Homework 3

Fragments of Structural and Algorithmic Graph Theory

University of Helsinki, September 2015

The homework is due **Friday September 11, 2015**. Solutions should be handed in pdf by e-mail to `tomescu@cs.helsinki.fi`. If you are taking the course for 2 credits, submit solutions to problems 1 and 2. If you are taking the course for 3 credits, submit solutions to all three problems. Write clearly. The solution sheet should contain the following data: first name, last name, university, student number, number of credits (2 or 3). If you are a student of a university other than the University of Helsinki, write also the name and contact details of the person responsible for your grade registration. You are supposed to solve the homework on your own.

1. *(An upper bound on the chromatic number.)*

   Let v be a vertex in a connected graph $G = (V, E)$, and let G_r denote the subgraph of G induced by the vertices at distance exactly r from v (for all $r \geq 0$). Prove that

   $$\chi(G) \leq \max_r (\chi(G_r) + \chi(G_{r+1})) \,.$$

2. *(Further polynomial cases of 3-coloring.)*

   Let k be a positive integer.

   Give a polynomial time algorithm for each of the following two decision problems:

   - 3-Colorability of graphs with small vertex cover number
     Input: A graph $G = (V, E)$ such that G has a vertex cover of size at most k.
     Question: Is G 3-colorable?
   - 3-Colorability of graphs with small independence number
     Input: A graph $G = (V, E)$ such that each independent set of G has size at most k.
     Question: Is G 3-colorable?

   Justify the correctness of your algorithms and analyze their time complexity.

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3. *(Problems related to coloring.)*

   (a) Show that the Clique Cover problem is NP-complete. (The problem takes an input a graph G and an integer k, and the question is whether $V(G)$ can be covered with k cliques.)

   Hint: Reduce the Chromatic Number *problem to this problem.*

   (b) An **edge dominating set** in a graph $G = (V, E)$ is a subset of edges $D \subseteq E$ such that: $(\forall e \in E \setminus D)(\exists f \in D)(e$ and f have an endpoint in common$)$. Show that for every positive integer k, the 3-Edge Colorability problem (*"Given a graph G, is G 3-edge-colorable?"*) can be solved in polynomial time for graphs that contain an edge dominating set of size at most k.