Homework 4

Fragments of Structural and Algorithmic Graph Theory
University of Helsinki, September 2015

The homework is due **Friday September 11, 2015**. Solutions should be handed in pdf by e-mail to `tomescu@cs.helsinki.fi`. If you are taking the course for 2 credits, submit solutions to problems 1 and 2. If you are taking the course for 3 credits, submit solutions to all three problems. Write clearly. The solution sheet should contain the following data: first name, last name, university, student number, number of credits (2 or 3). If you are a student of a university other than the University of Helsinki, write also the name and contact details of the person responsible for your grade registration. You are supposed to solve the homework on your own.

1. *(Hereditary graph classes.)*

   (a) Show the following statement:
   For every two sets of graphs M_1 and M_2 we have:
   $$\mathit{Free}(\mathcal{M}_1) \subseteq \mathit{Free}(\mathcal{M}_2) \text{ if and only if } (\forall G_2 \in M_2)(\exists G_1 \in M_1)(G_1 < G_2)\,.$$

   (b) Write a pseudocode for a polynomial time algorithm for computing the chromatic number of a given *cograph* G. Justify the correctness of your algorithm.

2. *(Squares of paths.)*

   For a positive integer n, the graph P_n^2 is the square of a path of order n. It can be defined by $V(P_n^2) = \{1, \ldots, n\}$ and $E(P_n^2) = \{\{i, j\} \mid 1 \le i < j \le n \ \wedge \ |i - j| \le 2\}$. Fig. 1 shows the graphs P_7^2 and P_8^2. Let $\mathcal{P}$ denote the set of all graphs isomorphic to some P_n^2 for $n \ge 1$.

   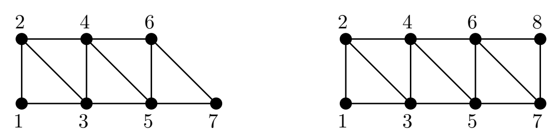

   Figure 1: Path squares of order 7 and 8

   (a) Is $\mathcal{P}$ a hereditary class? Justify.
   (b) Does each graph $\mathcal{P}$ have a perfect elimination ordering? Justify.
   (c) Is it true that every graph in $\mathcal{P}$ is chordal? Justify.
   (d) Is it true that every graph in $\mathcal{P}$ is split? Justify.
   (e) Is it true that every graph in $\mathcal{P}$ is perfect? Justify.
   (f) Determine the chromatic number $\chi(P_n^2)$ for all $n \ge 1$. Justify your answer.

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3. *(The dominating set problem in a subclass of perfect graphs.)*

   A **dominating vertex** in a graph G is a vertex adjacent to all other vertices.

   (a) Prove the following proposition.
   Every connected $\{P_4, C_4\}$-free graph has a dominating vertex.
   Hint: Prove that every vertex of maximum degree is dominating.

   (b) How can one compute in linear time the domination number $\gamma(G)$ (i.e., the minimum size of a dominating set) of a given $\{P_4, C_4\}$-free graph G? Justify.