Probabilistic models, Spring 2013
Exercise 1: Solutions

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1. a) The requirement for the probability table is that $P(X,Y) = P(X)P(Y)$ holds for all values of X and Y. One such table is shown below.

	X	$\neg X$
Y	$1/4$	$1/4$
$\neg Y$	$1/4$	$1/4$

b) A probability table where the independence criterion does not hold is shown below.

	X	$\neg X$
Y	$1/2$	0
$\neg Y$	0	$1/2$

c) We try to find a counterexample, by specifying a distribution $P(X,Y,Z)$ so that $P(X,Y) = P(X)P(Y)$ and $P(Y,Z) = P(Y)P(Z)$ but $P(X,Z) \neq P(X)P(Z)$.

The probability table below satisfies these criteria.

	Z		$\neg Z$	
	Y	$\neg Y$	Y	$\neg Y$
X	$1/4$	$1/4$	0	0
$\neg X$	0	0	$1/4$	$1/4$

That is proved by calculating the probability tables of the three marginal distributions $P(X,Y)$, $P(Y,Z)$ and $P(X,Z)$:

	Y	$\neg Y$
Z	$1/4$	$1/4$
$\neg Z$	$1/4$	$1/4$

	X	$\neg X$
Y	$1/4$	$1/4$
$\neg Y$	$1/4$	$1/4$

	Z	$\neg Z$
X	$1/2$	0
$\neg X$	0	$1/2$

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2. a) The marginal distribution of X:

$P(X=0) = P(X=0,Y=0) + P(X=0,Y=1) + P(X=0,Y=2) = \frac{1}{16} + \frac{2}{16} + 0 = \frac{3}{16} \quad (\approx 0.188)$
$P(X=1) = P(X=1,Y=0) + P(X=1,Y=1) + P(X=1,Y=2) = \frac{1}{16} + \frac{3}{16} + \frac{1}{16} = \frac{5}{16} \quad (\approx 0.312)$
$P(X=2) = P(X=2,Y=0) + P(X=2,Y=1) + P(X=2,Y=2) = \frac{1}{16} + \frac{3}{16} + \frac{2}{16} = \frac{6}{16} \quad (\approx 0.375)$
$P(X=3) = P(X=3,Y=0) + P(X=3,Y=1) + P(X=3,Y=2) = 0 + \frac{1}{16} + \frac{1}{16} = \frac{2}{16} \quad (\approx 0.125)$

The marginal distribution of Y:

$P(Y=0) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + 0 = \frac{3}{16} \quad (\approx 0.188)$
$P(Y=1) = \frac{2}{16} + \frac{3}{16} + \frac{3}{16} + \frac{1}{16} = \frac{9}{16} \quad (\approx 0.562)$
$P(Y=2) = 0 + \frac{1}{16} + \frac{2}{16} + \frac{1}{16} = \frac{4}{16} \quad (\approx 0.250)$

b) Conditional distribution $P(X|Y=1)$:

$$P(X=0|Y=1) = \frac{P(X=0,Y=1)}{P(Y=1)} = \frac{2/16}{9/16} = \frac{2}{9}$$
$$P(X=1|Y=1) = \frac{P(X=1,Y=1)}{P(Y=1)} = \frac{3/16}{9/16} = \frac{1}{3}$$
$$P(X=2|Y=1) = \frac{P(X=2,Y=1)}{P(Y=1)} = \frac{3/16}{9/16} = \frac{1}{3}$$
$$P(X=3|Y=1) = \frac{P(X=3,Y=1)}{P(Y=1)} = \frac{1/16}{9/16} = \frac{1}{9}$$

c) Conditional distribution $P(Y|X=2)$:

$$P(Y=0|X=2) = \frac{P(X=2,Y=0)}{P(X=2)} = \frac{1/16}{6/16} = \frac{1}{6}$$

$$P(Y=1|X=2) = \frac{P(X=2,Y=1)}{P(X=2)} = \frac{3/16}{6/16} = \frac{1}{2}$$
$$P(Y=2|X=2) = \frac{P(X=2,Y=2)}{P(X=2)} = \frac{2/16}{6/16} = \frac{1}{3}$$

d) X and Y are not independent, because for example $P(X=0)P(Y=0) = \frac{9}{256} \neq \frac{1}{16} = P(X=0,Y=0))$

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3. a) $P(Y|X)$:

	$X=0$	$X=1$	$X=2$	$X=3$
$Y=0$	$1/3$	$1/5$	$1/6$	0
$Y=1$	$2/3$	$3/5$	$3/6$	$1/2$
$Y=2$	0	$1/5$	$2/6$	$1/2$

$P(X)$:

$X=0$	$X=1$	$X=2$	$X=3$
$3/16$	$5/16$	$6/16$	$2/16$

b) $P(X|Y)$:

	$X=0$	$X=1$	$X=2$	$X=3$
$Y=0$	$1/3$	$1/3$	$1/3$	0
$Y=1$	$2/9$	$3/9$	$3/9$	$1/9$
$Y=2$	0	$1/4$	$2/4$	$1/4$

$P(Y)$:

$Y=0$	$Y=1$	$Y=2$
$3/16$	$9/16$	$4/16$

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4. Let's define the following events:
A = "the person in question is Alice"
W = "the person in question has a white car"
S = "the person in question likes sushi"

If Sally does not know if something is true or not (and no explicit probability was given) we will assume the prior probability of 0.5. Thus we get $P(A) = 0.5$, $P(W|A) = 0.8$, $P(W|\neg A) = 0.5$, $P(S|\neg A) = 0.9$, $P(S|A) = 0.5$. In addition, as hinted by "…knows some independent pieces of information …", we assume, that our prior beliefs concerning the car color and sushi are independent for both persons, that is, $W \perp S \mid A$ and $W \perp S \mid \neg A$. Now we get

$$P(A|\neg S, W) = \frac{P(A, \neg S, W)}{P(\neg S, W)} = \frac{P(A, \neg S, W)}{P(A, \neg S, W) + P(\neg A, \neg S, W)}$$

This means that we only need to be able to compute the joint distribution values $P(A, \neg S, W)$ and $P(\neg A, \neg S, W)$. Thanks to the chain rule, e.g. $P(A, \neg S, W)$ can be formulated as $P(A, \neg S, W) = P(A)P(\neg S|A)P(W|A, \neg S)$, which simplifies to $P(A, \neg S, W) = P(A)P(\neg S|A)P(W|A)$, because $W \perp S \mid A$.

Using the above expression the solution is calculated as below.

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\begin{aligned}
\frac{P(A,\neg S,W)}{P(A,\neg S,W)+P(\neg A,\neg S,W)} &= \frac{P(A)P(\neg S|A)P(W|A)}{P(A)P(\neg S|A)P(W|A)+P(\neg A)P(\neg S|\neg A)P(W|\neg A)} \\
&= \frac{P(A)(1-P(S|A))P(W|A)}{P(A)(1-P(S|A))P(W|A)+(1-P(A))(1-P(S|\neg A))P(W|\neg A)} \\
&= \frac{0.5\cdot(1-0.5)\cdot 0.8}{0.5\cdot(1-0.5)\cdot 0.8+(1-0.5)\cdot(1-0.9)\cdot 0.5} = 0.888\ldots
\end{aligned}
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