} \rangle \hookrightarrow T2e \Rightarrow \quad$ false; $\quad$ Discharge $\Rightarrow$ AUTO

y_0 cannot equal w_0 either, the reason being that the set $\{y_0, w_0\}$ is a block of a perfect matching and hence it cannot be a singleton. If follows, thanks to the assumption $y_0 \in x_0$ (entailing that $y_0 \neq x_0$), that y_0 does not belong to $\bigcup(m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\})$ either.

$\langle m, \{y_0, w_0\}, w_0 \rangle \hookrightarrow T\mathsf{perfectMatching}_1 \ \Rightarrow \quad y_0 \neq w_0$
$\langle m \setminus \{\{y_0, w_0\}\}, \{x_0, w_0\} \rangle \hookrightarrow T2e \ \Rightarrow \quad y_0 \notin \bigcup(m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\})$

We now know that the perfect matching $m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\}$ can be extended with the doubleton $\{y_0, z_0\}$, which readily leads us to the sought contradiction.

$\langle m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\}, z_0, y_0 \rangle \hookrightarrow T\mathsf{perfectMatching}_3 \ \Rightarrow$
$\quad \mathsf{PerfectMatching}(m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\} \cup \{\{y_0, z_0\}\})\ \&$
$\quad\quad m \setminus \{\{y_0, w_0\}\} \cup \{\{x_0, w_0\}\} \cup \{\{y_0, z_0\}\} =$
$\quad\quad\quad m \setminus \{\{y_0, w_0\}\} \cup \{\{y_0, z_0\}, \{x_0, w_0\}\}$
EQUAL $\Rightarrow$ false; Discharge $\Rightarrow$ QED

B.9 Each claw-free set admits a near-perfect matching

Every claw-free finite, transitive set admits a perfect matching perhaps omitting one of its elements.

THM clawFreeness$_2$: [Claw-free sets admit near-perfect matchings]
$\mathsf{Finite}(S)\ \&\ \mathsf{Trans}(S)\ \&\ \mathsf{ClawFree}(S) \rightarrow$
$\quad \langle \exists m, y \mid \mathsf{PerfectMatching}(m)\ \&\ S \setminus \{y\} = \bigcup m \rangle$. PROOF:
Suppose_not$(s_1) \Rightarrow$ AUTO

For, supposing the contrary, there would be an inclusion-minimal finite set s_0 which is transitive and claw-free, and such that no perfect matching m partitions the set s_0 possibly deprived of an element y_0.

APPLY $\langle \mathsf{fin}_\Theta : s_0 \rangle$ finiteInduction$\big(s_0 \mapsto s_1, P(S) \mapsto (\mathsf{Trans}(S)\ \&\ \mathsf{ClawFree}(S)\ \&$
$\quad \neg\langle \exists m, y \mid \mathsf{PerfectMatching}(m)\ \&\ S \setminus \{y\} = \bigcup m \rangle)\big) \Rightarrow$
$\quad Stat1: \langle \forall s \mid s \subseteq s_0 \rightarrow \mathsf{Finite}(s)\ \&\ \big(\mathsf{Trans}(s)\ \&\ \mathsf{ClawFree}(s)\ \&$
$\quad\quad \neg\langle \exists m, y \mid \mathsf{PerfectMatching}(m)\ \&\ s \setminus \{y\} = \bigcup m \rangle \leftrightarrow s = s_0)\rangle$
$\langle s_0 \rangle \hookrightarrow Stat1(Stat1\star) \Rightarrow$
$\quad Stat2: \neg\langle \exists m, y \mid \mathsf{PerfectMatching}(m)\ \&\ s_0 \setminus \{y\} = \bigcup m \rangle\ \&$
$\quad\quad \mathsf{Trans}(s_0)\ \&\ \mathsf{ClawFree}(s_0)\ \&\ \mathsf{Finite}(s_0)$

We observe that such an s_0 cannot equal $\emptyset$ or $\{\emptyset\}$, else the null perfect matching would cover it.

Suppose $\Rightarrow$ $Stat3: s_0 \cap \bigcup s_0 = \emptyset$
$\quad \langle s_0 \rangle \hookrightarrow T3a \Rightarrow$ AUTO
$\quad \langle s_0 \rangle \hookrightarrow T31d \Rightarrow$ AUTO
$\quad \langle \emptyset \rangle \hookrightarrow T31d \Rightarrow$ AUTO
$\quad \langle \emptyset, \emptyset \rangle \hookrightarrow Stat2 \Rightarrow \neg\mathsf{PerfectMatching}(\emptyset)$
$\langle\ \rangle \hookrightarrow T\mathsf{perfectMatching}_0(Stat3\star) \Rightarrow$ false; Discharge $\Rightarrow$ $s_0 \cap \bigcup s_0 \neq \emptyset$

Thanks to the finiteness of s_0, the THEORY pivotsForClawFreeness can be applied to s_0. We thereby pick an element x from the frontier of s_0 and an element y of x which is pivotal relative to s_0. This y will have at most two $\in$-predecessors (one of the two being x) in s_0. We denote by z a predecessor of y in s_0, such that z differs from x if possible.

APPLY $\langle x_\Theta : x, y_\Theta : y, z_\Theta : z, t_\Theta : t'\rangle$ pivotsForClawFreeness$(s_0 \mapsto s_0) \Rightarrow$
$$Stat4: \{v \in s_0 \mid y \in v\} = \{x,z\} \;\&\; z \in s_0 \;\&\; y \in z \;\&\; y \in x \;\&\; y, x \in s_0 \;\&\; y \notin \bigcup\bigcup s_0 \;\&\; t' = \{z \in s_0 \mid y \notin z\} \;\&\; \mathsf{Trans}(t') \;\&\; \mathsf{ClawFree}(t') \;\&\; s_0 \supseteq t' \;\&\; x \notin t' \;\&\; y \in t' \setminus \bigcup t' \;\&\; t' = s_0 \setminus \{x,z\} \;\&\; x \notin z \;\&\; z \notin x$$

Moreover, we take t' to be s_0 deprived of the predecessors of y and, if $x \neq z$, we take $t = t'$ else we take $t = t' \setminus \{y\}$.

Loc_def $\Rightarrow$ $t = \textbf{if } x = z \textbf{ then } t' \setminus \{y\} \textbf{ else } t' \textbf{ fi}$

Thus it turns out readily that t is transitive; hence, by the inductive hypothesis, there is a perfect matching m_0 for t.

$\langle t', t\rangle \hookrightarrow T\mathsf{clawFreeness}_a(Stat4\star) \Rightarrow$ $Stat5: \mathsf{ClawFree}(t) \;\&\; x \notin t \;\&\; s_0 \supseteq t \;\&\; t' \supseteq t \;\&\; t' = s_0 \setminus \{x,z\} \;\&\; y \in t'$
$\langle t', t\rangle \hookrightarrow T4c(Stat4\star) \Rightarrow$ $\mathsf{Trans}(t)$
$\langle t\rangle \hookrightarrow Stat1(Stat4\star) \Rightarrow$ $Stat6: \langle \exists m, y \mid \mathsf{PerfectMatching}(m) \;\&\; t \setminus \{y\} = \bigcup m\rangle$
$\langle m_0, y_0\rangle \hookrightarrow Stat6(Stat6\star) \Rightarrow$ $Stat7: \mathsf{PerfectMatching}(m_0) \;\&\; \bigcup m_0 = t \setminus \{y_0\}$

The possibility that y does not belong to $\bigcup m_0$ is then discarded; in fact, if this were the case, then by adding the pair $\{x,y\}$ to m_0 we would get a perfect matching for s_0, while we have assumed that such a matching does not exist. From the fact $y \in \bigcup m_0$ it follows that y belongs to t, hence that $t = t'$ and that x, z are distinct.

Suppose $\Rightarrow$ $Stat8: y \notin \bigcup m_0$
$\langle m_0, x, y\rangle \hookrightarrow T\mathsf{perfectMatching}_3(Stat4\star) \Rightarrow$
$Stat9: \mathsf{PerfectMatching}(m_0 \cup \{\{x,y\}\})$
Loc_def $\Rightarrow$ $Stat10: v = \textbf{if } x = z \textbf{ then } y \textbf{ else } z \textbf{ fi}$
$(Stat4\star)$ELEM $\Rightarrow$ $Stat11: s_0 = t \cup \{x, v\} \;\&\; \{x\} \cup \{y\} = \{x,y\}$
$(Stat5, Stat11, Stat8, Stat7\star)$ELEM $\Rightarrow$ $y = v \vee y = y_0$
$\langle m_0, t, y_0, s_0, \{x\}, v, y\rangle \hookrightarrow T31h(Stat4\star) \Rightarrow$
$Stat12: \langle \exists d \mid \bigcup(m_0 \cup \{\{x\} \cup \{y\}\}) = s_0 \setminus \{d\}\rangle$
$\langle d_0\rangle \hookrightarrow Stat12(Stat12\star) \Rightarrow$ $\bigcup(m_0 \cup \{\{x\} \cup \{y\}\}) = s_0 \setminus \{d_0\}$
EQUAL $\langle Stat11\rangle \Rightarrow$ $Stat13: \bigcup(m_0 \cup \{\{x,y\}\}) = s_0 \setminus \{d_0\}$
$\langle m_0 \cup \{\{x,y\}\}, d_0\rangle \hookrightarrow Stat2(Stat9, Stat13\star) \Rightarrow$ false; Discharge $\Rightarrow$
$Stat14: y \in \bigcup m_0$
Use_def$(\bigcup) \Rightarrow$ $Stat15: y \in \{h : p \in m_0, h \in p\} \;\&\; y \in x \cap z \;\&\; x \neq z \;\&\; t = t' \;\&\; x \notin z \;\&\; z \notin x \;\&\; \bigcup m_0 = \{h : p \in m_0, h \in p\}$

It also follows that y is the tail of that arc p_1 of m_0 to which it belongs. In fact, if y were instead the head of p_1, then the tail x_2 of p_1, which must belong to $\bigcup m_0$ would belong to $s_0 \setminus \{x,z\}$, hence would be inside s_0 but outside the set of predecessors of y, which is absurd.

Suppose $\Rightarrow$ $Stat16: \neg\langle \exists w \mid \{y,w\} \in m_0 \;\&\; w \in y\rangle$
$\langle p_1, h_1\rangle \hookrightarrow Stat15(Stat16\star) \Rightarrow$ $p_1 \in m_0 \;\&\; y \in p_1$

Use_def(PerfectMatching) $\Rightarrow$ *Stat17* : $\langle \forall p \in m_0, \exists x \in p, y \in x, \forall q \in m_0 \mid x \in q \vee y \in q \rightarrow \{x, y\} = q \rangle$

$\langle p_1, x_2, w_2, p_1 \rangle \hookrightarrow$ *Stat17*(*Stat16*$\star$) $\Rightarrow$ $x_2 \in p_1 \;\&\; w_2 \in x_2 \;\&\; p_1 \in m_0 \;\&\; \{x_2, w_2\} = p_1$

$\langle w_2 \rangle \hookrightarrow$ *Stat16*(*Stat16*$\star$) $\Rightarrow$ *Stat18* : $y \in x_2 \;\&\; \{y, x_2\} \in m_0$

Suppose $\Rightarrow$ *Stat19* : $x_2 \notin \{u : v \in m_0, u \in v\}$

$\langle \{y, x_2\}, x_2 \rangle \hookrightarrow$ *Stat19*(*Stat18*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

EQUAL $\langle$*Stat4*$\rangle$ $\Rightarrow$ *Stat20* : $\{v \in s_0 \mid y \in v\} = \{x, z\} \;\&\; x_2 \in \bigcup m_0$

(*Stat7*, *Stat5*, *Stat20*$\star$)ELEM $\Rightarrow$ *Stat21* : $x_2 \notin \{v \in s_0 \mid y \in v\} \;\&\; x_2 \in s_0$

$\langle x_2 \rangle \hookrightarrow$ *Stat21*(*Stat18*, *Stat21*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$

Stat22 : $\langle \exists w \mid \{y, w\} \in m_0 \;\&\; w \in y \rangle$

> Call w the head of the arc issuing from y in m_0. Then y, x, z, w form a potential claw; this implies, since s_0 is claw-free that either $w \in x$ or $w \in z$.

$\langle w \rangle \hookrightarrow$ *Stat22*(*Stat22*$\star$) $\Rightarrow$ *Stat23* : $w \in y \;\&\; \{y, w\} \in m_0$

$\langle s_0, y \rangle \hookrightarrow$ *T3c*(*Stat2*, *Stat4*, *Stat23*, *Stat4*, *Stat5*, *Stat15*$\star$) $\Rightarrow$

Stat24 : $w, y, x, z \in s_0$

(*Stat2*, *Stat7*, *Stat15*$\star$)ELEM $\Rightarrow$

ClawFree(s_0) & PerfectMatching(m_0) & $y \in x \cap z$

$\langle s_0, y, x, z, w \rangle \hookrightarrow$ *T*clawFreeness$_b$(*Stat15*$\star$) $\Rightarrow$ $w \in x \cup z$

> Obviously, $w \in \bigcup m_0$. Moreover, through elementary reasoning we derive $\bigcup m_0 \cup \{z, x\} = s_0 \setminus \{y_1\}$, where y_1 lies outside s_0 if both x and z has been covered by the matching m_0, otherwise y_1 equals the one of x, z (which might be the same set) left over by m_0.

Suppose $\Rightarrow$ *Stat25* : $w \notin \{u : v \in m_0, u \in v\}$

$\langle \{y, w\}, w \rangle \hookrightarrow$ *Stat25*(*Stat23*, *Stat23*$\star$) $\Rightarrow$ false; Discharge $\Rightarrow$ AUTO

(*Stat7*, *Stat5*$\star$)ELEM $\Rightarrow$ $x \notin \bigcup m_0 \;\&\; z \notin \bigcup m_0$

Loc_def $\Rightarrow$ *Stat26* : $y_1 = \textbf{if } y_0 \in \{x, z\} \textbf{ then } s_0 \textbf{ else } y_0 \textbf{ fi}$

(*Stat24*, *Stat26*$\star$)ELEM $\Rightarrow$ $s_0 \setminus \{x, z, y_0\} \cup \{z, x\} = s_0 \setminus \{y_1\}$

(*Stat7*, *Stat15*, *Stat5*$\star$)ELEM $\Rightarrow$ $\bigcup m_0 \cup \{z, x\} = s_0 \setminus \{x, z, y_0\} \cup \{z, x\}$

EQUAL $\langle$*Stat15*$\rangle$ $\Rightarrow$ *Stat27* : $\bigcup m_0 \cup \{z, x\} = s_0 \setminus \{y_1\} \;\&\; w \in \bigcup m_0$

> If there is an edge between w and x, then we deviate the perfect matching by replacing $\{y, w\}$ by $\{y, z\}$ and $\{x, w\}$; otherwise we replace $\{y, w\}$ by $\{y, x\}$ and $\{z, w\}$. Plainly we get a perfect matching for s_0 in either case, which leads us to the desired contradiction.

Suppose $\Rightarrow$ $w \in x$

$\langle m_0, y, w, x, z \rangle \hookrightarrow$ *T*perfectMatching$_4$(*Stat15*$\star$) $\Rightarrow$

Stat28 : PerfectMatching$(m_0 \setminus \{\{y, w\}\} \cup \{\{y, z\}, \{x, w\}\})$

$\langle m_0, \{y, w\}, \{y, z\}, \{x, w\}, \{z, x\} \rangle \hookrightarrow$ *T31f*(*Stat15*, *Stat27*$\star$) $\Rightarrow$

$\bigcup(m_0 \setminus \{\{y, w\}\} \cup \{\{y, z\}, \{x, w\}\}) = s_0 \setminus \{y_1\}$

$\langle m_0 \setminus \{\{y, w\}\} \cup \{\{y, z\}, \{x, w\}\}, y_1 \rangle \hookrightarrow$ *Stat2*(*Stat28*$\star$) $\Rightarrow$ false

Discharge $\Rightarrow$ AUTO

$\langle m_0, y, w, z, x \rangle \hookrightarrow$ *T*perfectMatching$_4$(*Stat15*$\star$) $\Rightarrow$

Stat29 : PerfectMatching$(m_0 \setminus \{\{y, w\}\} \cup \{\{y, x\}, \{z, w\}\})$

$\langle m_0, \{y, w\}, \{y, x\}, \{z, w\}, \{z, x\} \rangle \hookrightarrow$ *T31f*(*Stat15*, *Stat27*$\star$) $\Rightarrow$

$\bigcup(m_0 \setminus \{\{y, w\}\} \cup \{\{y, x\}, \{z, w\}\}) = s_0 \setminus \{y_1\}$

$\langle m_0 \setminus \{\{y,w\}\} \cup \{\{y,x\},\{z,w\}\}, y_1\rangle \hookrightarrow$ *Stat2*(*Stat29*⋆) ⇒ false;
Discharge ⇒ QED

B.10 From membership digraphs to general graphs

Let us now place the results presented so far under the more general perspective that motivates this work. We display in this section the interfaces of two THEORYs (not developed formally with Ref, as of today), explaining why we can work with membership as a convenient surrogate for the edge relationship of general graphs.

One of these, THEORY finGraphRepr, will implement the proof that any finite graph (v_0, e_0) is 'isomorphic', via a suitable orientation of its edges and an injection ϱ_Θ of v_0 onto a set ν_Θ, to a digraph $(\nu_\Theta, \{(x,y) : x \in \nu_\Theta,\ y \in x \cap \nu_\Theta\})$ that enjoys the *weak extensionality* property: "distinct non-sink vertices have different out-neighborhoods".

Although accessory, the weak extensionality condition is the clue for getting the desired isomorphism; in fact, for any weakly extensional digraph, acyclicity always ensures that a variant of *Mostowski's collapse* is well-defined: in order to get it, one starts by assigning a distinct set Mt to each sink t and then proceeds by putting recursively

$$Mw \quad = \quad \{\, Mu : (w,u) \text{ is an arc} \,\}$$

for all non-sink vertices w; plainly, injectivity of the function $u \mapsto Mu$ can be ensured globally by a suitable choice of the images Mt of the sinks t.

DISPLAY finGraphRepr

THEORY finGraphRepr(v_0, e_0)
  $\mathsf{Finite}(v_0)\ \&\ e_0 \subseteq \{\{x,y\} : x,y \in v_0 \mid x \neq y\}$
⇒ $(\varrho_\Theta,\ \nu_\Theta)$
  $1\text{–}1(\varrho_\Theta)\ \&\ \mathbf{domain}(\varrho_\Theta) = v_0\ \&\ \mathbf{range}(\varrho_\Theta) = \nu_\Theta$
  $\langle \forall x \in v_0, y \in v_0 \mid \{x,y\} \in e_0 \leftrightarrow \varrho_\Theta{\restriction}x \in \varrho_\Theta{\restriction}y \vee \varrho_\Theta{\restriction}y \in \varrho_\Theta{\restriction}x\rangle$
  $\{x \in \nu_\Theta \mid x \cap \nu_\Theta \neq \emptyset\} \subseteq \mathcal{P}(\nu_\Theta)$
END finGraphRepr

The other THEORY, cfGraphRepr, will specialize finGraphRepr to the case of a *connected, claw-free* (undirected, finite) graph—connectedness and claw-freeness are specified, respectively, by the second and by the third assumption of this THEORY . For these graphs, we can insist that the orientation be so imposed as to ensure *extensionality* in full: "distinct vertices have different out-neighborhoods". Consequently, the following will hold:

- there is a unique sink, $\emptyset$; moreover,
- the set ν_Θ of vertices underlying the image digraph is transitive. And, trivially,
- ν_Θ is a claw-free set, in an even stronger sense than the definition with which we have been working throughout this proof scenario.

(As regards the third of these points, it should be clear that none of the four non-isomorphic membership renderings of a claw are induced by any quadruple of elements of ν_Θ; our definition forbade only two of them, though!)

DISPLAY cfGraphRepr

THEORY cfGraphRepr$(\mathsf{v}_0, \mathsf{e}_0)$

$\mathsf{Finite}(\mathsf{v}_0)$ & $\mathsf{e}_0 \subseteq \{\{\mathsf{x},\mathsf{y}\} : \mathsf{x},\mathsf{y} \in \mathsf{v}_0 \mid \mathsf{x} \neq \mathsf{y}\}$

$\langle \forall \mathsf{x} \in \mathsf{v}_0, \mathsf{y} \in \mathsf{v}_0 \mid \mathsf{x} \neq \mathsf{y}$ & $\{\mathsf{x},\mathsf{y}\} \notin \mathsf{e}_0 \rightarrow \langle \exists \mathsf{p} \subseteq \mathsf{e}_0 \mid \mathsf{Cycle}(\mathsf{p} \cup \{\{\mathsf{y},\mathsf{x}\}\})\rangle\rangle$

$\langle \forall \mathsf{w} \in \mathsf{v}_0, \mathsf{x} \in \mathsf{v}_0, \mathsf{y} \in \mathsf{v}_0, \mathsf{z} \in \mathsf{v}_0 \mid \{\mathsf{w},\mathsf{y}\}, \{\mathsf{y},\mathsf{x}\}, \{\mathsf{y},\mathsf{z}\} \in \mathsf{e}_0 \rightarrow$

$\mathsf{x} = \mathsf{z} \vee \mathsf{w} \in \{\mathsf{z},\mathsf{x}\} \vee \{\mathsf{x},\mathsf{z}\} \in \mathsf{e}_0 \vee \{\mathsf{z},\mathsf{w}\} \in \mathsf{e}_0 \vee \{\mathsf{w},\mathsf{x}\} \in \mathsf{e}_0\rangle$

$\Rightarrow (\rho_\Theta, \nu_\Theta)$

$1\text{–}1(\rho_\Theta)$ & $\mathbf{domain}(\rho_\Theta) = \mathsf{v}_0$ & $\mathbf{range}(\rho_\Theta) = \nu_\Theta$

$\langle \forall \mathsf{x} \in \mathsf{v}_0, \mathsf{y} \in \mathsf{v}_0 \mid \{\mathsf{x},\mathsf{y}\} \in \mathsf{e}_0 \leftrightarrow \rho_\Theta\restriction\mathsf{x} \in \rho_\Theta\restriction\mathsf{y} \vee \rho_\Theta\restriction\mathsf{y} \in \rho_\Theta\restriction\mathsf{x}\rangle$

$\mathsf{Trans}(\nu_\Theta)$ & $\mathsf{ClawFree}(\nu_\Theta)$

END cfGraphRepr

Via the THEORY cfGraphRepr, the above-proved existence results about perfect matchings and Hamiltonian cycles can be transferred from a realm of special membership digraphs to the *a priori* more general realm of the connected claw-free graphs.

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