Probabilistic Models: Spring 2013
Exercise session 4 (exercises 14–18)

Instructions: All course participants are requested to submit their exercise solutions as follows:

- Deadline: before 12 o'clock noon of the Wednesday when the corresponding exercise session will be held
- Submission as a PDF file by email to joonas.paalasmaa at helsinki.fi
(cc: to petri.myllymaki at cs.helsinki.fi)
- If you write the solutions by hand, please scan your paper. However, we strongly recommend that you type your solution by using a word processor. LaTeX is of course especially suitable for typesetting math, and it also has a convenient front-end LyX.
- Use as the title of your paper and subject of the email:
"ProMo-2013, Exercise session n, yourlastname"
- Use as the file name:
"ProMo-2013, Exercise session n, yourlastname.pdf"
- In all the exercises, do not just give the answer, but also the derivation of how you obtained it.
- Participants are encouraged to write computer programs to derive solutions whenever appropriate. In this case, please enclose the program source code too as a separate file.

After the exercise session, you are allowed to send a modified version of one of the solutions you sent before the exercise session:

- You can only modify a solution that you submitted before the 12 o'clock deadline.
- You can only send a modified version for one solution, and only if you attended the exercise session.
- Deadline: midnight after the exercise session.
- Submission as before, but this time please send only the modified solution, not the other (unchanged) solutions. Enclose the original solution first and then continue with the new material: first explain what you did wrong in the first time and then continue with modifications.
- As the title of the submission, please use:
"ProMo-2013, Exercise session n, yourlastname, modified exercise x."

14. **Naive Bayes**. Let us consider the following Naive Bayes classifier: the hidden node Y has three possible values 1, 2 and 3, and the four predictors X_1, X_2, X_3 and X_4 are binary, and the parameters are as follows:

$$
\begin{aligned}
P(Y=1) &= 0.6 \\
P(Y=2) &= 0.3 \\
P(Y=3) &= 0.1 \\
P(X_1=1|Y=1) &= 0.6 \\
P(X_2=1|Y=1) &= 0.1 \\
P(X_3=1|Y=1) &= 0.1 \\
P(X_4=1|Y=1) &= 0.9 \\
P(X_1=1|Y=2) &= 0.3 \\
P(X_2=1|Y=2) &= 0.9 \\
P(X_3=1|Y=2) &= 0.7 \\
P(X_4=1|Y=2) &= 0.2 \\
P(X_1=1|Y=3) &= 0.95 \\
P(X_2=1|Y=3) &= 0.6 \\
P(X_3=1|Y=3) &= 0.3 \\
P(X_4=1|Y=3) &= 0.4
\end{aligned}
$$

a) Compute the classification distribution $P(Y|X_1, X_2, X_3, X_4)$ for each of the 16 cases (X_1, X_2, X_3, X_4).

b) Compute the conditional distributions $P(X_1, X_2, X_3, X_4|Y)$ for each value of Y (3 cases, 16 values each).

c) Compute probability $P(X_1, X_2, X_3, X_4)$ for each of the 16 cases (by marginalizing over Y).

d) What is $P(X_1|X_2 = 0, X_3 = 0)$?

15.-16. **Location estimation with a HMM**. Mobile devices can be located indoors using WiFi positioning. The mobile device measures signal strengths received from different WiFi access points ("fingerprints"). The observed signal strengths depend (to some degree) on the location. However, the environment is constantly changing (people moving etc.) and thus the signal strengths at a given location vary over time.

Assume that we have 4 possible locations labeled A, B, C and D. Our observations consist of measurements of signal strengths X and Y with possible values $\{low, medium, high\}$. We model this domain by a first-order HMM where the location is the hidden variable, and X and Y are assumed to be independent given the location. The transition probabilities between the hidden state L_t at time t and the hidden state L_{t+1} at time $t+1$ are as follows:

$$
\begin{aligned}
P(L_{t+1}=A|L_t=A) &= 1/3 \\
P(L_{t+1}=B|L_t=A) &= 1/3 \\
P(L_{t+1}=C|L_t=A) &= 1/3 \\
P(L_{t+1}=B|L_t=B) &= 1/3 \\
P(L_{t+1}=A|L_t=B) &= 1/3 \\
P(L_{t+1}=D|L_t=B) &= 1/3
\end{aligned}
$$

$$
\begin{aligned}
P(L_{t+1} = C | L_t = C) &= 1/3 \\
P(L_{t+1} = A | L_t = C) &= 1/3 \\
P(L_{t+1} = D | L_t = C) &= 1/3 \\
P(L_{t+1} = D | L_t = D) &= 1/3 \\
P(L_{t+1} = B | L_t = D) &= 1/3 \\
P(L_{t+1} = C | L_t = D) &= 1/3
\end{aligned}
$$

The emission probabilities of X and Y are as follows:

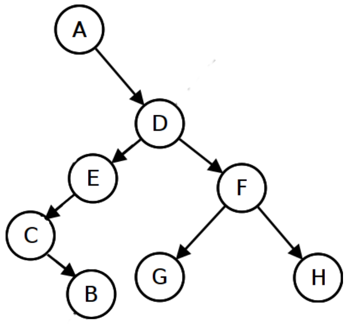
$$
\begin{aligned}
P(X = low | L = A) &= 0.1 \\
P(X = medium | L = A) &= 0.2 \\
P(X = high | L = A) &= 0.7 \\
P(X = low | L = B) &= 0.3 \\
P(X = medium | L = B) &= 0.4 \\
P(X = high | L = B) &= 0.3 \\
P(X = low | L = C) &= 0.3 \\
P(X = medium | L = C) &= 0.4 \\
P(X = high | L = C) &= 0.3 \\
P(X = low | L = D) &= 0.7 \\
P(X = medium | L = D) &= 0.2 \\
P(X = high | L = D) &= 0.1 \\
P(Y = low | L = A) &= 0.7 \\
P(Y = medium | L = A) &= 0.2 \\
P(Y = high | L = A) &= 0.1 \\
P(Y = low | L = B) &= 0.3 \\
P(Y = medium | L = B) &= 0.4 \\
P(Y = high | L = B) &= 0.3 \\
P(Y = low | L = C) &= 0.2 \\
P(Y = medium | L = C) &= 0.3 \\
P(Y = high | L = C) &= 0.5 \\
P(Y = low | L = D) &= 0.1 \\
P(Y = medium | L = D) &= 0.2 \\
P(Y = high | L = D) &= 0.7
\end{aligned}
$$

We make the following observations (X, Y) during 6 time steps: $O_1 = (high, high)$, $O_2 = (medium, medium)$, $O_3 = (low, high)$, $O_4 = (low, medium)$, $O_5 = (high, medium)$, $O_6 = (medium, medium)$. Let us assume that a priori (before making the observations) all states have the following probabilities: $P(L_0 = A) = 0.2$, $P(L_0 = B) = 0.3$, $P(L_0 = C) = 0.3$ and $P(L_0 = D) = 0.2$. (Hint: treat the pair (X, Y) as a single variable)

a) Compute the smoothed distribution over the hidden states at each moment of time, i.e., $P(L_1 | O_1, \ldots, O_6), P(L_2 | O_1, \ldots, O_6), \ldots$, and $P(L_6 | O_1, \ldots, O_6)$.

b) Use the Viterbi algorithm to compute the most probable path between the hidden states $L_1 \rightarrow L_2 \rightarrow \ldots \rightarrow L_6$. What is the path and what is its probability?

17.-18. **Belief propagation**. We are given a Bayesian network whose structure is the following DAG:



All variables are binary and conditional probabilities are as follows:

$$
\begin{aligned}
P(A=0) &= 0.3 \\
P(D=0|A=0) &= 0.9 \\
P(D=0|A=1) &= 0.7 \\
P(E=0|D=0) &= 0.6 \\
P(E=0|D=1) &= 0.8 \\
P(C=0|E=0) &= 0.3 \\
P(C=0|E=1) &= 0.9 \\
P(B=0|C=0) &= 0.4 \\
P(B=0|C=1) &= 0.9 \\
P(F=0|D=0) &= 0.5 \\
P(F=0|D=1) &= 0.7 \\
P(G=0|F=0) &= 0.4 \\
P(G=0|F=1) &= 0.7 \\
P(H=0|F=0) &= 0.8 \\
P(H=0|F=1) &= 0.1
\end{aligned}
$$

Compute $P(B|A=0,H=0)$, $P(C|A=0,H=0)$, $P(D|A=0,H=0)$, $P(E|A=0,H=0)$, $P(F|A=0,H=0)$, and $P(G|A=0,H=0)$ using belief propagation.