Probabilistic models, Spring 2013
Exercise session 3: Solutions

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10. a) We want to prove that any node X is conditionally independent of any other node Y given its Markov blanket $MB(X)$ (assuming $Y \notin \{X\} \cup MB(X)$). Let's look at an arbitrary trail from X to Y. We divide in three separate cases (in the following "$-$" means arc with any direction, that is, both "$\rightarrow$" or "$\leftarrow$" are allowed):

1) The trail is of form $X \leftarrow Z - \cdots - Y$. Node Z blocks the connection along the trail. Since $Z \in MB(X)$, X and Y are not d-connected by $MB(X)$ along the trail.
2) The trail is of form $X \rightarrow Z \rightarrow \cdots - Y$. Node Z blocks the connection along the trail. Since $Z \in MB(X)$, X and Y are not d-connected by $MB(X)$ along the trail.
3) The trail is of form $X \rightarrow Z \leftarrow W - \cdots - Y$. Node W blocks the connection along the trail. Since $W \in MB(X)$, X and Y are not d-connected by $MB(X)$ along the trail.

In all three cases the connection was blocked by $MB(X)$. Since the trail was arbitrary this holds for all trails. As X and Y are not d-connected by $MB(X)$ along any trail, they are d-separated by $MB(X)$. Therefore X is independent of Y given $MB(X)$.

b) $A \perp C \mid B$
$A \perp D \mid B$
$A \perp E \mid B$
$A \perp F \mid B$
$B \perp E \mid C$
$B \perp F \mid \{C, D\}$
$C \perp F \mid \{D, E\}$
$D \perp E \mid C$

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11. a) $A \perp B$
$A \perp D \mid C$
$A \perp D \mid \{B, C\}$
$B \perp D \mid C$
$B \perp D \mid \{A, C\}$

b) i) Based on the above independencies we get

$$\begin{aligned} P(A,B,C,D) &= P(A)\,P(B \mid A)\,P(C \mid A,B)\,P(D \mid A,B,C) \\ &= P(A)\,P(B)\,P(C \mid A,B)\,P(D \mid C). \end{aligned}$$

Using this we can calculate the following joint probabilities:

A	B	C	D	$P(A,B,C,D)$
0	0	0	0	0.07840
0	0	0	1	0.03360
0	0	1	0	0.01200
0	0	1	1	0.03600
0	1	0	0	0.00420
0	1	0	1	0.00180
0	1	1	0	0.00850
0	1	1	1	0.02550
1	0	0	0	**0.24640**
1	0	0	1	0.10560
1	0	1	0	0.07200
1	0	1	1	0.21600
1	1	0	0	0.03920
1	1	0	1	0.01680
1	1	1	0	0.02600
1	1	1	1	0.07800

ii) From the table we see that $\max P(A,B,C,D) = 0.24640$.

iii)
$P(A=0) = 0.2$
$P(A=1) = 0.8$

$P(B=0) = 0.8$
$P(B=1) = 0.2$

$P(C=0) = \sum_{a,b,d} P(A=a, B=b, C=0, D=d) \approx 0.5260$
$P(C=1) = \sum_{a,b,d} P(A=a, B=b, C=1, D=d) \approx 0.4740$

$P(D=0) = \sum_{a,b,c} P(A=a, B=b, C=c, D=0) \approx 0.4867$
$P(D=1) = \sum_{a,b,c} P(A=a, B=b, C=c, D=1) \approx 0.5133$

iv)
In general we can calculate all these from the joint probabilities calculated in i). For example:

$$P(A=1 \mid D=1) = \frac{P(A=1, D=1)}{P(D=1)} = \frac{\sum_{c,b} P(A=1, B=b, C=c, D=1)}{\sum_{a,c,d} P(A=a, B=b, C=c, D=1)} = \ldots \approx 0.8112$$

In many of these cases we can also get the result easier way. For example, since $A \perp B$:

$$P(A=1 \mid B=1) = P(A=1) = 0.8.$$

$P(A=1 \mid D=1) \approx 0.8112$
$P(B=1 \mid D=1) \approx 0.2379$
$P(A=1 \mid D=0) \approx 0.7882$
$P(B=1 \mid D=0) \approx 0.1601$
$P(A=1 \mid C=1) \approx 0.8270$
$P(B=1 \mid C=1) \approx 0.2911$
$P(A=0 \mid C=1) \approx 0.1730$
$P(B=0 \mid C=1) \approx 0.7089$
$P(C=1 \mid D=0) \approx 0.2435$
$P(C=1 \mid A=1) \approx 0.4900$
$P(D=1 \mid A=1) \approx 0.5205$
$P(A=1 \mid C=1, D=1) \approx 0.8270$
$P(B=1 \mid C=1, D=1) \approx 0.2911$
$P(A=1 \mid C=1, D=0) \approx 0.8270$
$P(B=1 \mid C=1, D=0) \approx 0.2911$
$P(A=1 \mid C=1, B=1) \approx 0.7536$
$P(A=1 \mid C=1, B=0) \approx 0.8571$
$P(A=1 \mid B=1) \approx 0.8000$
$P(A=1 \mid B=0) \approx 0.8000$

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12. We get the following equivalence classes:

Class	Networks
1	$\{\}$
2	$\{XY\}$, $\{YX\}$
3	$\{XZ\}$, $\{ZX\}$
4	$\{YZ\}$, $\{ZY\}$
5	$\{XY,YZ\}$, $\{ZY,YX\}$, $\{YX,YZ\}$
6	$\{XZ,ZY\}$, $\{YZ,ZX\}$, $\{ZX,ZY\}$
7	$\{YX,XZ\}$, $\{ZX,XY\}$, $\{XY,XZ\}$
8	$\{XY,ZY\}$
9	$\{XZ,YZ\}$
10	$\{YX,ZX\}$
11	$\{XY,YZ,XZ\}$, $\{YX,YZ,XZ\}$, $\{XZ,ZY,XY\}$, $\{ZX,ZY,XY\}$, $\{YZ,ZX,YX\}$, $\{ZY,ZX,YX\}$

Total: 25 networks in 11 equivalence classes.

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13. a) Let's assume that G is a DAG (implicit assumption). To prove that G and G' are Markov equivalent we need to show that 1) they have the same skeleton, 2) they have the same v-structures and 3) G' is a DAG.

1) Since we only reversed the arc $X \to Y$, the skeleton did not change.
2) We want to show that no v-structures were added or removed.
   - If a v-structure was removed, it needed to be of form $X \to Y \leftarrow Z$. But since $Z \in Pa_G(Y)$ and the arc $X \to Y$ is covered, $Z \in Pa_G(X)$, that is, there is an arc $Z \to X$. Therefore $X \to Y \leftarrow Z$ is not a v-structure in G, which is a contradiction. Thus no v-structures were removed.
   - Likewise, if a v-structure was added, it needed to be of form $Y \to X \leftarrow Z$. But since $Z \in Pa_G(X)$ and the arc $X \to Y$ is covered, $Z \in Pa_G(Y)$, that is, there is an arc $Z \to Y$. Therefore $Y \to X \leftarrow Z$ is not a v-structure in G', which is a contradiction. Thus no v-structures were added.

   Therefore G and G' have the same v-structures.
3) We need to show that there are not cycles in G'. If we had introduced a cycle by reversing the arc $X \to Y$, the cycle would need to go through the new arc $Y \to X$ and therefore be of form $X \to \cdots \to Z \to Y \to X$. But since arc $X \to Y$ was covered and $Z \in Pa_G(Y)$, there must also be an arc $Z \to X$ in G (and in G'). By using this arc as a shortcut on the above cycle we get another cycle $X \to \cdots \to Z \to X$. But since this shorter cycle does not contain arc between X and Z, it must have existed in the original network G. This is a contradiction with our assumption of G being a DAG, so G' must be acyclic.

b) Consider the following network G of three nodes: $X \to Y \leftarrow Z$. Now $Pa_G(Y) = \{X,Z\} \neq \{X\} = Pa_G(X) \cup \{X\}$, so the arc $X \to Y$ is not covered. And indeed, if the arc $X \to Y$ is reversed, the resulting network $X \leftarrow Y \leftarrow Z$ has a different set of v-structures and is not Markov equivalent to G.