582206 Models of Computation (Autumn 2009)
Exercise 1 (8–11 September)

This set of problems is a brief recap of the main prerequisites from courses *Introduction to Discrete Mathematics* and *Data Structures*. Chapter 0 of Sipser’s book gives a good summary of this material.

1. Consider $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, which are subsets of the set of integers $\mathbb{Z}$. We use $\overline{C}$ to denote the complement of $C$:

   $\overline{C} = \mathbb{Z} - C = \{x \in \mathbb{Z} | x \notin C\}$.

   What are the elements of the set
   
   (a) $(A \cap B) \cup (A \cap \overline{B})$?
   
   (b) $A \cap \overline{B}$?

2. Prove by induction that $n^3 - n$ is divisible by three for all natural numbers $n$.

3. Consider a directed graph $G = (V, E)$, where

   $$
   V = \{a, b, c, d, e\},
   $$

   $$
   E = \{(a, b), (b, c), (c, a), (b, d), (d, e), (e, b), (c, d)\}.
   $$

   Apply breadth-first search to determine the shortest paths from node $a$ to all other nodes.

4. Recall that a relation $\sim$ in set $X$ is an *equivalence relation*, if for all $x$, $y$ and $z$ the following conditions hold:

   - reflexivity: $x \sim x$
   - symmetricity: if $x \sim y$, then $y \sim x$
   - transitivity: if $x \sim y$ and $y \sim z$, then $x \sim z$.

   (a) Let $G = (V, E)$ be an undirected graph. We write $u \sim v$ to denote that there is path from node $u$ to node $v$ in the graph. Is $\sim$ an equivalence relation in $V$? Justify your answer briefly.

   (b) Let $G = (V, E)$ be a directed graph. We write $u \sim v$ to denote that there is path from node $u$ to node $v$ in the graph. Is $\sim$ an equivalence relation in $V$? Justify your answer briefly.

5. [Sipser Problem 0.12] Prove that if an undirected graph has at least two nodes, then it has two nodes that have the same degree (ie number of adjacent edges).