Probabilistic models, Spring 2013
Exercise 1: Solutions

1. a) The requirement for the probability table is that $P(X, Y) = P(X)P(Y)$ holds for all values of $X$ and $Y$. One such table is shown below.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$\neg X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>$1/4$</td>
</tr>
<tr>
<td>$\neg Y$</td>
<td>$1/4$</td>
</tr>
</tbody>
</table>

b) A probability table where the independence criterion does not hold is shown below.

<table>
<thead>
<tr>
<th>$X$</th>
<th>$\neg X$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y$</td>
<td>$1/2$</td>
</tr>
<tr>
<td>$\neg Y$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

b) Conditional distribution $P(X|Y)$:

- $P(X = 0|Y = 1) = \frac{P(X = 0, Y = 1)}{P(Y = 1)} = \frac{2/16}{9/16} = \frac{2}{9}$
- $P(X = 1|Y = 1) = \frac{P(X = 1, Y = 1)}{P(Y = 1)} = \frac{3/16}{9/16} = \frac{1}{3}$
- $P(X = 2|Y = 1) = \frac{P(X = 2, Y = 1)}{P(Y = 1)} = \frac{3/16}{9/16} = \frac{1}{3}$
- $P(X = 3|Y = 1) = \frac{P(X = 3, Y = 1)}{P(Y = 1)} = \frac{1/16}{9/16} = \frac{1}{9}$

2. a) The marginal distribution of $X$:

- $P(X = 0) = P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 0, Y = 2) = \frac{1}{16} + \frac{2}{16} + 0 = \frac{3}{16} \approx 0.188$
- $P(X = 1) = P(X = 1, Y = 0) + P(X = 1, Y = 1) + P(X = 1, Y = 2) = \frac{1}{16} + \frac{3}{16} + \frac{1}{16} = \frac{5}{16} \approx 0.312$
- $P(X = 2) = P(X = 2, Y = 0) + P(X = 2, Y = 1) + P(X = 2, Y = 2) = \frac{1}{16} + \frac{3}{16} + \frac{2}{16} = \frac{6}{16} \approx 0.375$
- $P(X = 3) = P(X = 3, Y = 0) + P(X = 3, Y = 1) + P(X = 3, Y = 2) = 0 + \frac{1}{16} + \frac{1}{16} = \frac{2}{16} \approx 0.125$

The marginal distribution of $Y$:

- $P(Y = 0) = \frac{1}{16} + \frac{1}{16} + 0 = \frac{3}{16} \approx 0.188$
- $P(Y = 1) = \frac{3}{16} + \frac{3}{16} + \frac{1}{16} = \frac{7}{16} \approx 0.562$
- $P(Y = 2) = 0 + \frac{1}{16} + \frac{2}{16} = \frac{3}{16} \approx 0.250$

b) Conditional distribution $P(Y|X)$:

- $P(Y = 0|X = 1) = \frac{P(Y = 0, X = 1)}{P(X = 1)} = \frac{\frac{2}{16}}{\frac{3}{16}} = \frac{2}{3}$
- $P(Y = 1|X = 1) = \frac{P(Y = 1, X = 1)}{P(X = 1)} = \frac{\frac{1}{16}}{\frac{3}{16}} = \frac{1}{3}$
- $P(Y = 2|X = 1) = \frac{P(Y = 2, X = 1)}{P(X = 1)} = \frac{\frac{1}{16}}{\frac{3}{16}} = \frac{1}{3}$
- $P(Y = 0|X = 3) = \frac{P(Y = 0, X = 3)}{P(X = 3)} = \frac{\frac{1}{16}}{\frac{2}{16}} = \frac{1}{2}$
- $P(Y = 1|X = 3) = \frac{P(Y = 1, X = 3)}{P(X = 3)} = \frac{\frac{1}{16}}{\frac{2}{16}} = \frac{1}{2}$
- $P(Y = 2|X = 3) = \frac{P(Y = 2, X = 3)}{P(X = 3)} = \frac{\frac{1}{16}}{\frac{2}{16}} = \frac{1}{2}$
Let’s define the following events:

- $W$: "the person in question has a white car"
- $S$: "the person in question likes sushi"
- $A$: "the person in question is Alice"

**Using the above expression the solution is calculated as below.**

The prior beliefs concerning the car color and sushi are independent for both persons, that is, $W \perp S | A$. This means that we only need to be able to compute the joint distribution values $P(A, \neg S, W)$ and $P(\neg A, \neg S, W)$. Thanks to the chain rule, e.g., $P(A, \neg S, W)$ can be formulated as $P(A, \neg S, W) = P(A)P(\neg S | A)P(W | A, \neg S)$, which simplifies to $P(A, \neg S, W) = P(A)P(\neg S | A)P(W | A)$, because $W \perp S | A$.

Using the above expression the solution is calculated as below.
\[
\frac{P(A, \neg S, W)}{P(A, \neg S, W) + P(\neg A, \neg S, W)} = \frac{P(A)P(\neg S|A)P(W|A)}{P(A)P(\neg S|A)P(W|A) + P(\neg A)P(\neg S|\neg A)P(W|\neg A)}
\]
\[
= \frac{P(A)(1 - P(S|A))P(W|A) + (1 - P(A))(1 - P(S|\neg A))P(W|\neg A)}{0.5 \cdot (1 - 0.5) \cdot 0.8}
\]
\[
= 0.5 \cdot 0.8 + (1 - 0.5) \cdot (1 - 0.9) \cdot 0.5 = 0.888\ldots
\]