# Homework 1

## Fragments of Structural and Algorithmic Graph Theory
## University of Helsinki, September 2015

The homework is due **Friday September 11, 2015**. Solutions should be handed in pdf by e-mail to `tomescu@cs.helsinki.fi`. If you are taking the course for 2 credits, submit solutions to problems 1 and 2. If you are taking the course for 3 credits, submit solutions to all three problems. Write clearly. The solution sheet should contain the following data: first name, last name, university, student number, number of credits (2 or 3). If you are a student of a university other than the University of Helsinki, write also the name and contact details of the person responsible for your grade registration. You are supposed to solve the homework on your own.

1. *(Basics of graphs.)*

   (a) Let $n$ be a positive integer. How many graphs are there with vertex set $\{1, \ldots, n\}$? Justify your answer.

   (b) Show that a graph $T$ is a tree if and only if every pair of points in $T$ is connected by a unique path.

   (c) Show that a connected graph on $n$ vertices has at least $n - 1$ edges.

   *(Hint: Use induction on n.)*

2. *(Polynomial equivalence of problems.)*

   Prove that the following problems are all polynomial-time equivalent, that is, if any of these problems can be solved in polynomial time, then all of them can.

   - Clique: Given a graph $G$ and an integer $k$, does there exist a clique of size at least $k$ in $G$?
   - FindClique: Given a graph $G$ and an integer $k$, find a clique of size at least $k$ in G if one exists.
   - MaxClique: Given a graph $G$, find the size of the largest clique in the graph.
   - FindMaxClique: Given a graph $G$, find a clique of maximum size in $G$.

***********************************

3. *(Proving NP-completeness.)*

   (a) Hitting Set is the following decision problem:
   **Input**: A finite set $U$, a family $\mathcal{S}$ of nonempty subsets of $U$, an integer $k$.
   **Question**: Is there a set $H \subseteq U$ of at most $k$ elements such that $(\forall S \in \mathcal{S})(H \cap S \neq \emptyset)$?

   Show that the Hitting Set problem is NP-complete.

   (b) Connected Dominating Set is the following decision problem:
   **Input**: A connected graph $G$, an integer $k$.
   **Question**: Is there a dominating set $D \subseteq V(G)$ of at most $k$ vertices such that the subgraph of $G$ induced by $D$ is connected?

   Show that the Connected Dominating Set problem is NP-complete.