Let us analyze the average case time complexity of the verification phase.

- The best pattern partitioning is as even as possible. Then each pattern factor has length at least \( r = \lfloor m/(k+1) \rfloor \).
- The expected number of exact occurrences of a random string of length \( r \) in a random text of length \( n \) is at most \( n/\sigma^r \).
- The expected total verification time is at most
  \[
  O \left( \frac{n^2 (k+1)}{\sigma} \right) \leq O \left( \frac{m+k}{\sigma} \right). 
  \]
  This is \( O(n) \) if \( r \geq 3 \log_2 m \).
- The condition \( r \geq 3 \log_2 m \) is satisfied when \( k + 1 \leq m/(3 \log_2 m + 1) \).

**Theorem 3.24:** The average case time complexity of the Baeza-Yates–Perleberg algorithm is \( O(n) \) when \( k \leq m/(3 \log_2 m + 1) - 1 \).

### 4. Suffix Trees and Arrays

Let \( T = T[0..n] \) be the text. For \( i \in [0..n] \), let \( T_i \) denote the suffix \( T[i..n] \). Furthermore, for any subset \( C \subseteq [0..n] \), we write \( T_C = \{ T_i \mid i \in C \} \). In particular, \( T_{[0..n]} \) is the set of all suffixes of \( T \).

**Suffix tree and suffix array** are search data structures for the set \( T_{[0..n]} \).

- **Suffix tree** is a compact trie for \( T_{[0..n]} \).
- **Suffix array** is an ordered array for \( T_{[0..n]} \).

They support fast exact string matching on \( T \):

- A pattern \( P \) has an occurrence starting at position \( i \) if and only if \( P \) is a prefix of \( T_i \).
- Thus we can find all occurrences of \( P \) by a prefix search in \( T_{[0..n]} \).

A data structure supporting fast string matching is called a **text index**. There are numerous other applications too, as we will see later.

#### Suffix Tree

The suffix tree of a text \( T \) is the compact trie of the set \( T_{[0..n]} \) of all suffixes of \( T \).

We assume that there is an extra character \( \$ \notin \Sigma \) at the end of the text. That is, \( T[n] = \$ \) and \( T_i = T[i..n] \) for all \( i \in [0..n] \). Then:

- No suffix is a prefix of another suffix, i.e., the set \( T_{[0..n]} \) is prefix free.
- All nodes in the suffix tree representing a suffix are leaves.

This simplifies algorithms.

**Example 4.1:** \( T = \text{banana}\$ \).

![Suffix Tree Diagram](image)

**Brute Force Construction**

Let us now look at algorithms for constructing the suffix tree. We start with a brute force algorithm with time complexity \( \Theta(n + \Sigma \text{LCP}(T_{[0..n]})) \). Later we will modify this algorithm to obtain a linear time complexity.

The idea is to add suffixes to the trie one at a time starting from the longest suffix. The insertion procedure is essentially the same as we saw in Algorithm 1.2 (insertion into trie) except it has been modified to work on a compact trie instead of a trie.
Let $S_u$ denote the string represented by a node $u$. The suffix tree representation uses four functions:

- $child(u,c)$ is the child $v$ of node $u$ such that the label of the edge $(u,v)$ starts with the symbol $c$, and $\bot$ if $u$ has no such child.
- $parent(u)$ is the parent of $u$.
- $depth(u)$ is the length of $S_u$.
- $start(u)$ is the starting position of some occurrence of $S_u$ in $T$.

Then

- $S_u = T[start(u) \ldots start(u) + depth(u)]$.
- $T[start(u) + depth(parent(u)) \ldots start(u) + depth(u)]$ is the label of the edge $(parent(u), u)$.

A locus in the suffix tree is a pair $(u,d)$ where $depth(parent(u)) < d \leq depth(u)$. It represents

- the uncompact trie node that would be at depth $d$ along the edge $(parent(u), u)$, and
- the corresponding string $S_{(u,d)} = T[start(u) \ldots start(u) + d]$.

Every factor of $T$ is a prefix of a suffix and thus has a locus along the path from the root to the leaf representing that suffix.

During the construction, we need to create nodes at an existing locus in the middle of an edge, splitting the edge into two edges.

**CreateNode**$(u,d)$ // / $d < depth(u)$
(1) $v \leftarrow start(u); p \leftarrow parent(u)$
(2) create new node $v$
(3) $start(v) \leftarrow i; depth(v) \leftarrow d$
(4) $child(v, T[i + depth(v)]) \leftarrow u; parent(u) \leftarrow v$
(5) $child(p, T[i + depth(p)]) \leftarrow v; parent(v) \leftarrow p$
(6) return $v$

Suffices links are well defined for all nodes except the root.

**Lemma 4.4**: If the suffix tree of $T$ has a node $u$ such that $S_u$ is the longest proper suffix of $S_v$, i.e., if $S_u = T[i..j)$ then $S_v = T[i+1..j]$. □

Usually, suffix links are needed only for internal nodes. For root, we define $slink(root) = root$.

Now we are ready to describe the construction algorithm.

**Algorithm 4.2**: Brute force suffix tree construction

Input: text $T[0..n]$ ($|T| = n$)
Output: suffix tree of $T$: root, child, parent, depth, start
(1) create new node root; depth(root) ← 0
(2) $u \leftarrow root; d \leftarrow 0$ // / $(u,d)$ is the active locus
(3) for $i = 0$ to $n - 1$ do // insert suffix $T_i$
(4) while $d = depth(u)$ and $child(u, T[i+d]) \neq \bot$ do
(5) $u \leftarrow child(u, T[i+d]); d \leftarrow d + 1$
(6) while $d < depth(u)$ and $T[start(u) + d] = T[i+d]$ do $d \leftarrow d + 1$
(7) if $d < depth(u)$ then // $(u,d)$ is in the middle of an edge
(8) $u \leftarrow CreateNode(u,d)$
(9) $CreateLeaf(i,u)$ // Create leaf representing suffix $T_i$
(10) $u \leftarrow root; d \leftarrow 0$

$CreateLeaf(i,u)$ // Create leaf representing suffix $T_i$
(1) create new leaf $v$
(2) $start(v) \leftarrow i; depth(v) \leftarrow n - i + 1$
(3) $child(v, T[i+d]) \leftarrow u; parent(v) \leftarrow u$ // Set $u$ as parent
(4) return $w$

Suffix links are used to compute the suffix links during or after the brute force construction.

**ComputeSlink**$(u,d)$
(1) $d \leftarrow depth(u)$
(2) $v \leftarrow slink(parent(u))$
(3) while $depth(v) < d - 1$ do
(4) $v \leftarrow child(v, T[start(v) + depth(v) + 1])$
(5) if $depth(v) > d - 1$ then // no node at $(v,d-1)$
(6) $v \leftarrow CreateNode(v,d - 1)$
(7) $slink(v) \leftarrow v$

The procedure $CreateNode(v,d - 1)$ sets $slink(v) = \bot$.

The algorithm uses the suffix link of the parent, which must have been computed before. Otherwise the order of computation does not matter.

The same idea can be used for computing the suffix links during or after the brute force construction.

**Suffix Links**

The key to efficient suffix tree construction are suffix links:

- $slink(u)$ is the node $v$ such that $S_v$ is the longest proper suffix of $S_u$, i.e., if $S_u = T[i..j)$ then $S_v = T[i+1..j]$.

**Example 4.3**: The suffix tree of $T = banana$ with internal node suffix links.

```
  +--- b
  |   +--- a
  |   |   +--- n
  |   |   |   +--- a
  |   |   +--- n
  +--- a

slink(b) = a, slink(a) = n, slink(n) = root
```

Suffix links are the same as Aho–Corasick failure links but Lemma 4.4 ensures that $depth(slink(u)) = depth(u) - 1$. This is not the case for an arbitrary trie or a compact trie.

Suffix links are used to compute the suffix links during or after the brute force construction.

**Example 4.4**: The creation of a new node on line (6) is never needed in a fully constructed suffix tree, but during the brute force algorithm the necessary node may not exist yet.

- If a new internal node $u_i$ was created during the insertion of the suffix $T_i$, there exists an earlier suffix $T_j$, $j < i$ that branches at $u_i$ into a different direction than $T_i$.
- Then $slink(u_i)$ represents a prefix of $T_{j+1}$ and thus exists at least as a locus on the path labelled $T_{j+1}$. However, it might not become a branching node until the insertion of $T_{j+1}$.
- If in such a case, $CreateSLink(u_i)$ creates $slink(u_i)$ a moment before it would otherwise be created by the brute force construction.
McCreight’s Algorithm

McCreight’s suffix tree construction is a simple modification of the brute force algorithm that computes the suffix links during the construction and uses them as short cuts:

- Consider the situation, where we have just added a leaf $w_i$ representing the suffix $T_i$ as a child to a node $u_i$. The next step is to add $w_{i+1}$ as a child to a node $u_{i+1}$.
- The brute force algorithm finds $u_{i+1}$ by traversing from the root. McCreight’s algorithm takes a short cut to $\text{slink}(u_i)$.

\[ \text{slink}(u_i) \]

\[ u_i \]

\[ w_i \]

\[ w_{i+1} \]

\[ u_{i+1} \]

- This is safe because $\text{slink}(u_i)$ represents a prefix of $T_{i+1}$.

\[ \text{Theorem 4.6:} \text{ Let } T \text{ be a string of length } n \text{ over an alphabet of constant size. McCreight’s algorithm computes the suffix tree of } T \text{ in } O(n) \text{ time.} \]

**Proof.** Insertion of a suffix $T_i$ takes constant time except in two points:

- The while loops on lines (4)–(6) traverse from the node $\text{slink}(u_i)$ to $u_{i+1}$. Every round in these loops increments $d$. The only place where $d$ decreases is on line (11) and even then by one. Since $d$ can never exceed $n$, the total time on lines (4)–(6) is $O(n)$.
- The while loop on lines (3)–(4) during a call to $\text{ComputeSlink}(u_i)$ traverses from the node $\text{slink}(\text{parent}(u_i))$ to $\text{slink}(u_i)$. Let $d'_i$ be the depth of $\text{parent}(u_i)$. Clearly, $d'_{i+1} \geq d'_i - 1$, and every round in the while loop during $\text{ComputeSlink}(u_i)$ increases $d'_{i+1}$. Since $d'_i$ can never be larger than $n$, the total time on lines (3)–(4) in $\text{ComputeSlink}$ is $O(n)$.

Applications of Suffix Tree

Let us have a glimpse of the numerous applications of suffix trees.

Exact String Matching

As already mentioned earlier, given the suffix tree of the text, all occurrences of a pattern $P$ can be found in time $O(|P| + \text{occ})$.

Even if we take into account the time for constructing the suffix tree, this is asymptotically as fast as Knuth–Morris–Pratt for a single pattern and Aho–Corasick for multiple patterns.

However, the primary use of suffix trees is in indexed string matching, where we can afford to spend a lot of time in preprocessing the text, but must then answer queries very quickly.

Text Statistics

Suffix tree is useful for computing all kinds of statistics on the text. For example:

- Every node in the suffix tree represents a factor of the text and, vice versa, every factor is represented by some node. Thus the number of distinct factors in the text is exactly the number of distinct locuses, which can be computed by a traversal of the suffix tree in $O(n)$ time even though the resulting value is typically $\Theta(n^2)$.
- The longest repeating factor of the text is the longest string that occurs at least twice in the text. It is represented by the deepest internal node in the suffix tree.

Approximate String Matching

Several approximate string matching algorithms achieving $O(kn)$ worst case complexity are based on suffix trees.

Filtering algorithms that reduce approximate string matching to exact string matching such as partitioning the pattern into $k + 1$ factors, can use suffix trees in the filtering phase.

Another approach is to generate all strings in the $k$-neighborhood of the pattern, i.e., all strings within edit distance $k$ from the pattern and search for them in the suffix tree.

The best practical algorithms for indexed approximate string matching are hybrids of the last two approaches. For example, partition the pattern into $\ell \leq k + 1$ factors and find approximate occurrences of the factors with edit distance $[k/\ell]$ using the neighborhood method in the filtering phase.

Generalized Suffix Tree

A generalized suffix tree of two strings $S$ and $T$ is the suffix tree of the string $SLT^S$, where $S$ and $T$ are symbols that do not occur elsewhere in $S$ and $T$.

Each leaf is marked as an $S$-leaf or a $T$-leaf according to the starting position of the suffix it represents. Using a depth first traversal, we determine for each internal node if its subtree contains only $S$-leaves, only $T$-leaves, or both. The deepest node that contains both represents the longest common factor of $S$ and $T$. It can be computed in linear time.

The generalized suffix tree can also be defined for more than two strings.
**AC Automaton for the Set of Suffixes**

As already mentioned, a suffix tree with suffix links is essentially an Aho–Corasick automaton for the set of all suffixes.

- We saw that it is possible to follow suffix link / failure transition from any locus, not just from suffix tree nodes.
- Following such an implicit suffix link may take more than a constant time, but the total time during the scanning of a string with the automaton is linear in the length of the string. This can be shown with a similar argument as in the construction algorithm.

Thus suffix tree is asymptotically as fast to operate as the AC automaton, but needs much less space.

---

**Longest Common Extension**

The longest common extension (LCE) query asks for the length of the longest common prefix of two suffixes of a text $T$:

$$\text{LCE}(i,j) := lcp(T_i, T_j)$$

- The lowest common ancestor (LCA) of two nodes $v$ and $w$ in a tree is the deepest node that is an ancestor of both $v$ and $w$. Any tree can be preprocessed in linear time so that the LCA of any two nodes can be computed in constant time. The details are omitted here.
- A LCE query can be implemented as a LCA query on the suffix tree of $T$:

$$\text{LCE}(i,j) = \text{LCA}(v_i, v_j)$$

where $v_i$ and $v_j$ are the leaves that represent the suffixes $T_i$ and $T_j$. Thus, given the suffix tree augmented with a precomputed LCA data structure, LCE queries can be answered in constant time.

Some $O(n \log n)$ worst case time approximate string matching algorithms are based on LCE queries.

---

**Matching Statistics**

The matching statistics of a string $S[0..n)$ with respect to a string $T$ is an array $M[S[0..n)]$, where $M[S[i]]$ is a pair $(i, p_i)$ such that

1. $S[i..i+\ell_i)$ is the longest prefix of $S_i$ that is a factor of $T$, and
2. $T[p_i..p_i+\ell_i) = S[i..i+\ell_i)$.

Matching statistics can be computed by using the suffix tree of $T$ as an AC-automaton and scanning $S$ with it.

- If before reading $S[i]$ we are at the locus $(v,d)$ in the automaton, then $S[i..i+d) = T[j..j+d)$, where $j = \text{start}(v)$. If reading $S[i]$ causes a failure transition, then $MS[i] = (d, j)$.
- Following the failure transition decrements $d$ and thus increments $i - d$ by one. Following a normal transition/edge, increments both $i$ and $d$ by one, and thus $i - d$ stays the same. Thus all entries are computed.

From the matching statistics, we can easily compute the longest common factor of $S$ and $T$. Because we need the suffix tree only for $T$, this saves space compared to a generalized suffix tree.

Matching statistics are also used in some approximate string matching algorithms.

---

**Longest Palindrome**

A palindrome is a string that is its own reverse. For example, saippuakauppias is a palindrome.

We can use the LCA preprocessed generalized suffix tree of a string $T$ and its reverse $T^R$ to find the longest palindrome in $T$ in linear time.

- Let $k_i$ be the length of the longest common extension of $T_{i+1}$ and $T_{n-i}$, which can be computed in constant time. Then $T[i..i+k_i]$ is the longest odd length palindrome with the middle at $i$.
- We can find the longest odd length palindrome by computing $k_i$ for all $i \in [0..n)$ in $O(n)$ time.
- The longest even length palindrome can be found similarly in $O(n)$ time. The longest palindrome overall is the longer of the two.