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Assume that $G = (V, E)$ is a bipartite graph with $V = L \cup R$ and $|L| = |R|$. Assume further that G satisfies Hall's condition, i.e. $\forall A \subseteq L\ |A| \leq |N(A)|$. We prove that G admits a perfect matching.

Proof Straight from Hall's condition, we see that $\forall v \in L\ v$ is connected to at least one vertex in R, i.e. $\forall A \subseteq L\ |A| = 1$, $\exists A' \subseteq R\ |A'| = 1$ and there is a perfect matching in $A \cup A'$. We consider a non-empty subset L_p of L for which there exists a subset R_p of R $|L_p| = |R_p|$ and there exits a perfect matching in $L_p \cup R_p$. Apparently, the A and A' just mentioned is an instance of L_p and R_p. Besides, we denote $L \setminus L_p$ as L_{left}, similarly, $R \setminus R_p$ as R_{left}. Our strategy is to expand A and A' by taking vertices (and corresponding edges) from L_{left} and R_{left}, at the same time ensuring that there exists a perfect mathing in $L_p \cup R_p$. We will show that this expansion eventually exhausts L_{left} and R_{left}, which proves the claim.

When expanding L_p and R_p, there are two possible cases to consider:

1. There does not exist any edge connecting vertices in L_{left} and R_{left}, that is, all the vertices in L_{left} are connected to vertices in R_p and all the vertices in R_{left} are connected to vertices in L_p;

2. There exists at least one edge connecting a vertex in L_{left} and a vertex in R_{left}.

For Case 2, we just take that pair of vertices out of L_{left} and R_{left} and add them to L_p and R_p respectively. The union of the prefect matching existing in L_p and R_p before the expansion and the edge between the two vertices added is still a perfect matching.

Note that Case 1 is impossible until $|L_p| \geq |L_{left}|$. Otherwise, it contradicts Hall's condition, which requires that $(|L_p| =)|R_p| \geq |N(L_{left})| \geq |L_{left}|$. Besides, once Case 1 appears, what follows will be always Case 1.

For Case 1, let $R_1 = N(L_{left})$, therefore $R_1 \subseteq R_p$. In addition, every vertex in R_1 is connected to a unique vertex in L_p (due to the existence of a perfect matching), all of which together is denoted by L_1. Let $B = L_1 \cup L_{left}$. According to Hall's condition, $|N(B)| \geq |B| > |L_1| = |R_1|$, which means that there is at least one vertex w_2 in $N(B)$ $w_2 \notin R_1$. w_2 can be either in R_{left} or $R_p \setminus R_1$. If w_2 is in R_{left}, we can trace back one of its neighbours v_1 in L_1, which must be connected to a vertex w_1 in R_1 within the perfect matching and w_1 must be connected to a vertex v_2 in L_{left}. Then, we can extend the perfect matching by including the edges (v_1, w_2) and (v_2, w_1) and excluding the edge (v_1, w_1). Thus, v_2 is added to L_p and w_2 is added to R_p. On the other hand, if w_2 is in $R_p \setminus R_1$, then let $R_2 = N(B)$. Clearly $R_1 \subset R_2$. Also, let L_2 be the set of vertices in L_p which connect to R_2. We have $L_1 \subset L_2$. We now set $B = L_2 \cup L_{left}$. Then, we face the exactly same situation as before and apply the same reasoning. If the iteration continues, the options for w_2 (in R_p) is reduced by one vertex at a time and B ($\subset L_p$) is expanded by one vertex at a time. Eventually, R_p (and L_p) will be exhausted, which gives way to the senario that w_2 has to be in R_{left}. Then, L_p and R_p can be expanded as described above.