Inference in Bayesian Networks

How to generate random vectors from a Bayesian network?

	Cloudy=no	Cloudy=yes
	0.5	0.5

Cloudy	Sprinkler=on	Sprinkler=off
no	0.5	0.5
yes	0.9	0.1

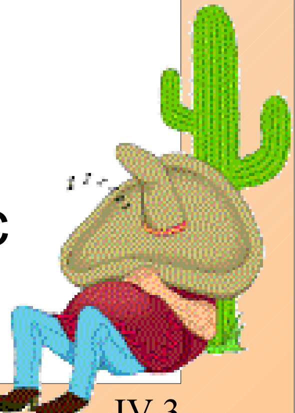
Cloudy	Rain=yes	Rain=no
no	0.2	0.8
yes	0.8	0.2

Sprinkler	Rain	WetGrass=yes	WetGrass=no
on	no	0.90	0.10
on	yes	0.99	0.01
off	no	0.01	0.99
off	yes	0.90	0.10

- Sample parents first
  - P(C)
    - (0.5, 0,5) → yes
  - P(S|C=yes)
    - (0.9, 0.1) → on
  - P(R | C=yes)
    - (0.8, 0.2) → no
  - P(W | S=on, R=no)
    - (0.9, 0.1) → yes
- P(C,S,R,W) = P(yes,on,no,yes) = 0.5 x 0.9 x 0.2 x 0.9 = 0.081

Two types of probabilistic reasoning

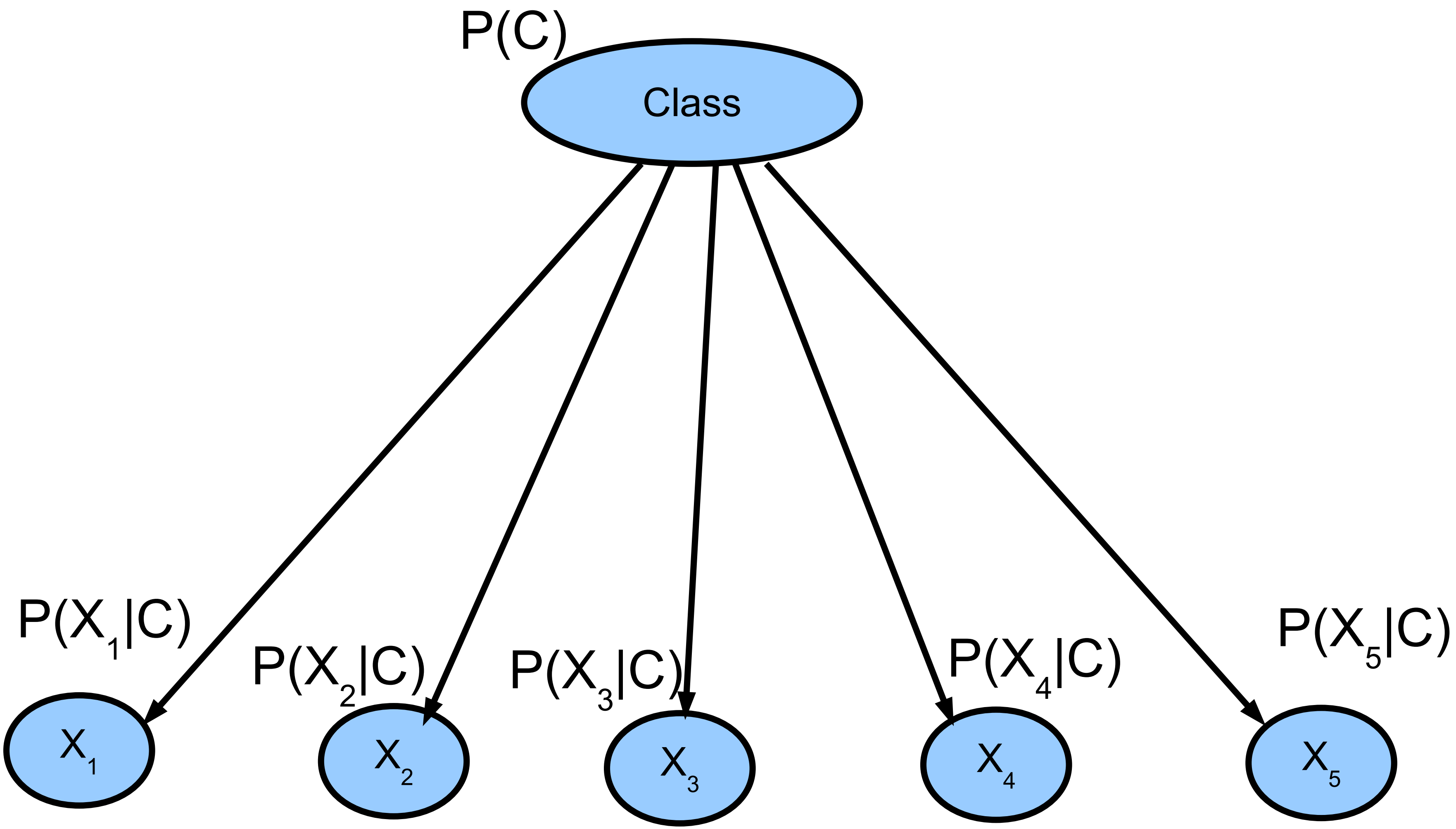
- n (discrete) random variables $X_1,\ldots,X_n$
- joint probability distribution $P(X_1,\ldots,X_n)$
- Input: a partial value assignment Ω,
  $\Omega = < X_1, X_2=x_2, X_3, X_4=x_4, X_5=x_5, X_6,\ldots,X_n >$
- **Probabilistic reasoning, type I (marginal distribution):**
  - compute $P(X=x \mid \Omega)$ for some X not instantiated in Ω, and for all values x of X.
- **Probabilistic reasoning, type II (MAP assignment):**
  - Given Ω, find a maximum a posterior probability value assignment jointly for all the X_i not instantiated in Ω
- N.B. These are not the same thing!
- Bayesian networks: a family of probabilistic models and algorithms enabling computationally efficient probabilistic reasoning



Some famous (simple) Bayesian network models

- Naïve Bayes classifier
- Finite mixture model
- Tree Augmented Naïve Bayes
- Hidden Markov Models (HMMs)

Naïve Bayes classifier



- X_i are called predictors or indicators

Naïve Bayes Classifier

- Structure tailored for efficient diagnostics $P(C|x_1,x_2,\ldots,x_n)$.
  - Obs! Does NOT try to model directly the target probability distribution $P(C|x_1,x_2,\ldots,x_n)$
- Unrealistic conditional independence assumptions, but OK for the particular query $P(C|x_1,x_2,\ldots,x_n)$.
- Because of wrong independence assumptions, NB is often poorly calibrated:
  - Probabilities $P(C|x_1,x_2,\ldots,x_n)$ may be way off, but $\mathrm{argmax}_c\ P(c|x_1,x_2,\ldots,x_n)$ still often correct.

Calculating $P(C|x_1,x_2,\ldots,x_n,NB)$

- Boldly calculate through joint probability

$$P(C|x_{1,}\ldots,x_n)\propto P(C,x_{1,}\ldots,x_n)=P(C)\prod_{i=1}^{n}P(x_i|C)$$

- No need to have all the predictors. Having just set X_A of predictors (and not X_B):

$$\begin{aligned}P(C|x_A)\propto P(C,x_A)&=\sum_{x_B}P(C,x_A,x_B)\\&=\sum_{x_B}P(C)\prod_{i\in A}P(x_i|C)\prod_{j\in B}P(x_j|C)\\&=P(C)\prod_{i\in A}P(x_i|C)\sum_{x_B}\prod_{j\in B}P(x_j|C)\\&=P(C)\prod_{i\in A}P(x_i|C)\prod_{j\in B}\sum_{x_j}P(x_j|C)=P(C)\prod_{i\in A}P(x_i|C)\end{aligned}$$

Example

$P(X_i = 0 \mid C)$	X_1	X_2	X_3	X_4	X_5
$C = 0$	0.8	0.5	0.4	0.7	0.9
$C = 1$	0.2	0.7	0.6	0.2	0.6

6 binary variables: C, X_1,...X_5, P(C=0)=0.4

$P(C=0 \mid X_1=0,X_2=1,X_3=0,X_4=1,X_5=0)$

$\alpha\ 0.4 \times 0.8 \times 0.5 \times 0.4 \times 0.3 \times 0.9 = 0.017$ 17/27=63%

$P(C=1 \mid X_1=0,X_2=1,X_3=0,X_4=1,X_5=0)$

$\alpha\ 0.6 \times 0.2 \times 0.3 \times 0.6 \times 0.8 \times 0.6 = 0.010$ 10/27=37%

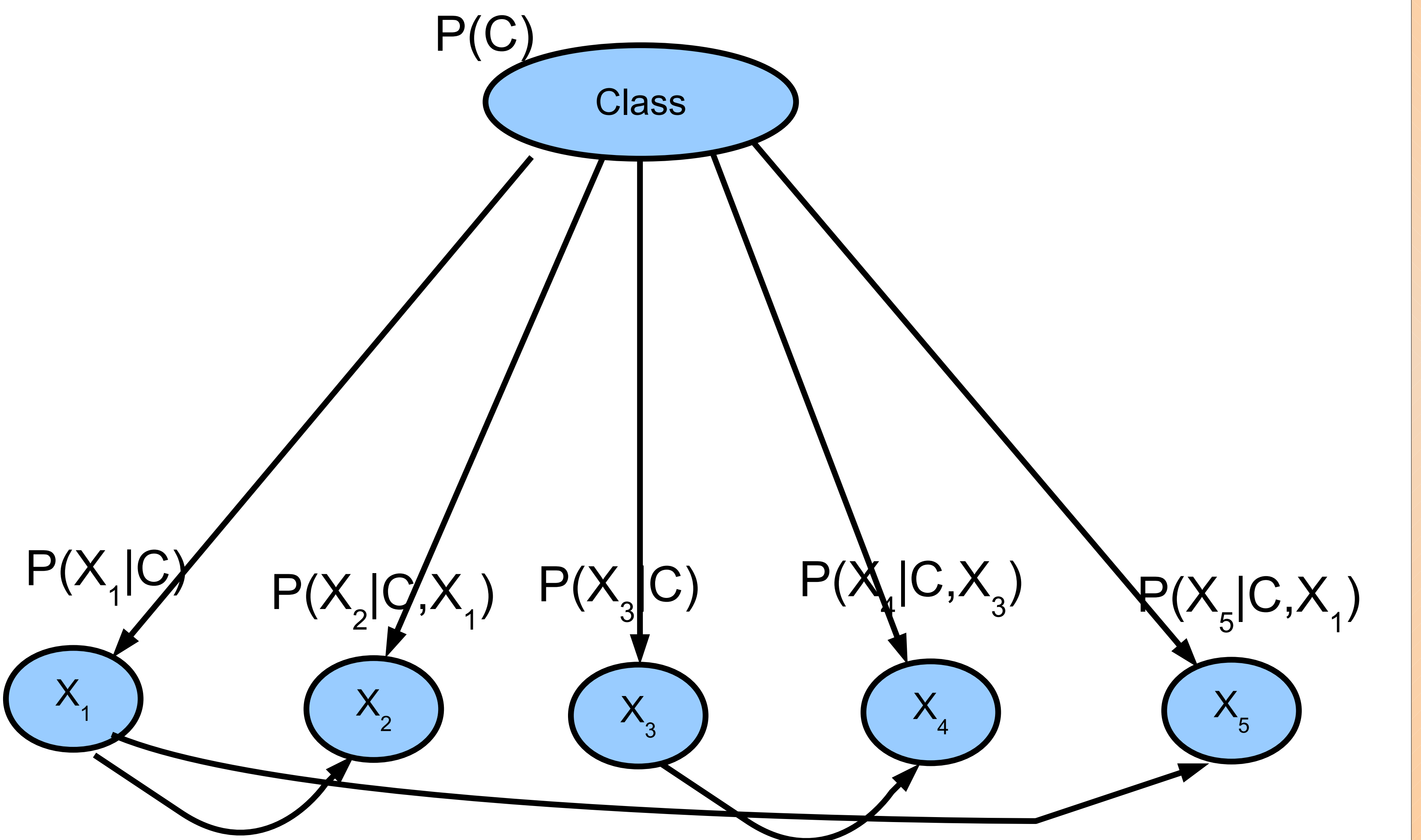
$P(C=0 \mid X_2=1,X_3=0,X_4=1,X_5=0)$

$\alpha\ 0.4 \times 0.5 \times 0.4 \times 0.3 \times 0.9 = 0.022$ 22/74=30%

$P(C=1 \mid X_2=1,X_3=0,X_4=1,X_5=0)$

$\alpha\ 0.6 \times 0.3 \times 0.6 \times 0.8 \times 0.6 = 0.052$ 52/74=70%

Tree Augmented Naïve Bayes (TAN)



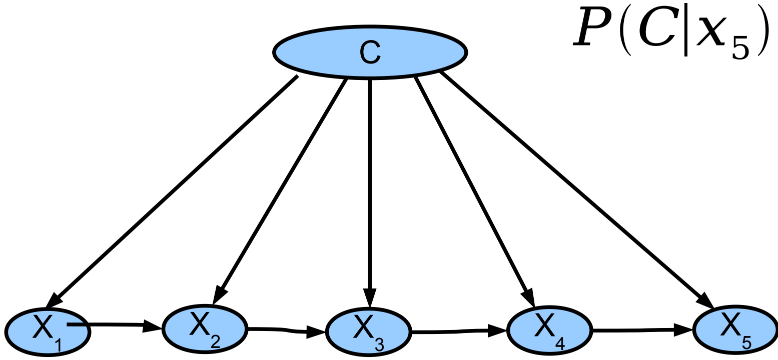
- X_i may have at most one other X_j as an extra parent.

Calculating $P(C|x_1,x_2,\ldots,x_n,TAN)$

- Again, boldly calculate via joint probability

$$P(C|x_{1,}\ldots,x_n) \propto P(C,x_{1,}\ldots,x_n) = P(C)\prod_{i=1}^{n} P(x_i|C,Pa(x_i))$$

- But missing predictors may hurt more. For example:



$$P(C|x_5) \propto P(C)P(x_5|C) = P(C)\sum_{x_4} P(x_4|C)P(x_5|x_{4,}C)$$
$$= P(C)\sum_{x_4} P(x_5|C,x_4)P(x_4|C)$$
$$= P(C)\sum_{x_4} P(x_5|C,x_4)\sum_{x_3} P(x_4|C,x_3)P(x_3|C)$$
$$= \ldots$$

NB as a Finite Mixture Model

- When the Naive Bayes structure is reasonable, it also makes a nice (marginal) joint probability model $P(X_1,X_2,...,X_n)$ for “predictors”.
- A computationally effective alternative for building a Bayesian network for $X_1,X_2,...,X_n$.
- Joint probability $P(X_1,X_2,...,X_n)$ is represented as a mixture of K joint probability distributions $P_k(X_1,X_2,...,X_n) = P_k(X_1)P_k(X_2)...P_k(X_n)$, where $P_k(\cdot) = P(\cdot|C=k)$.

Calculating with $P(X_1,X_2,\ldots,X_n|NB)$

- Joint probability a simple marginalization:

$$P(X_1,\ldots,X_n)=\sum_{k=1}^{K} P(X_1,\ldots,X_n,C=k)$$
$$=\sum_{k=1}^{K} P(C=k)\prod_{i=1}^{n} P(X_i|C=k)$$

- Inference

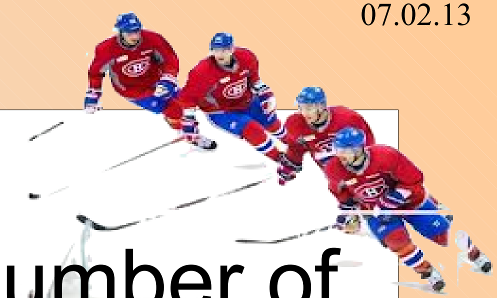
$$P(X|e)\propto P(e,X)=\sum_{k=1}^{K} P(e,X,C=k)$$
$$=\sum_{k=1}^{K} P(C=k)P(e,X|C=k)$$
$$=\sum_{k=1}^{K} P(C=k)\prod_{X_i\in X} P(X_i|C=k)\prod_{e_i\in e} P(e_i|C=k)$$

Example

- Consider the previous example (the NB model).
- What is $P(X_4 \mid X_5=0)$?
- $P(X_4=0, X_5=0 \mid C=0) = 0.7 \times 0.9 = 0.63$
- $P(X_4=1, X_5=0 \mid C=0) = 0.3 \times 0.9 = 0.27$
- $P(X_4=0, X_5=0 \mid C=1) = 0.2 \times 0.6 = 0.12$
- $P(X_4=1, X_5=0 \mid C=1) = 0.8 \times 0.6 = 0.48$
- $P(X_4=0, X_5=0) = P(X_4=0, X_5=0 \mid C=0)P(C=0) + P(X_4=0, X_5=0 \mid C=1)P(C=1) = 0.63 \times 0.4 + 0.12 \times 0.6 = 0.324$
- $P(X_4=1, X_5=0) = P(X_4=1, X_5=0 \mid C=0)P(C=0) + P(X_4=1, X_5=0 \mid C=1)P(C=1) = 0.27 \times 0.4 + 0.48 \times 0.6 = 0.396$
- $P(X_4=0 \mid X_5=0) = P(X_4=0,X_5=0)/P(X_5=0) = 0.45$
- $P(X_4=1 \mid X_5=0) = P(X_4=1,X_5=0)/P(X_5=0) = 0.55$

Hidden Markov Models

- Temporal/sequential probabilistic models
- States of the process are hidden but an output dependent on the hidden state is observable
- Frequently applied in e.g. speech recognition, robot navigation, and other pattern recognition tasks

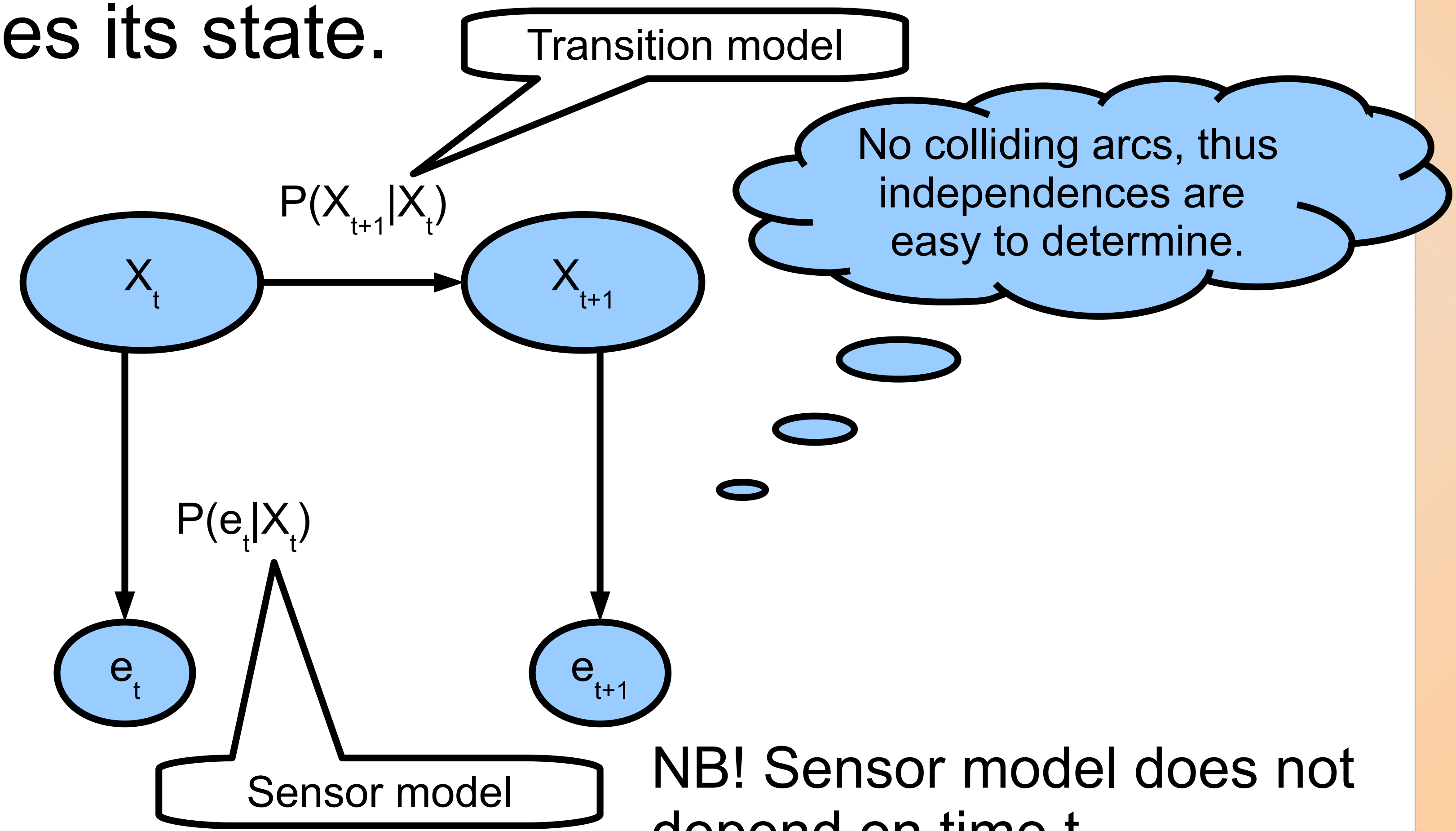


Markov chains

- Assume that the world has a finite number of states, and the changes in the world are caused by a stationary process:
  - The process does not change over time
- The wold has a *Markov property*:
  - The current state depends only on a finite history of previous states
- *A Markov chain* is a sequence of random variables $X_0, X_1, X_1, \ldots$ with the Markov property
  - We mainly consider first-order Markov chains where $P(X_t \mid X_{0:t-1}) = P(X_t \mid X_{t-1})$

Hidden Markov Models

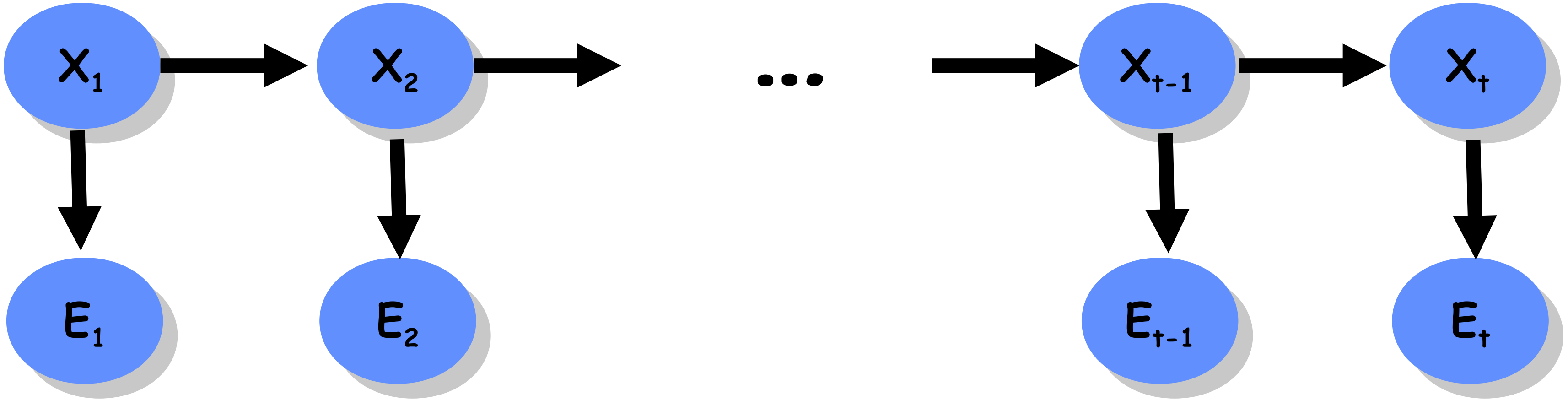
- Models observations about a system that changes its state.



NB! Sensor model does not depend on time t.

Hidden Markov Model as a BN

- For inference, easier to think of as a long chain of variables
- (For learning, the two-state model more fitting)
- No head-to-head nodes!
- Node X_t represents the (hidden) state at time t, and E_t is the observation at time t



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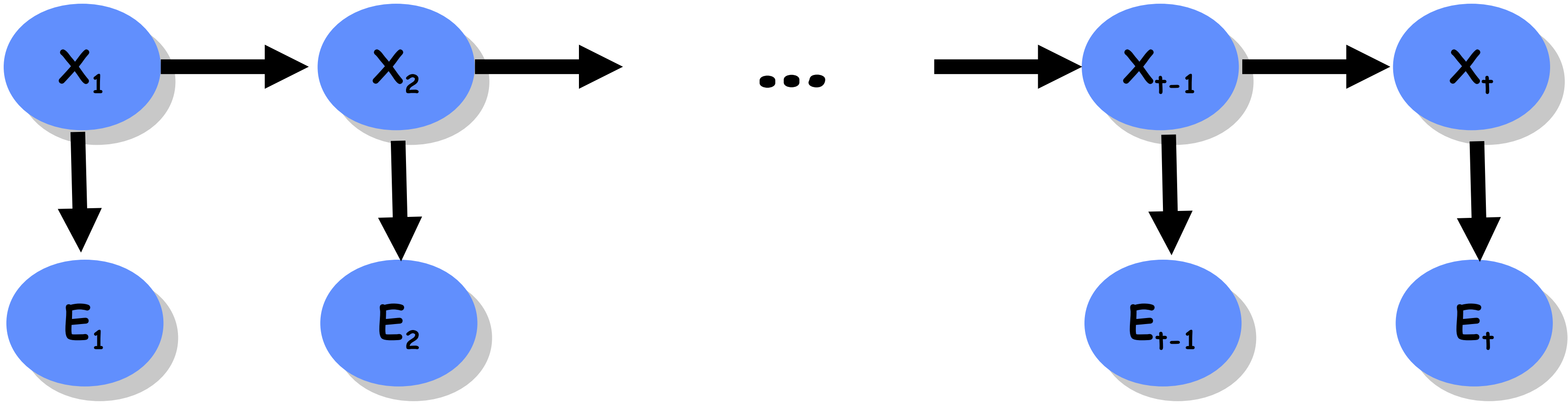
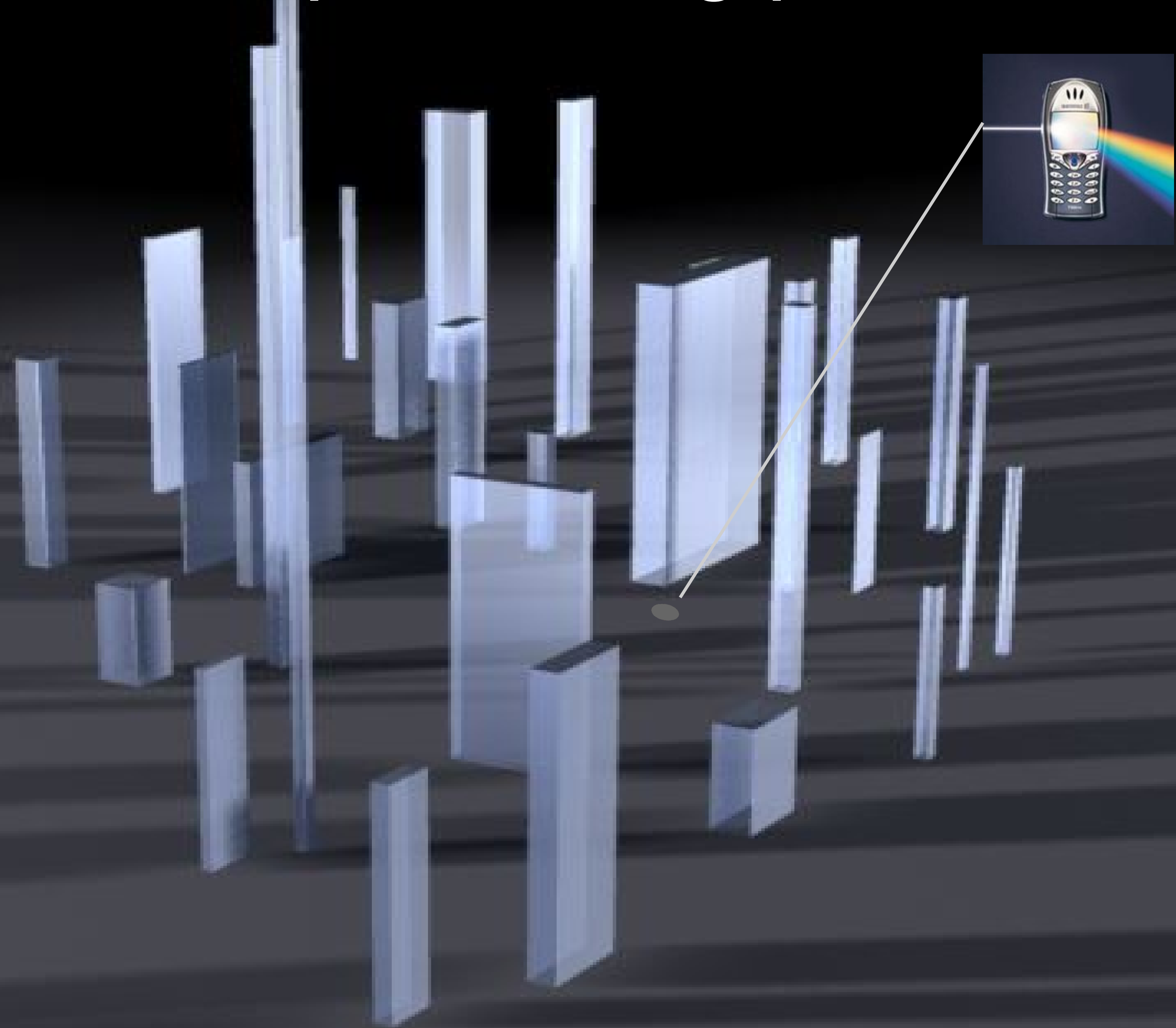


Graphical models on Manhattan – A probabilistic approach to mobile device positioning

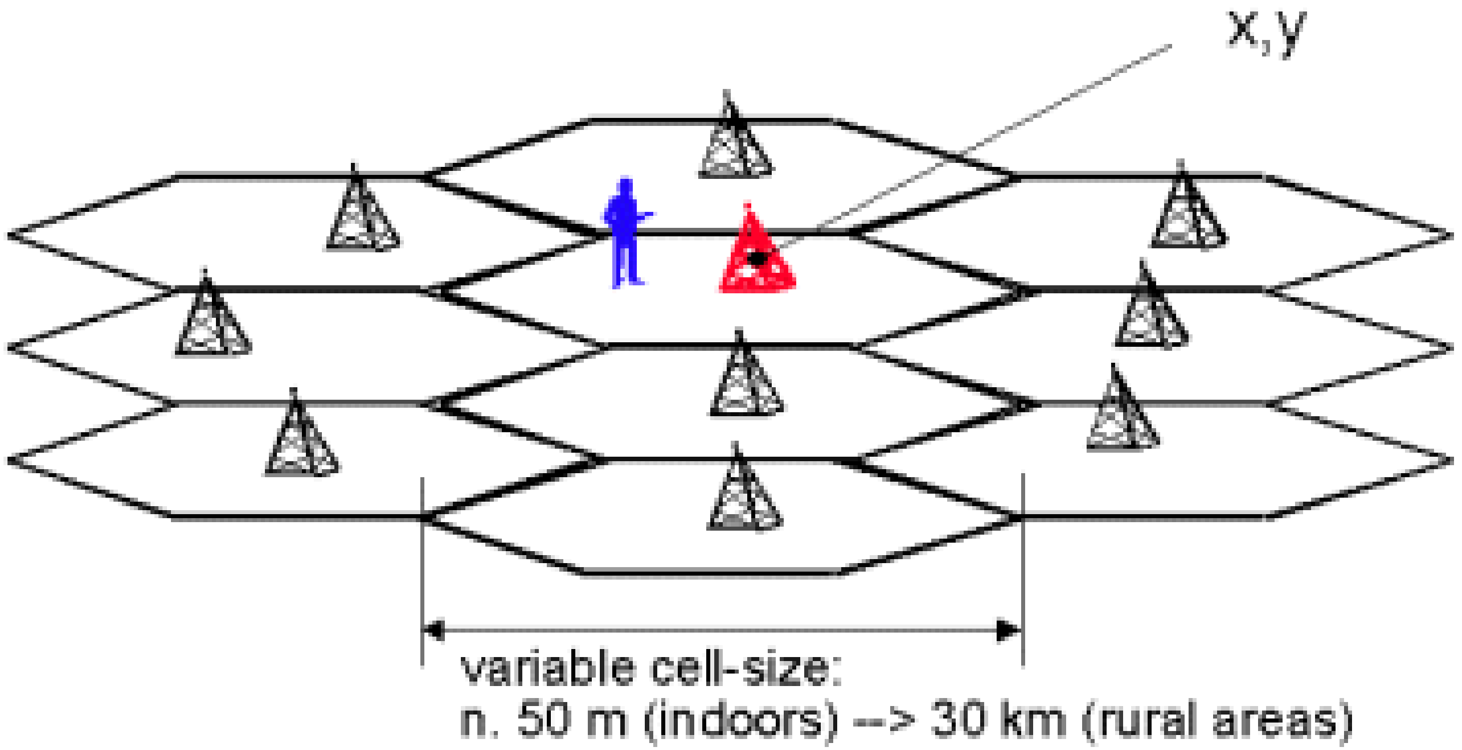
Location positioning problem



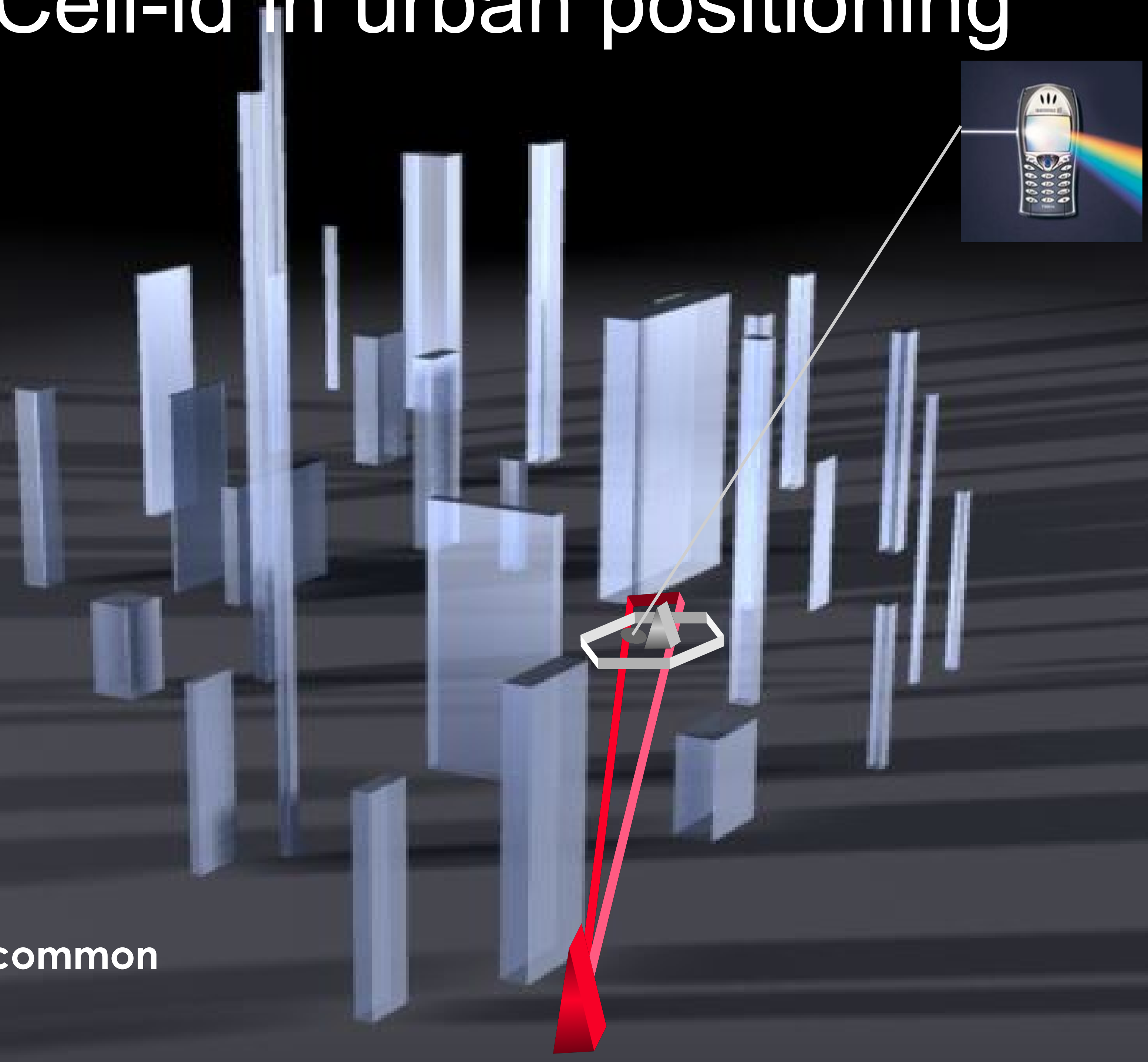
The positioning problem

- Given some location-dependent observations *O*, measured by a mobile device, determine the location *L* of the device
- Why is this a good research problem?
  - The goodness of different solutions is extremely easy to validate (just go to a known location and test)
  - The results have immediate practical applications
    - Location-based services (LBS)
    - FCC Enhanced 911:
      - Network-based solutions: error below 100 meters for 67 percent of calls, 300 meters for 95 percent of calls
      - Handset-based solutions: error below 50 meters for 67 percent of calls, 150 meters for 95 percent of calls

Cell ID

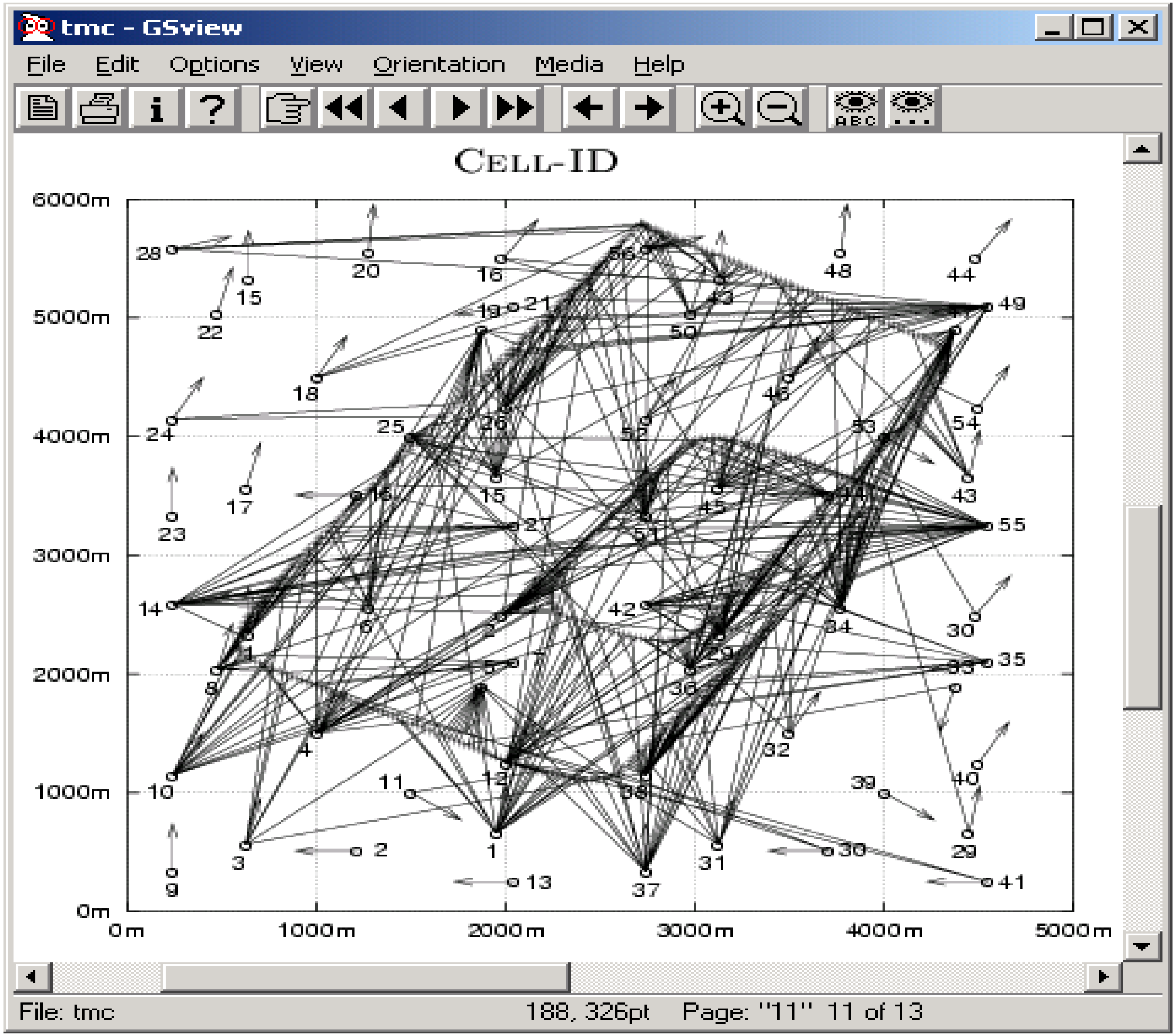


Cell-id in urban positioning

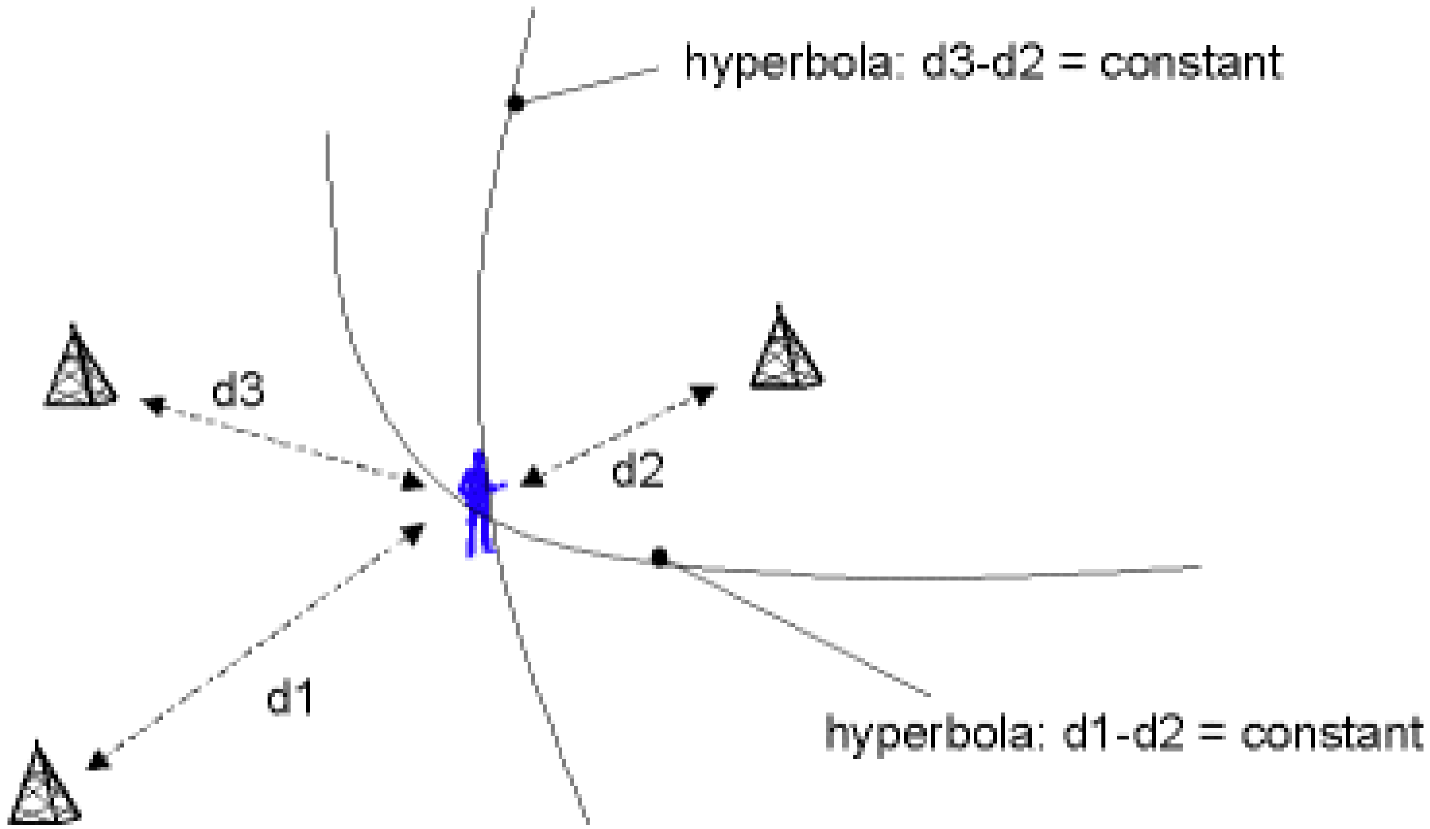


- errors > 500m common
+ simple

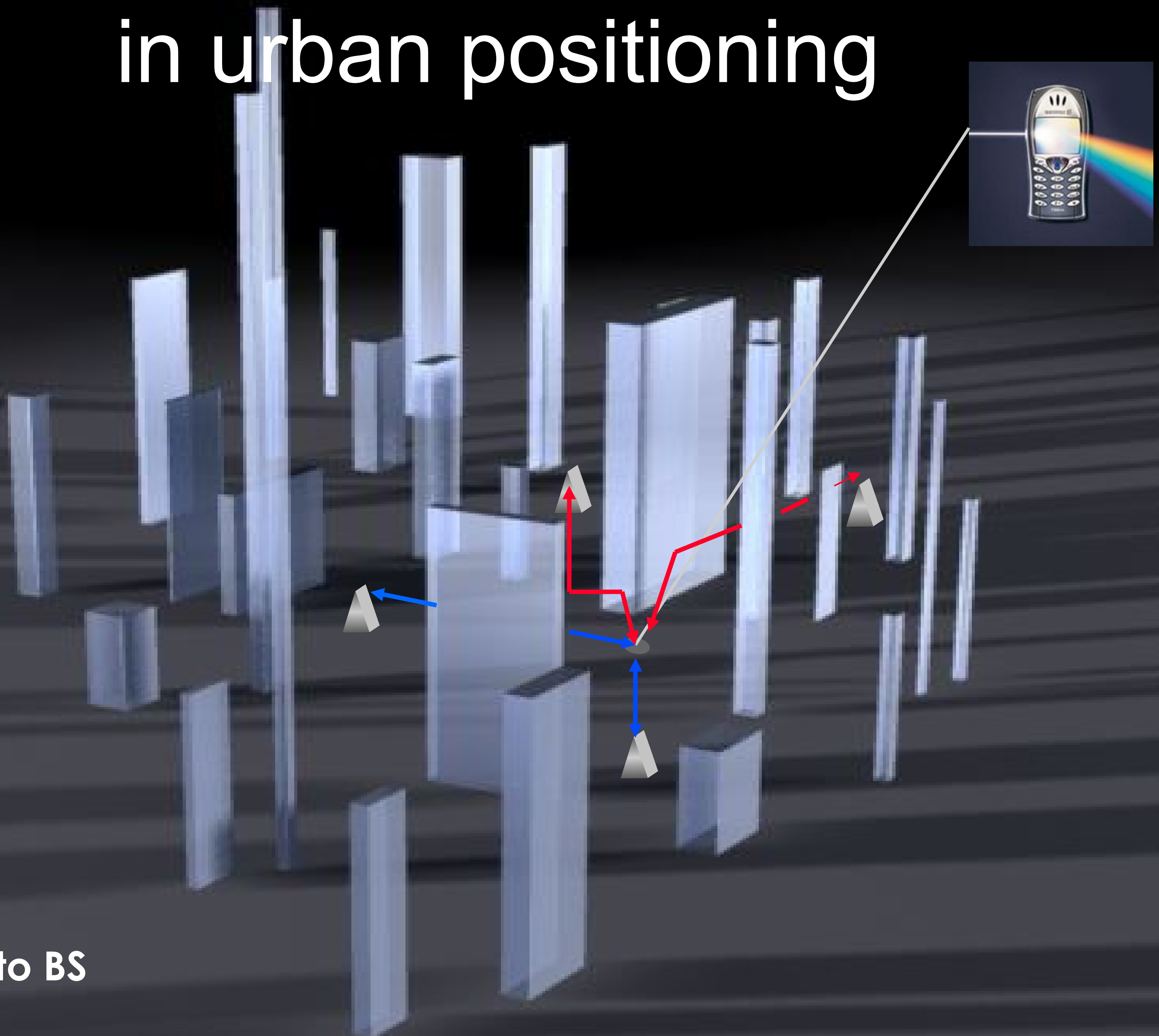
Cell ID errors



Enhanced Observed Time Difference (E-OTD)

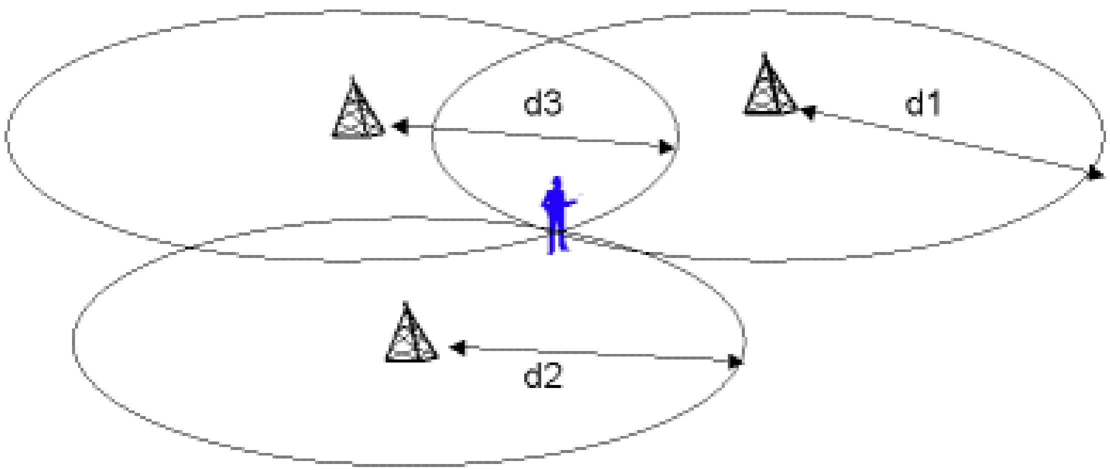


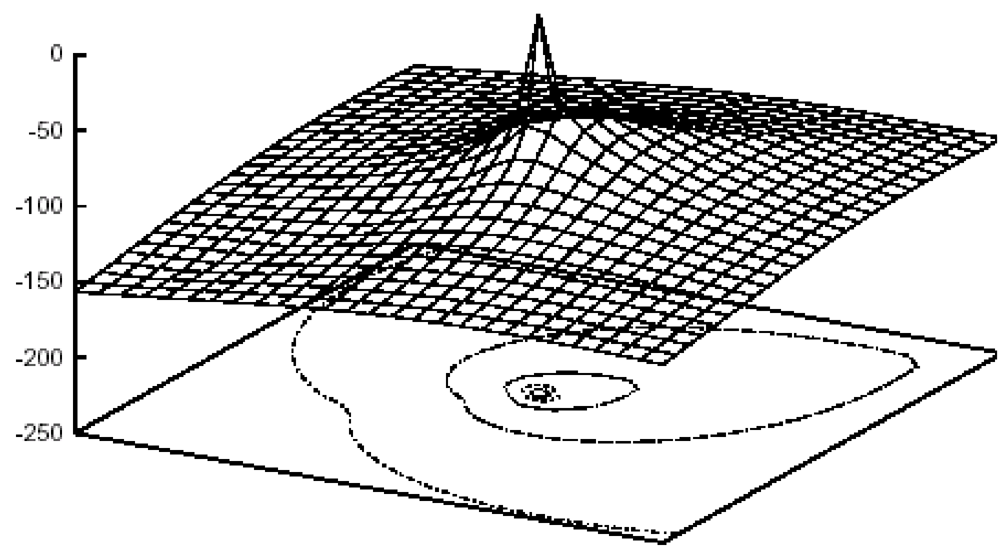
Problems with E-OTD in urban positioning



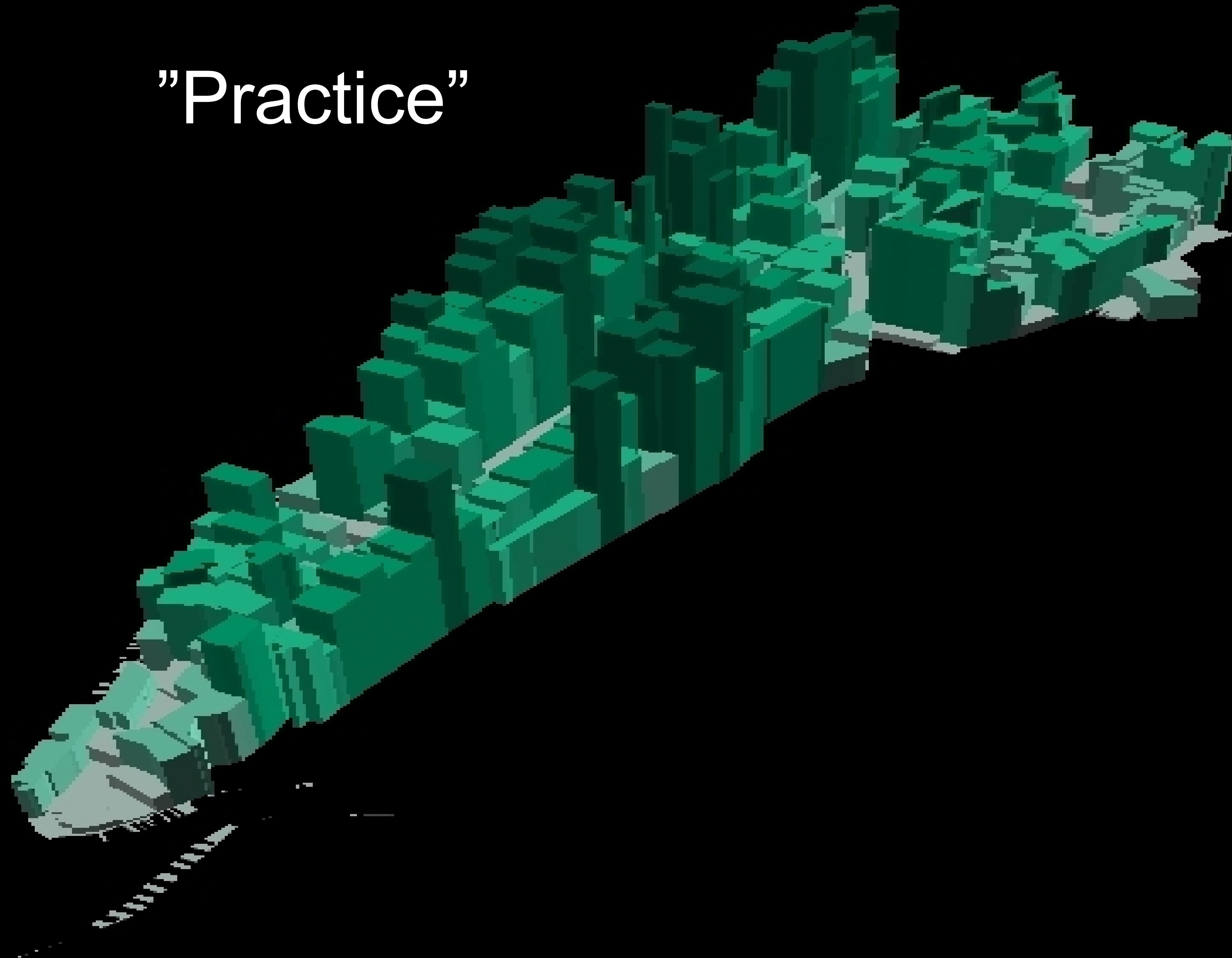
- multi-paths
- no line of sight to BS
- extra hardware

"Theory"



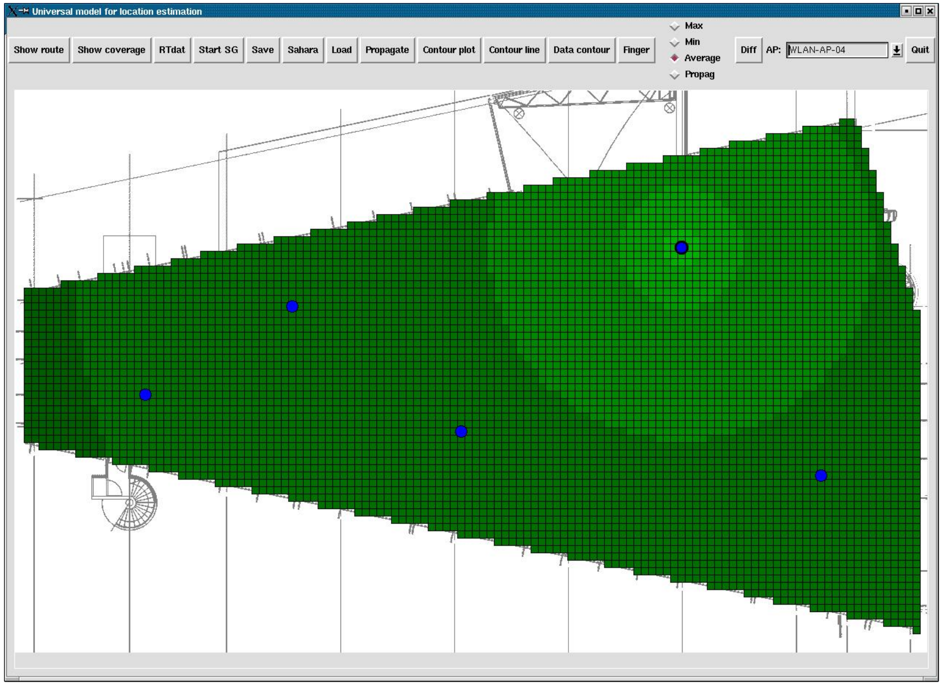


”Practice”

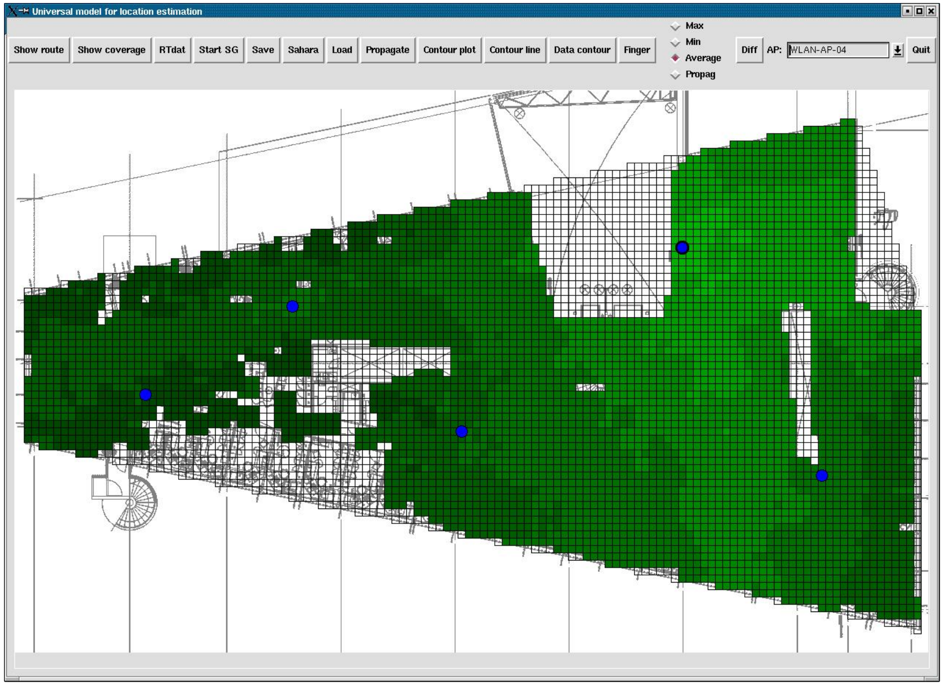


The signal propagation approach

Theory



Reality



Empirical modeling in urban positioning

+accurate
+handset or network based
-calibration measurements required

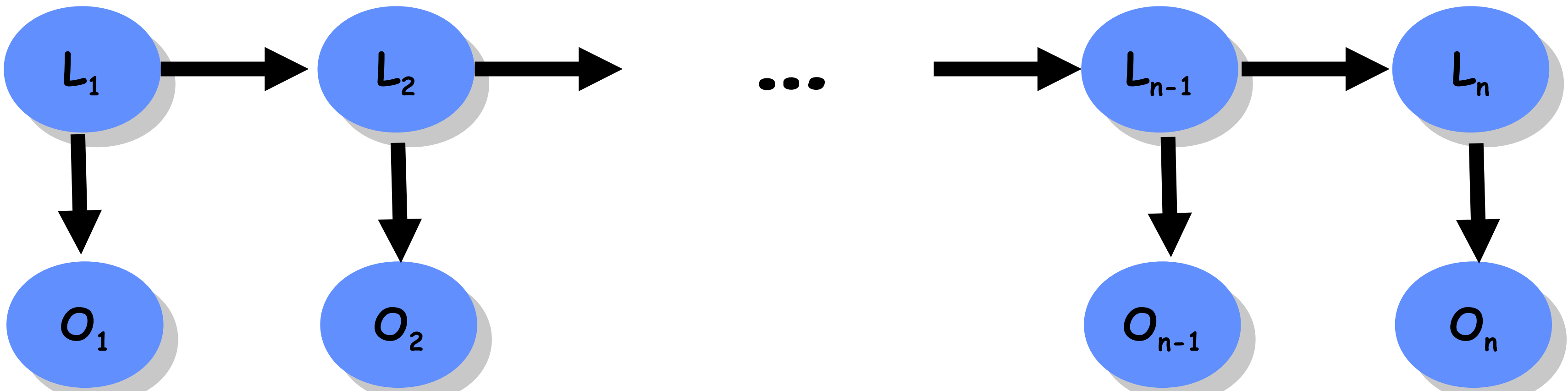
A probabilistic approach to positioning

Bayes rule: $P(L \mid O) = \dfrac{P(O \mid L)\, P(L)}{P(O)}$

- A probabilistic model assigns a probability for each possible location L given the observations O.
  - P(O | L) is the conditional probability of obtaining observations O at location L.
  - P(L) is the prior probability of location L. (Could be used to exploit user profiles, rails etc.)
  - P(O) is just a normalizing constant.
- How to obtain P(O | L)? ⇨ Empirical observations + machine learning

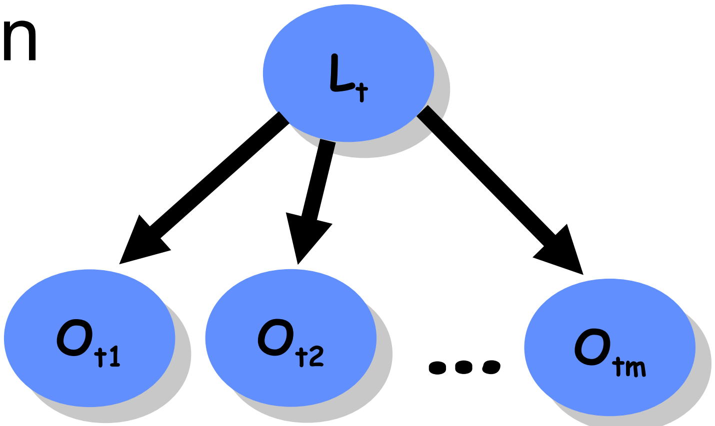
Tracking with Markov models

- Typically we have a sequence (history) of observations $O_1,\ldots,O_n$, and wish to determine $P(L_n \mid O^n)$
- Assumption: $P(O_t \mid L_t)$ are known, and given location L_t, the observation O_t is independent of the rest of the history
- The model: a hidden Markov model (HMM) where the locations L_t are the hidden unobserved states
- The transition probabilities $P(L_t \mid L_{t-1})$ can be easily determined from the physical properties of the moving object



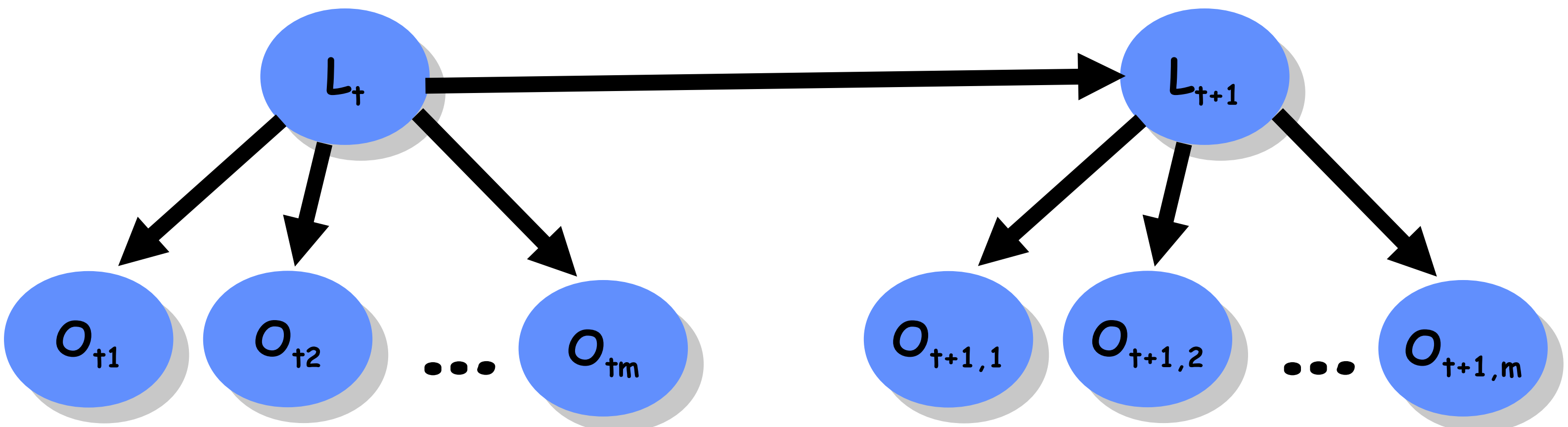
One more assumption

- The observation at time *t* typically consists of several measurements (e.g., strengths of signals from all the transmitters that can be heard)
- If the wireless network is designed in a reasonable manner (the transmitters are far from each other), it makes sense to assume that the individual observations are independent, given the location
- The “Naïve Bayes” model



The Model

First-order "semi-hidden" Markov model



Tracking as probabilistic inference

- As our hidden Markov model is a tree, we can compute the marginal of any L_t, given the history O^n, in linear time by using a simple forward-backward algorithm
- Alternatively, we can compute the maximum probability path $L_1,\ldots,L_n$ given the history (this is known as the **Viterbi** algorithm)
- **Kalman filter**: all the conditional distributions of the HMM model are normal distributions (linear dependencies with Gaussian noise)

Recursive tracking

- Assume that $\mathbf{P(L_{n-1} \mid O^{n-1})}$ has been computed.
- Our model defines the transition probabilities $\mathbf{P(L_t \mid L_{t-1})}$ and the local observation probabilities $\mathbf{P(O_t \mid L_t)}$
- Now $\mathbf{P(L_n \mid O^n) \propto P(L_n, O^n)}$

$$= \mathbf{P(O_n \mid L_n, O^{n-1})\, P(L_n , O^{n-1})}$$

$$= \mathbf{P(O_n \mid L_n) \sum_{L_{n-1}} P(L_n , L_{n-1} , O^{n-1})}$$

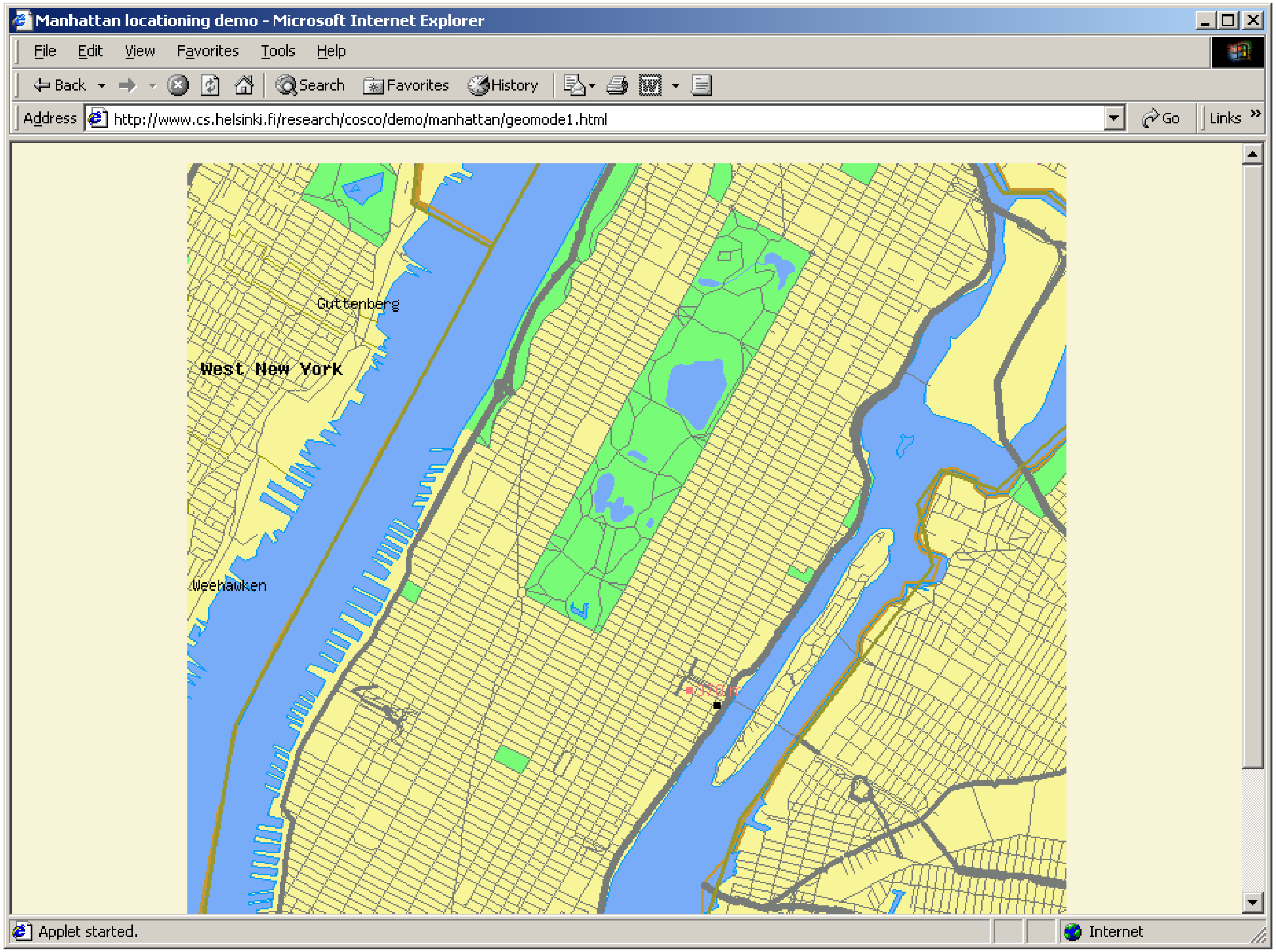
$$\propto \mathbf{P(O_n \mid L_n) \sum_{L_{n-1}} P(L_n \mid L_{n-1}) P(L_{n-1} \mid O^{n-1})}$$

- With a Kalman filter, the recursive process operates all the time with Gaussians

GSM-positioning trials

NYC Trial 2001

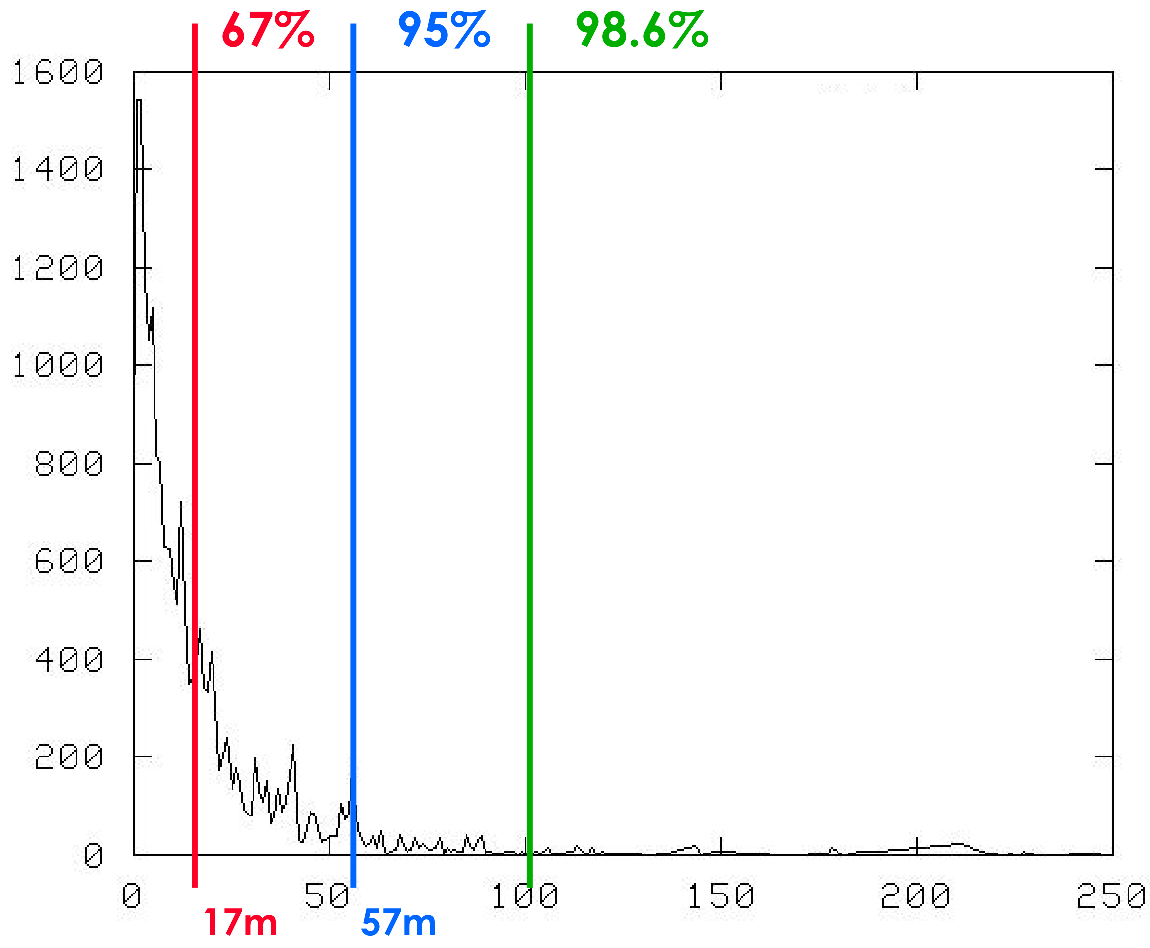
http://cosco.hiit.fi/demo/manhattan/



Details

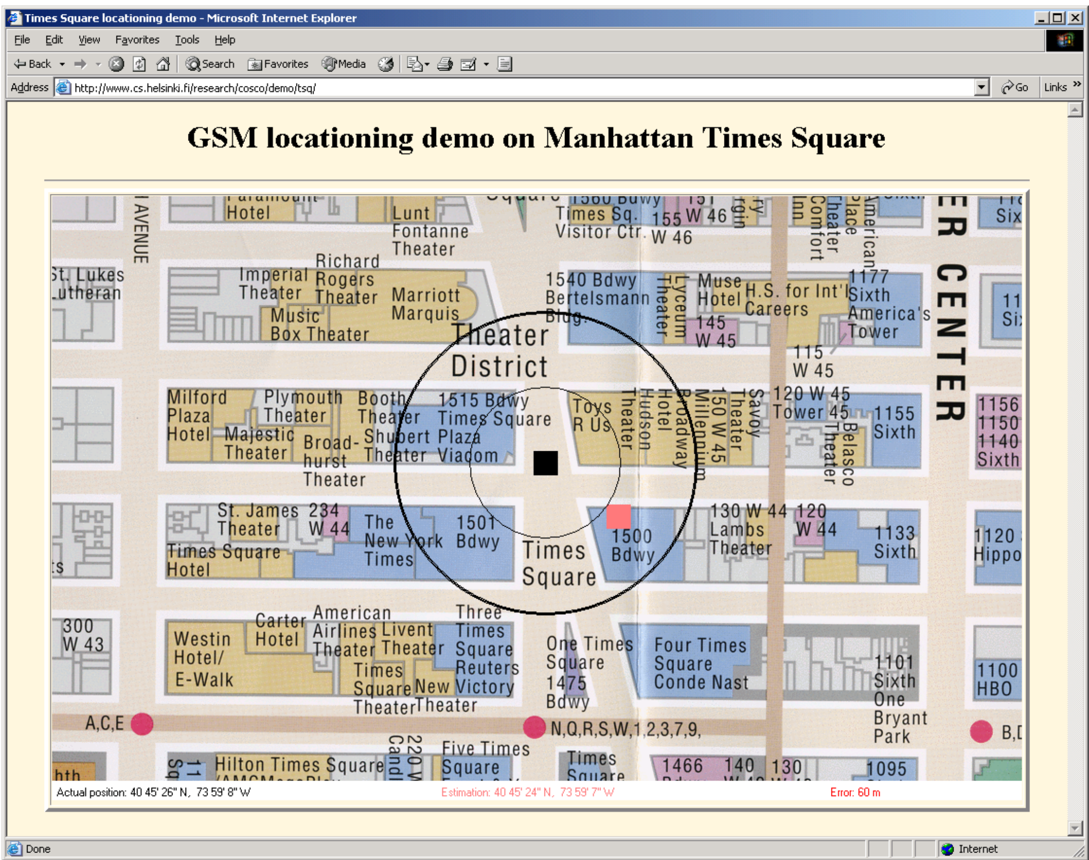
- Covering downtown Manhattan (10th - 114th St)
- Data gathering by car
- Modeling: 10 person days
- Target accuracy: less than 911 handset requirements
- Tests using cars

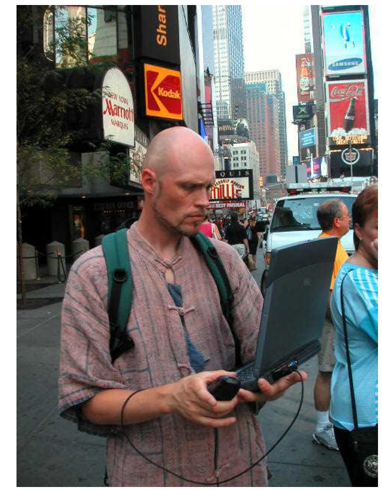
Accuracy of NYC Trial 2001



- 20166 points
- tracking; testing done in a car;

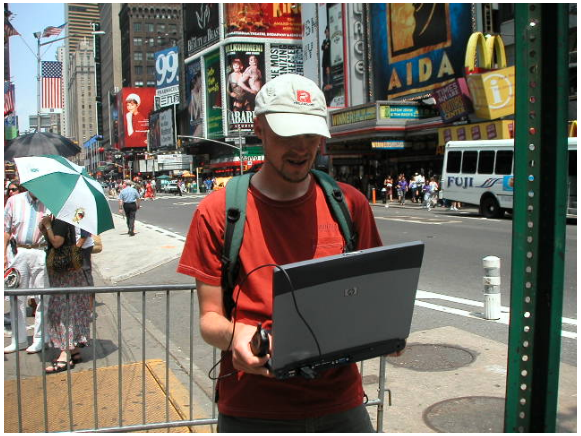
Trials: Manhattan 2002



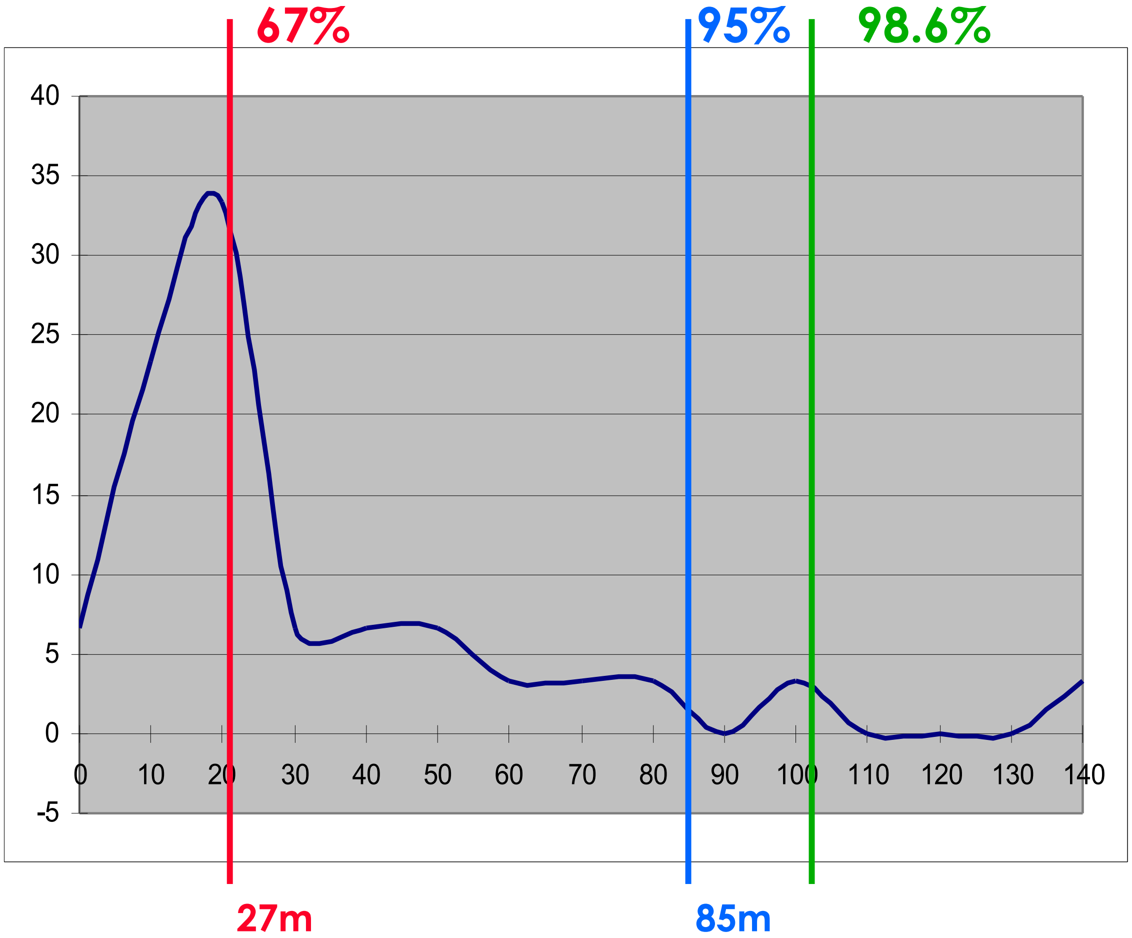


Challenges

- “real 911” simulation
  - No tracking information
  - Only up to 60 seconds of signal measurements
- Target accuracy: “theater level”
- Indoor testing (without indoor modeling)



Accuracy NYC Trial 2002



- 30 points
- static; testing done by walking;

WiFi-positioning

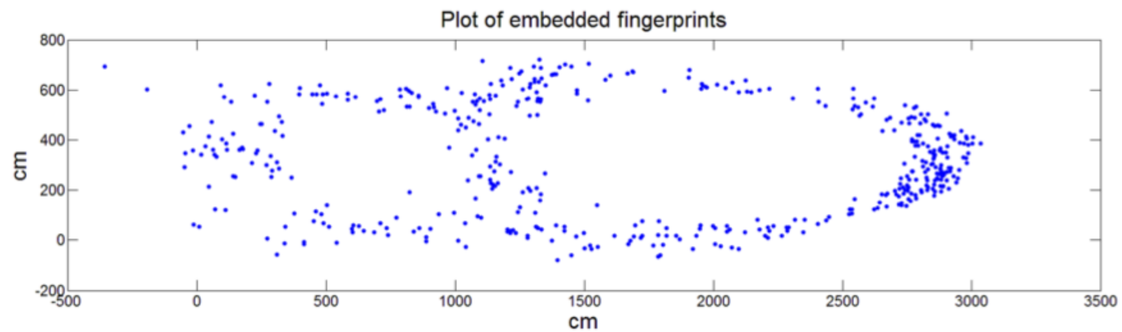
P.Myllymäki, T. Roos, H.Tirri, P.Misikangas and J.Sievänen. *A Probabilistic Approach to WLAN User Location Estimation.* International Journal of Wireless Information Networks, Vol. 9, No. 3, July 2002.

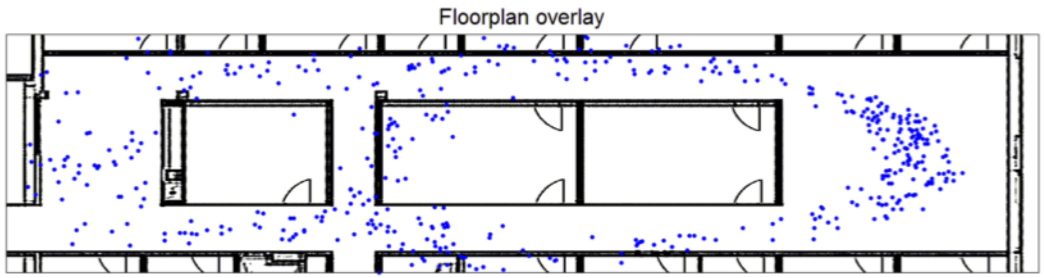
More information: www.ekahau.com

ekahau

Thesis topic: semi-supervised modeling in positioning

- ”automatic calibration”
- T. Pulkkinen, T. Roos, and P. Myllymäki: Semi-supervised learning for WLAN positioning.

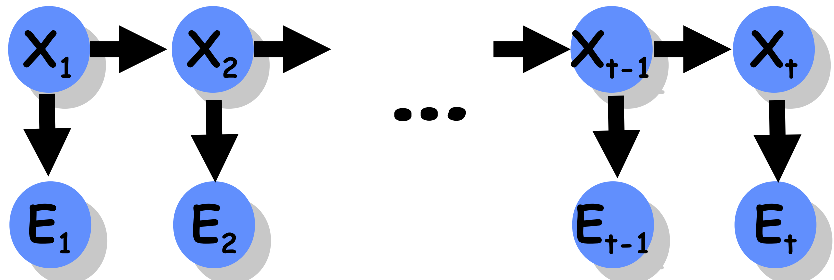




Joint probability of a HMM

- Joint probability factorizes like a BN
  - HMM is a Bayesian network!

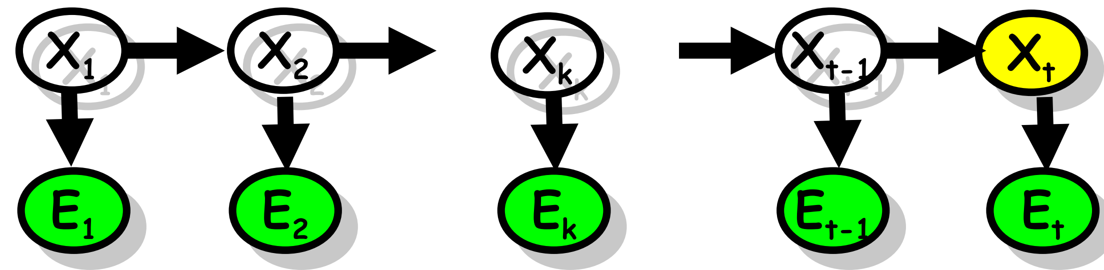
$$P(X_0, X_1, E_1, X_2, E_2, \ldots, X_t, E_t) = P(X_0) \prod_{i=1}^{t} P(X_i | X_{i-1}) P(E_i | X_i)$$



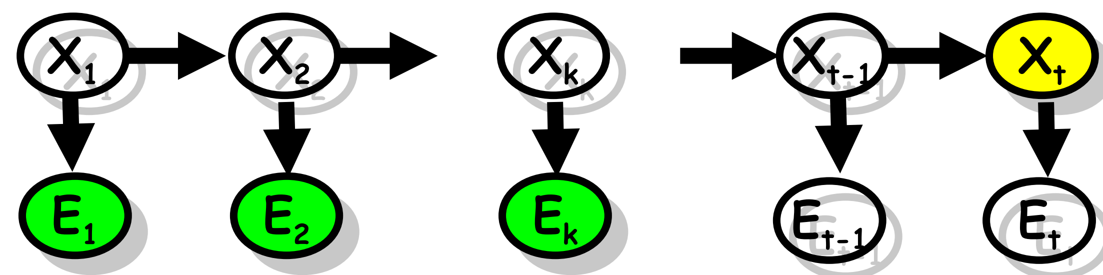
- Common inference tasks:
  - Filtering / monitoring: $P(X_t \mid e_{1:t})$
  - Prediction: $P(X_{t+k} \mid e_{1:t})$, $k>0$
  - Smoothing: $P(X_k \mid e_{1:t})$, $k<t$
  - Explanation: $\arg\max_{x_{1:t}} P(X_{1:t} \mid e_{1:t})$

Inference tasks visualized

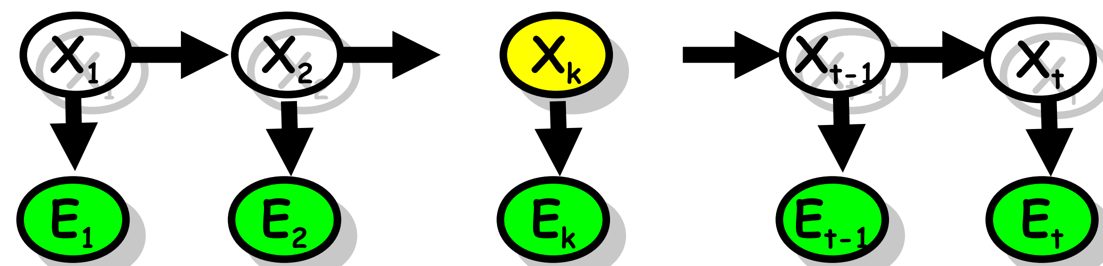
Filtering



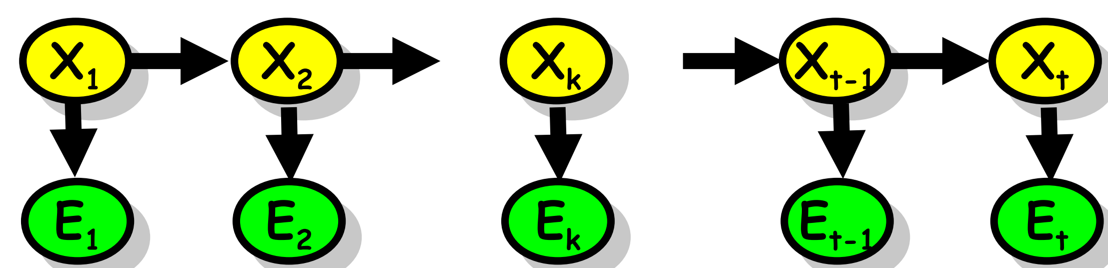
Prediction



Smoothing



Most likely sequence



Calculating $P(X_t \mid e_{1:t})$ in HMM

- Lets shoot for a recursive formula:

$$
\begin{aligned}
P(X_{t+1}|e_{1:t+1}) &= P(X_{t+1}|e_{t+1}, e_{1:t}) \\
&\propto P(e_{t+1}|X_{t+1}, e_{1:t}) P(X_{t+1}|e_{1:t}) \\
&= P(e_{t+1}|X_{t+1}) \underline{P(X_{t+1}|e_{1:t})}
\end{aligned}
$$

- and

$$
\begin{aligned}
P(X_{t+1}|e_{1:t}) &= \sum_{x_t} P(X_{t+1}, x_t|e_{1:t}) \\
&= \sum_{x_t} P(X_{t+1}|x_t, e_{1:t}) P(X_t|e_{1:t}) \\
&= \sum_{x_t} P(X_{t+1}|x_t) \underline{P(x_t|e_{1:t})}
\end{aligned}
$$

Forward algorithm for $P(X_t \mid e_{1:t})$

- Combining formulas we get a recursion

$$P(X_{t+1}|e_{1:t+1}) \propto P(e_{t+1}|X_{t+1}) \sum_{x_t} P(X_{t+1}|x_t) \underline{P(x_t|e_{1:t})}$$

- So first calculate

$$P(X_1|e_1) \propto P(e_1|X_1) \sum_{x_0} P(X_1|x_0) P(x_0)$$

- and then

$$P(X_2|e_{1,}e_2) \propto P(e_2|X_2) \sum_{x_1} P(X_2|x_1) P(x_1|e_1)$$

$$P(X_3|e_{1,}e_{2,}e_3) \propto P(e_3|X_3) \sum_{x_2} P(X_3|x_2) P(x_2|e_{1,}e_2)$$

Prediction: $P(X_{t+k} \mid e_{1:t})$, k>0

- $P(X_{t+1} \mid e_{1:t})$ part of the forward algorithm
- and from that on evidence does not count, and one can just calculate forward:

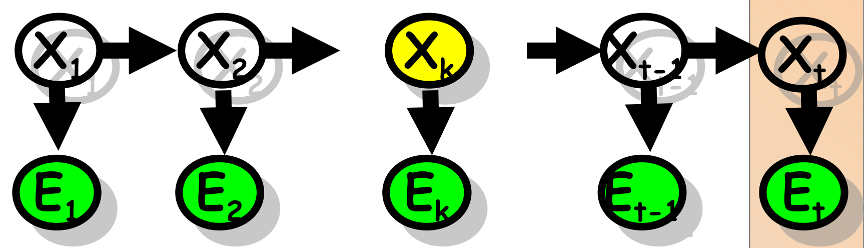
$$P(X_{t+2}|e_{1:t}) = \sum_{x_{t+1}} P(X_{t+2}|x_{t+1}, e_{1:t}) P(x_{t+1}|e_{1:t})$$
$$= \sum_{x_{t+1}} P(X_{t+2}|x_{t+1}) P(x_{t+1}|e_{1:t})$$
$$P(X_{t+3}|e_{1:t}) = \sum_{x_{t+2}} P(X_{t+3}|x_{t+2}, e_{1:t}) P(x_{t+2}|e_{1:t})$$
$$= \sum_{x_{t+2}} P(X_{t+3}|x_{t+2}) P(x_{t+2}|e_{1:t})$$

...

Smoothing: $P(X_k \mid e_{1:t}),\ k<t$

- Obvious move: divide $e_{1:t}$ to $e_{1:k}$ and $e_{k+1:t}$.

$$P(X_k|e_{1:t}) = P(X_k|e_{1:k}, e_{k+1:t})$$
$$\propto P(X_k|e_{1:k}) P(e_{k+1:t}|X_k, e_{1:k})$$
$$= P(X_k|e_{1:k}) \underline{P(e_{k+1:t}|X_k)}$$



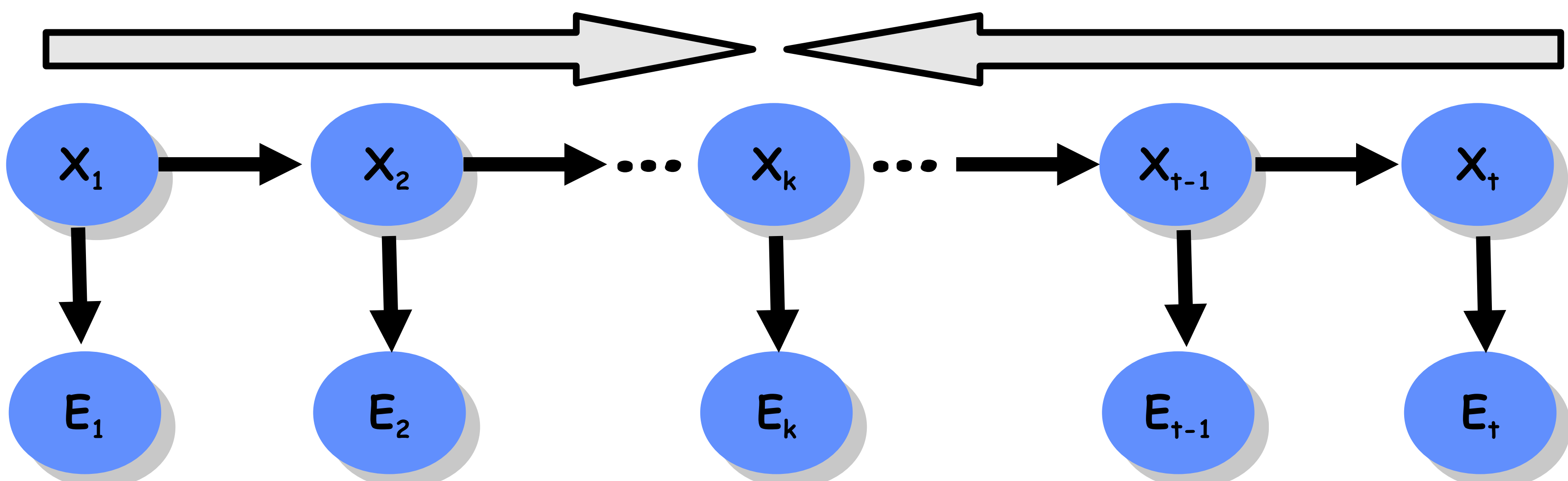
$$P(e_{k+1:t}|X_k) = \sum_{x_{k+1}} P(x_{k+1}, e_{k+1:t}|X_k)$$
$$= \sum_{x_{k+1}} P(x_{k+1}|X_k) P(e_{k+1:t}|x_{k+1}, X_k)$$
$$= \sum_{x_{k+1}} P(x_{k+1}|X_k) P(e_{k+1}, e_{k+2:t}|x_{k+1})$$
$$= \sum_{x_{k+1}} P(x_{k+1}|X_k) P(e_{k+1}|x_{k+1}) \underline{P(e_{k+2:t}|x_{k+1})}$$

- and the first (last) step:

$$P(e_t|X_{t-1}) = \sum_{x_t} P(x_t, e_t|X_{t-1}) = \sum_{x_t} P(e_t|x_t, X_{t-1}) P(x_t|X_{t-1})$$
$$= \sum_{x_t} P(e_t|x_t) P(x_t|X_{t-1})$$

Back and forth

- ”Brute-force” smoothing of the whole sequence takes $O(t^2)$ time
- *Forward-backward* algorithm: O(t)
- Finding the most probable sequence works in the same manner (the Viterbi algorithm / Viterbi path)



Finding the most probable sequence

- Want to compute:

$$max_{X_1,\dots X_n} P(X_1,\dots,X_n|e_1,\dots,e_n)$$
$$=max_{X_n} max_{X_1,\dots,X_{n-1}} P(X_1,\dots,X_{n-1},X_n,e_1,\dots,e_n)$$

- Recursion:

$$max_{X_1,\dots,X_{n-1}} P(X_1,\dots,X_{n-1},X_n|e_1,\dots,e_n)=max_{X_1,\dots,X_{n-1}} P(X_1,\dots,X_{n-1},X_n,e_1,\dots,e_n)$$
$$=max_{X_1,\dots,X_{n-1}} P(e_n|X_n,X_1,\dots,X_{n-1},e_1,\dots,e_{n-1})P(X_n,X_1,\dots,X_{n-1},e_1,\dots,e_{n-1})$$
$$=max_{X_1,\dots,X_{n-1}} P(e_n|X_n)P(X_n|X_1,\dots,X_{n-1},e_1,\dots,e_{n-1})P(X_1,\dots,X_{n-1},e_1,\dots,e_{n-1})$$
$$=P(e_n|X_n)max_{X_{n-1}} P(X_n|X_{n-1})max_{X_1,\dots,X_{n-2}} P(X_1,\dots,X_{n-2},X_{n-1}|e_1,\dots,e_{n-1})$$

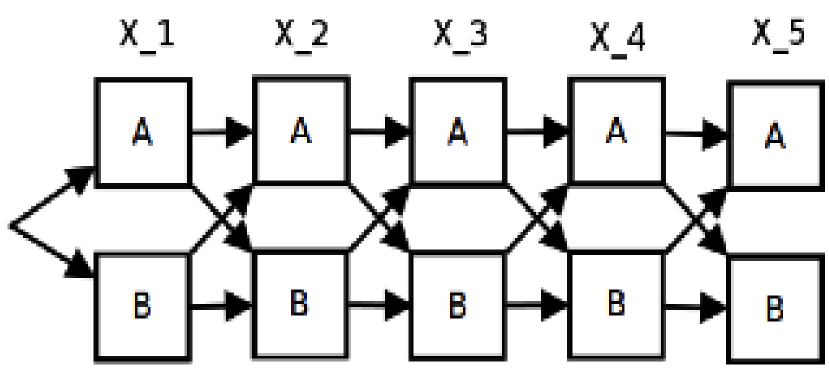
- More:
  - see e.g. Russel & Norvig, Chapter 15.2.

The Viterbi algorithm

Let $p(X,i) = max_{X_1,\ldots,X_{i-1}} P(X_1,\ldots,X_{i-1},X|e_1,\ldots,e_i)$

denote the probability of the most probable sequence of length i ending in state X.

$$p(X,1) = P(e_1|X)P(X) = P(e_1|X)\sum_{X_0}(P(X|X_o)P(X_0))$$
$$p(X,i) = P(e_i|X) max_Y [p(Y,i-1)P(X|Y)], for\, i>1.$$



The Viterbi algorithm

Let $p(X,i) = max_{X_1,\dots,X_{i-1}} P(X_1,\dots,X_{i-1},X|e_1,\dots,e_i)$

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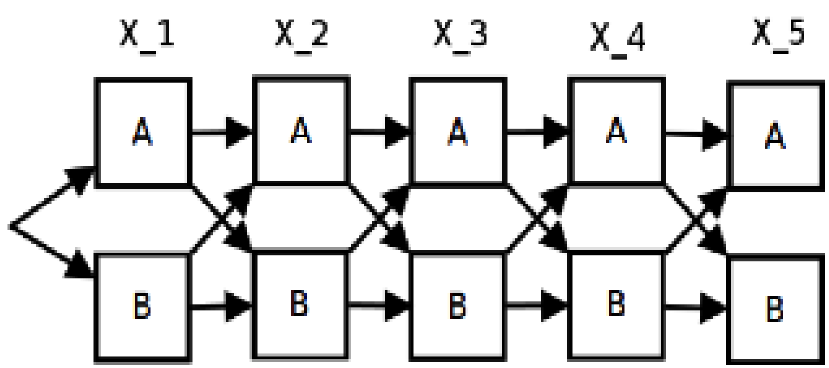
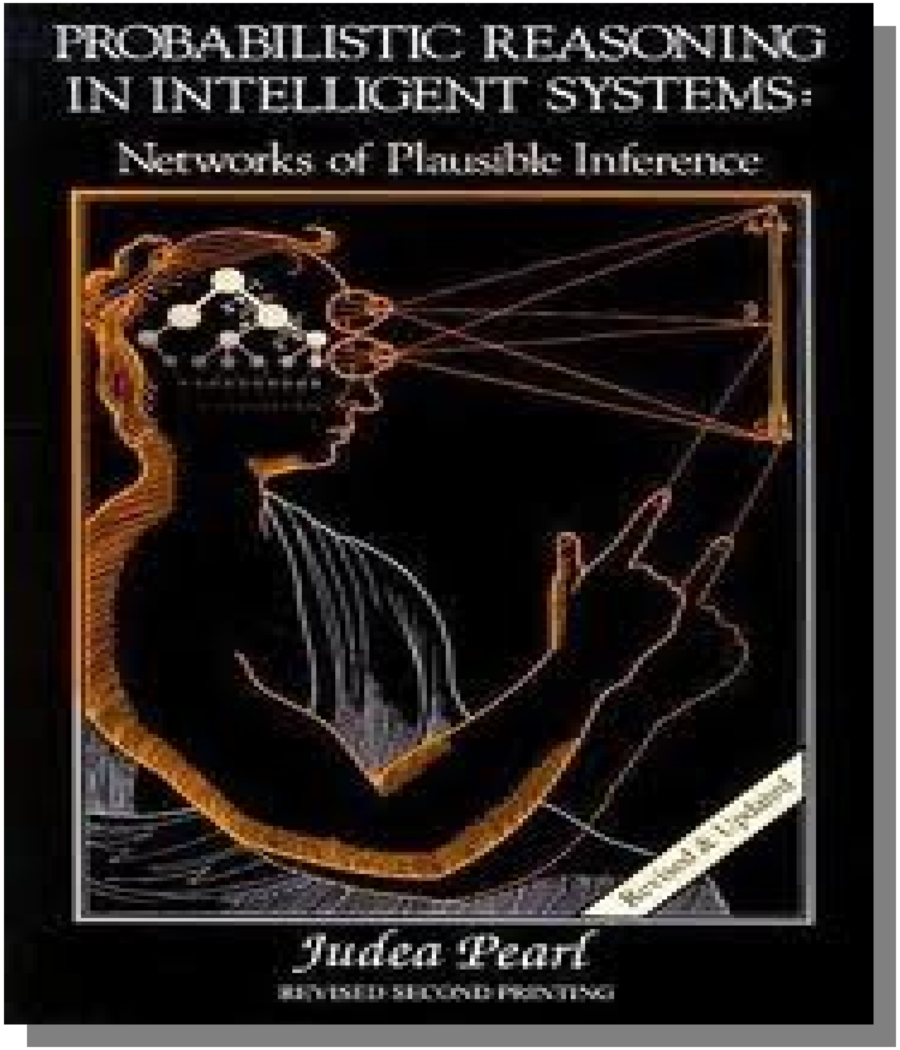


Probabilistic inference in DAGs

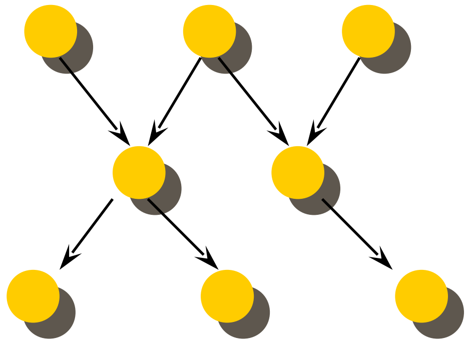


Types of inference

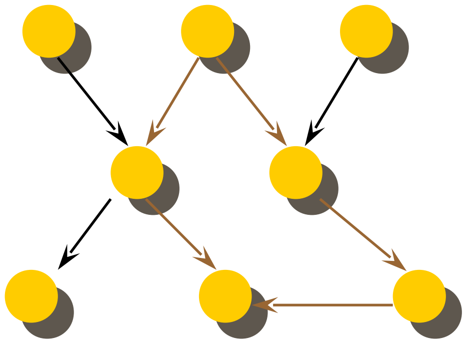
- Assume that both the structure of the model (the DAG), and the parameters (local probability tables) are fixed
- Recall the two types of inference task: either compute the conditional probability of a (set of) variables, given the values of others, or compute the maximum probability assignment
- Inference can be either exact or approximative

Exact inference in singly-connected BNs

- a singly connected BN = polytree (disregarding the arc directions, no two nodes can be connected with more than one path).

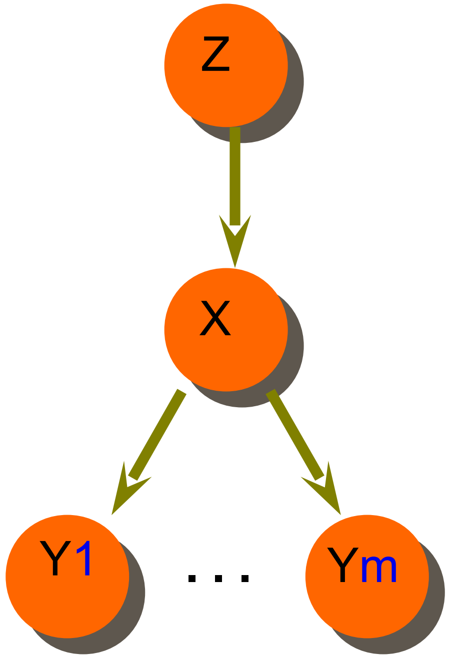


singly-connected



multi-connected

Probabilistic reasoning in singly-connected BNs



$$P(X|E) \propto P(X, E_+, E_-) \propto P(E_-|X) P(X|E_+)$$

$$P(E_-|X) = \prod_Y P(E_{Y-}|X)$$

$$P(E_{Y-}|X) = \sum_Y P(E_{Y-}|Y) P(Y|X)$$

$$P(X|E_+) = \sum_Z P(X|Z) P(Z|E_{Z+})$$

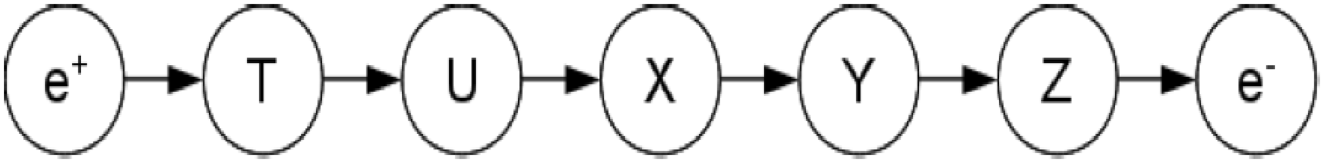
- a computationally efficient **message-passing** scheme: time requirement linear in the number of conditional probabilities in **θ**.

Belief propagation

- A message passing algorithm developed by Judea Pearl
- Computes the marginal distribution of an unobserved variable given the observed ones
- Each node maintains a *belief* of its state (the conditional probability distribution, given the evidence)
- Nodes pass messages to their neighbors and update their beliefs based on received messages

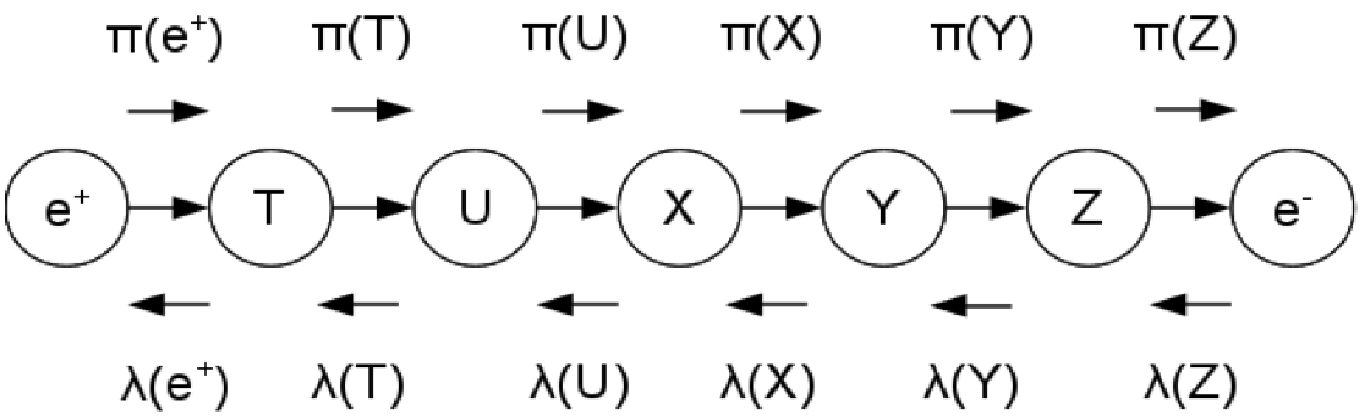
Belief propagation in chains

- A node can have at most one parent and child, no loops.
- We want to compute the marginal probability P(X | e), where the evidence e is an instantiation of node set E.
- Let us partition the evidence e into evidence from "upstream" e^+ and evidence from "downstream" e^-.



$$
\begin{aligned}
P(X \mid e) &= P(X \mid e^+, e^-) \\
&\propto P(e^- \mid X, e^+) P(X \mid e^+) \\
&= P(e^- \mid X) P(X \mid e^+)
\end{aligned}
$$

Message passing in chains



$$
\begin{aligned}
\lambda(U=u) &= P(e^- \mid U=u) \\
&= \sum_x P(e^- \mid X=x) P(X=x \mid U=u) \\
&= \sum_x \lambda(X=x) P(X=x \mid U=u)
\end{aligned}
$$

$$
\begin{aligned}
\pi(X=x) &= P(X=x \mid e^+) \\
&= \sum_u P(X=x \mid U=u) P(U=u \mid e^+) \\
&= \sum_u P(X=x \mid U=u) \pi(U=u)
\end{aligned}
$$

Initialization

- For nodes E with evidence e:

$$\lambda(E=e)=1, \textit{otherwise}\ \lambda(E=x)=0$$
$$\pi(E=e)=1, \textit{otherwise}\ \pi(E=x)=0$$

- Nodes with no parents:

$$\pi(x)=P(x) \qquad (\textit{prior probabilities})$$

- Nodes with no children:

$$\lambda(x)=1, \textit{for all } x$$

Belief propagation in trees

- Every node has at most one parent.
- Differences compared to chains:
  - Each node must combine impacts of the λ-messages obtained from its children.
  - Each node should distribute a separate π-message to each of its children.

Message passing in trees

Initialization like with chains. Then (in any order):

- *Belief updating:*

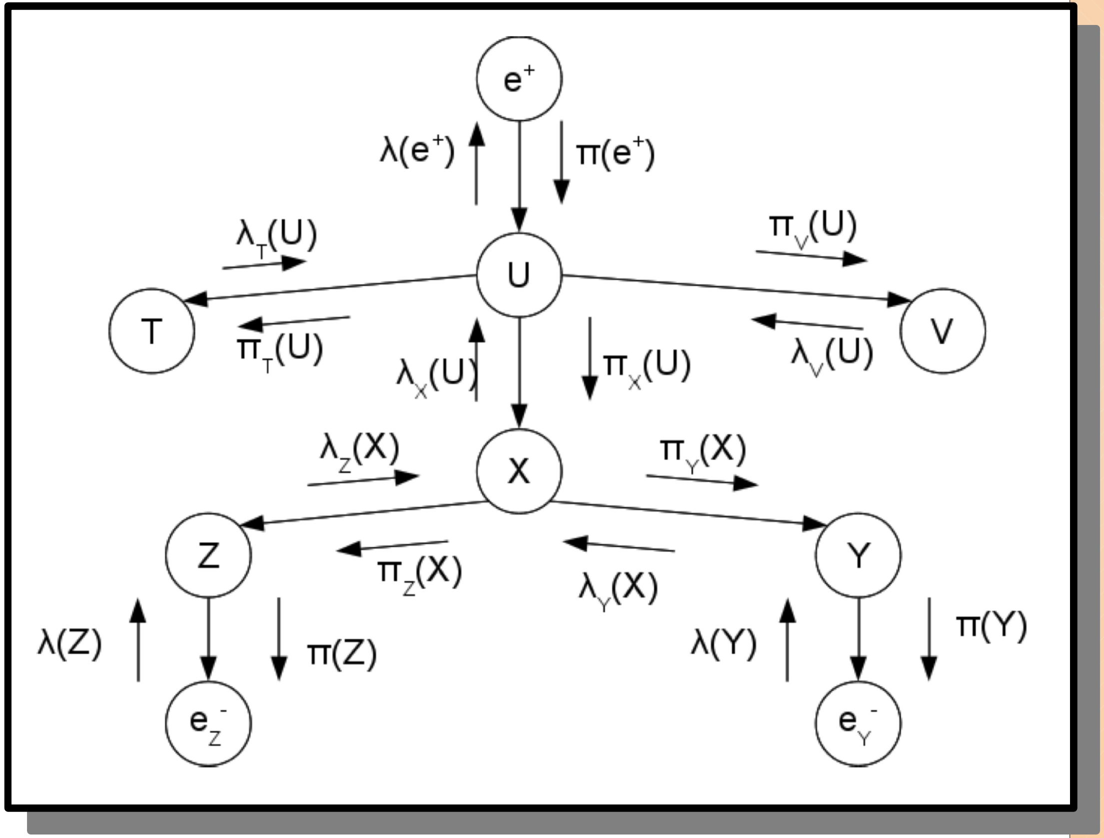
$$BEL(x) = P(x|e) \propto \lambda(x)\pi(x).$$
$$\lambda(x) = \prod_j \lambda_{Y_j}(x).$$
$$\pi(x) = \sum_u P(x|u)\pi_X(u).$$

- *Bottom-up propagation:*

$$\lambda_X(u) = \sum_x \lambda(x) P(x|u).$$

- *Top-down propagation:*

$$\pi_{Y_j}(x) \propto \pi(x) \prod_{k \neq j} \lambda_{Y_k}(x).$$

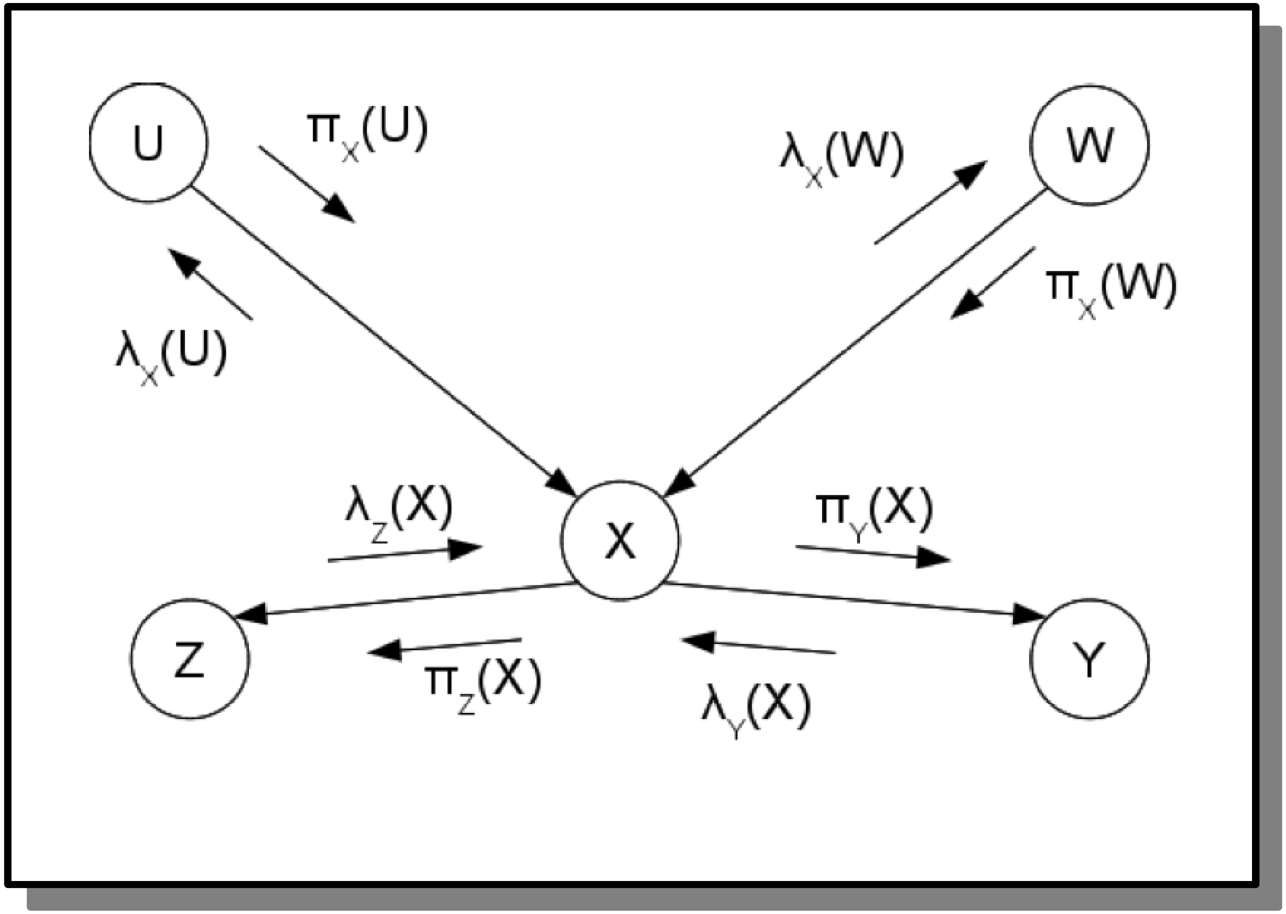


Belief propagation in polytrees

- Nodes can have multiple parents
- No loops
- Differences compared to trees:
  - Each node must combine impacts of the π-messages obtained from its parents.
  - Each node should distribute a separate λ-message to each of its parents.

Message passing in polytrees

- For details, see e.g. Neapolitan (Chapter 3.2.), or Pearl (Chapter 4.2.)

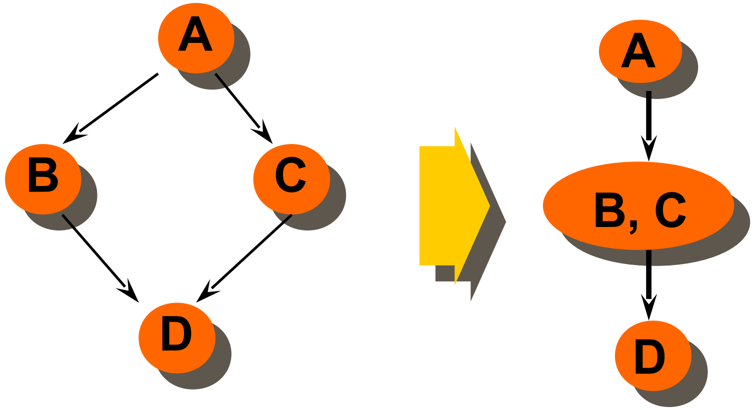


Complexity

- Number of messages sent depends linearly on the diameter of the network
- The time needed to compute a message is linear with respect to the size of the local probability table
  - But note that this means that the time (and size) is exponential with respect to the number of parents!
- The message-passing algorithm does not work with multi-connected networks

Probabilistic reasoning in multi-connected BNs

- Generally not computationally feasible as the problem has been shown to be NP-hard (Cooper 1990, Shimony 1994).
- Exact methods:
  - clustering
  - conditioning
  - variable elimination

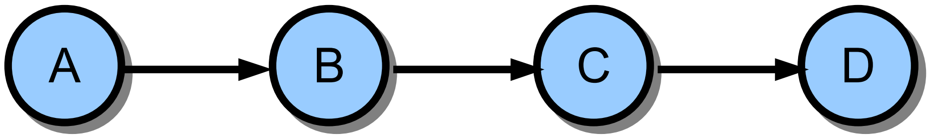


- Approximative methods:
  - stochastic sampling algorithms
  - loopy belief propagation
- Even approximative inference (both in terms of absolute and relative error) is NP-hard

Variable elimination

- Idea: eliminate (marginalize) one variable at a time
- Usually, each step depends on a limited number of variables only
- Time (and space) complexity of the algorithm depends on the structure of the network, and on the elimination order

Variable elimination: a simple example



$$P(D) = \sum_{A,B,C} P(A,B,C,D)$$

$$= \sum_C \sum_B \sum_A P(A)P(B|A)P(C|B)P(D|C)$$

$$= \sum_C \sum_B P(C|B)P(D|C) \sum_A P(A)P(B|A)$$

$$= \sum_C P(D|C) \sum_B P(C|B) \sum_A P(A)P(B|A)$$

Approximate inference in Bayesian networks

- How to estimate how probably it rains next day, if the previous night temperature is above the month average?
  - count rainy and non rainy days after warm nights (and count relative frequencies).
- Rejection sampling for P(**X**|**e**) :
  1. Generate random vectors ($\mathbf{x}_r$,$\mathbf{e}_r$,$\mathbf{y}_r$).
  2. Discard those those that do not match **e**.
  3. Count frequencies of different $\mathbf{x}_r$ and normalize.

Rejection sampling, bad news

- Good news first:
  - super easy to implement
- Bad news:
  - if evidence **e** is improbable, generated random vectors seldom conform with **e**, thus it takes a long time before we get a good estimate P(**X**|**e**).
  - With long **E**, all **e** are improbable.
- So called likelihood weighting can alleviate the problem a little bit, but not enough.

Gibbs sampling

- A **Markov Chain Monte Carlo (MCMC)** method that approximates the probability distribution by sampling from a ”cleverly” selected Markov Chain
- Given a Bayesian network for n variables **X** U **E** U **Y,** calculate P(**X**|**e**) as follows:

```
N = (associative) array of zeros
Generate random vector x,y.
While not enough samples:
   for V in X,Y:
      generate v from P(V | MarkovBlanket(V))
      replace v in x,y.
      N[x] +=1
      print normalize(N[x])
```

Sampling from the Markov blanket

$$
\begin{aligned}
& P(X|mb(X)) \\
& \quad = P(X|mb(x), Rest) \\
& \quad = \frac{P(X, mb(X), Rest)}{P(mb(X), Rest)} \\
& \quad \propto P(All) \\
& \quad = \prod_{X_i \in \mathbf{X}} P(X_i|Pa(X_i)) \\
& \quad = P(X|Pa(X)) \prod_{C \in ch(X)} P(C|Pa(C)) \prod_{R \notin \{X \cup ch(X)\}} P(R|Pa(R)) \\
& \quad \propto P(X|Pa(X)) \prod_{C \in ch(X)} P(C|Pa(C))
\end{aligned}
$$

Why does it work

- All decent Markov Chains have a unique stationary distribution P* that can be estimated by simulation.
- Detailed balance of transition function q and state distribution P* implies stationarity of P*.
- Proposed q = P(V|mb(V)), and P(**X**|**e**) form a detailed balance, thus P(**X**|**e**) is a stationary distribution, so it can be estimated by simulation.

Markov Chains: stationary distribution

- Defined by transition probabilities $q(x \rightarrow x')$ between states, where x and x' belong to a set of states X.
- Distribution P* over X is called stationary distribution for the Markov Chain q, if $P^*(x')=\sum_x P^*(x)q(x \rightarrow x')$.
- P*(X) can be found out by simulating Markov Chain q starting from a random state x_r.

Markov Chains: detailed balance

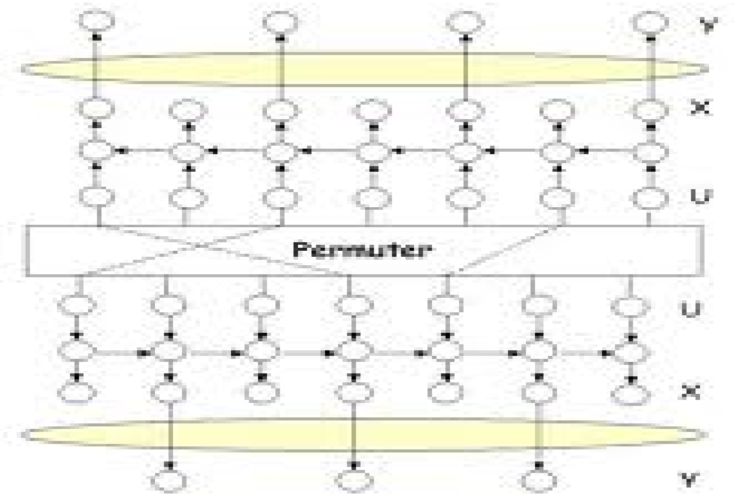
- Distribution P over X and a state transition distribution q are said to form a detailed balance, if for any states x and x', $P(x)q(x \to x') = P(x')q(x' \to x)$, i.e. it is equally probable to witness transition from x to x' as it is to witness transition from x' to x.
- If P and q form a detailed balance, $\sum_x P(x)q(x \to x') = \sum_x P(x')q(x' \to x) = P(x')\sum_x q(x' \to x) = P(x')$, thus P is stationary.

Gibbs sampler as Markov Chain

- Consider **Z**=(**X**,**Y**) to be states of a Markov chain, and $q((v,\mathbf{z}_{-V}))\rightarrow(v',\mathbf{z}_{-V}))=P(v'|\mathbf{z}_{-V}, \mathbf{e})$, where $\mathbf{Z}_{-V} = \mathbf{Z}-\{V\}$. Now $P^*(\mathbf{Z})=P(\mathbf{Z}|\mathbf{e})$ and q form a detailed balance, thus P* is a stationary distribution of q and it can be found with the sampling algorithm.
  - $P^*(\mathbf{z})q(\mathbf{z}\rightarrow\mathbf{z}') = P(\mathbf{z}|\mathbf{e})P(v'|\mathbf{z}_{-V}, \mathbf{e})$
    $= P(v,\mathbf{z}_{-V}|\mathbf{e})P(v'|\mathbf{z}_{-V}, \mathbf{e})$
    $= P(v|\mathbf{z}_{-V},\mathbf{e})P(\mathbf{z}_{-V}|\mathbf{e})P(v'|\mathbf{z}_{-V}, \mathbf{e})$
    $= P(v|\mathbf{z}_{-V},\mathbf{e})P(v', \mathbf{z}_{-V}|\mathbf{e}) = q(\mathbf{z}'\rightarrow\mathbf{z})P^*(\mathbf{z}')$, thus balance.

Loopy belief propagation

- What happens if you just keep iterating the message passing algorithm in a multi-connected network?
- In some cases it produces the right results, or at least a good approximation
- Turbo codes



So let us play….