582206 Models of Computation (Autumn 2009)
Exercise 1 (8–11 September)

This set of problems is a brief recap of the main prerequisites from courses *Introduction to Discrete Mathematics* and *Data Structures*. Chapter 0 of Sipser's book gives a good summary of this material.

1. Consider $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, which are subsets of the set of integers $\mathbb{Z}$. We use $\overline{C}$ to denote the complement of C:

$$\begin{aligned}\overline{C} &= \mathbb{Z} - C \\ &= \{x \in \mathbb{Z} \mid x \notin C\}.\end{aligned}$$

   What are the elements of the set

   (a) $(\overline{A} \cap B) \cup (A \cap \overline{B})$?

   (b) $\overline{\overline{A} \cap \overline{B}}$?

2. Prove by induction that $n^3 - n$ is divisible by three for all natural numbers n.

3. Consider a directed graph $G = (V, E)$, where

$$\begin{aligned}V &= \{a, b, c, d, e\} \\ E &= \{(a, b), (b, c), (c, a), (b, d), (d, e), (e, b), (c, d)\}.\end{aligned}$$

   Apply breadth-first search to determine the shortest paths from node a to all other nodes.

4. Recall that a relation $\sim$ in set X is an *equivalence relation*, if for all x, y and z the following conditions hold:

   reflexivity: $x \sim x$
   symmetricity: if $x \sim y$, then $y \sim x$
   transitivity: if $x \sim y$ and $y \sim z$, then $x \sim z$.

   (a) Let $G = (V, E)$ be an *undirected* graph. We write $u \leadsto v$ to denote that there is path from node u to node v in the graph. Is $\leadsto$ an equivalence relation in V? Justify your aswer briefly.

   (b) Let $G = (V, E)$ be a *directed* graph. We write $u \leadsto v$ to denote that there is path from node u to node v in the graph. Is $\leadsto$ an equivalence relation in V? Justify your aswer briefly.

5. [Sipser Problem 0.12] Prove that if an undirected graph has at least two nodes, then it has two nodes that have the same degree (ie number of adjacent edges).